Math 206B: Algebra
Final Exam Practice Problems

The goal of this document is to provide you with some practice problems for the Final exam about the material of Sections 11.1, 11.2, and 12.1 of Dummit and Foote.

Vector Spaces: Dummit and Foote Exercises

1. Exercise 2 Section 11.1 and Exercise 1 Section 11.2.
   In this pair of related exercises you show that two certain sets of polynomials give bases for the space of polynomials in $\mathbb{Q}[x]$ of degree at most 5. In the next exercise you compute the transition matrix between these bases.

2. Exercise 3 Section 11.2.
   In this exercise you describe the transition matrix between two particular bases for the space of polynomials in $F[x]$ of degree at most n.

3. Exercise 3 Section 11.1.
   In this exercise you compute the value of a linear transformation at any point starting from the values taken at a basis.

4. Exercise 7 Section 11.1.
   This exercise gives some properties of a linear transformation that satisfies $\varphi^2 = 0$.
   Note: This exercise came up as Algebra Comprehensive Exam Spring 2011: #2 and also as Algebra Comprehensive Exam Spring 2006: #L10.

5. Exercise 7 Section 11.2.
   In this exercise you prove a basic property of similar matrices.

6. Exercise 8 Section 11.2.
   In this exercise you give a relation between the matrix representing a linear transformation and the eigenvectors of that linear transformation. This kind of fact is pointing in the direction of Canonical Forms, which we will discuss at the beginning of Math 206C.

7. Exercise 11 Section 11.2.
   In this exercise you prove some things about a matrix representing an idempotent linear transformation.

Vector Spaces: Comprehensive Exam Problems

1. Algebra Advisory Exam, Fall 2009: #9
   Suppose V is a vector space, and let $\mathrm{GL}(V)$ be the group of all invertible linear transformations from V to itself. Suppose G is a subgroup of $\mathrm{GL}(V)$, and define

$$\begin{aligned} R \;=\; & \{\text{all linear transformations } T\colon V \to V \\ & \quad \text{such that } T(g(v)) = g(T(v)) \quad \text{for every } g \in G \text{ and } v \in V\}. \end{aligned}$$

   (a) Show that R is a ring.

(b) Suppose further that if W is any subspace of V such that $g(W) \subseteq W$ for every $g \in G$, then either $W = 0$ or $W = V$. Prove that if $T \in R$ and T is not the zero transformation, then T is invertible and $T^{-1} = R$.
Hint: If $T \in R$, what can you say about the kernel and image of T?

2. Algebra Comprehensive Exam Fall 2005: #L7
If V is a vector space and U, W are subspaces of V, then W is a complement of U if and only if $V = U \oplus W$. That is, V is the internal direct sum of U and W:

   - $V = U + W$;
   - $U \cap W = \{0\}$.

   Prove that any subspace of a vector space has a complement.

3. Algebra Comprehensive Exam Spring 2019: #5
Let R be the ring of $n \times n$ matrices with entries in a field F. Prove that every nonzero element of R is either a unit or a zero divisor.

4. Algebra Comprehensive Exam Spring 2018: #5
Assume that V is an n-dimensional vector space and $T\colon V \to V$ is a linear transformation. Assume $T^m = 0$ for some integer $m \geq 1$. Prove that $T^n = 0$.

5. Algebra Comprehensive Exam Spring 2018: #6
Prove or give a counter-example: If U_1, U_2 are subspaces of a vector space V and if $v_1, v_2 \in V$ are such that
$$v_1 + U_1 = v_2 + U_2,$$
then $U_1 = U_2$.

6. Algebra Comprehensive Exam Spring 2017: #8
Let V be a vector space and let W_1, W_2 be subspaces of V. Prove that $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

7. Algebra Comprehensive Exam Spring 2015: #6
Let V be a vector space and let W_1, W_2 be subspaces of V. Suppose $\dim V = n$, $\dim W_1 = r$, and $\dim W_2 = s$. What are the possible values of $\dim(W_1 \cap W_2)$? Prove your answers.
Note: This problem also came up in a slightly different form as Algebra Comprehensive Exam Spring 2013: #6.

8. Algebra Comprehensive Exam Spring 2008: #L8
Let V be a finite dimensional vector space and let $T\colon V \to V$ be a linear transformation. Prove that T is injective if and only if it is surjective.

Modules over a PID: Dummit and Foote Exercises

The following problems come from Section 12.1 of Dummit and Foote on 'Modules over a PID: The Basic Theory'. Solving some of these problems will be useful not only for this material from the very end of the course but will also be helpful more generally for solving problems about R-modules.

1. Exercise 1 Section 12.1.
   In this exercise you prove a basic fact about the rank of a torsion module.

2. Exercise 2 Section 12.1.
   In this exercise you consider a module of rank n and its quotient by a free module of rank n. This shows that if you can determine the rank of a module, you can use it to show that a particular quotient is a torsion module.

3. Exercise 3 Section 12.1.
   This is the exercise I mentioned in lecture about the rank of a direct sum of two modules.

4. Exercise 4 Section 12.1.
   This exercise relates the rank of a module, the rank of submodule, and the rank of a quotient. Note that we recently considered the analogous question in the special case of vector spaces over a field.

5. Exercise 7 Section 12.1.
   I mentioned this exercise in lecture about taking the quotient of a direct sum of some modules but the direct sum of some submodules.