Math 206B: Algebra
Final Exam: Things to Know

This document continues the ones from earlier in the course giving a list of definitions, examples, and results that you should know for the Final Exam on Thursday, 3/18. Here I will include only things that were not covered on the 'Things to Know' documents for Midterms 1 and 2.

Tensor Products

Definitions

1. The tensor product of two R-modules.
   The universal mapping property of tensor products.
2. Elementary tensors.

Examples

1. $\mathbb{Q} \otimes_{\mathbb{Z}} A = 0$ for a finite abelian group A.
2. $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z} = 0$. (Same proof idea.)
3. Example of a tensor that is provably not an elementary tensor. $e_1 \otimes e_1 + e_2 \otimes e_2$ in $F \otimes_R F$ where F is a freed R-module of rank $n \geq 2$. (Example 4.11 in Conrad's notes)
4. $R \otimes_R M \cong M,\ R \otimes_R R \cong R$. (See Theorem 4.5 in Conrad's notes below for $I = 0$.)
5. Let A be an abelian group. $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} A \cong A/nA$. (See Theorem 4.5 in Conrad's notes below for $R = \mathbb{Z},\ I = n\mathbb{Z}$.)
6. Take $K = V$ in Theorem 4.21 below. $K \otimes_R K \cong K$.

Theorems

1. The tensor product of two R-modules exists. (Theorem 3.2 in Conrad's 'Tensor Products' notes.) You will not be asked for the full proof of a result like this, but you should be familiar with the main ideas (the tensor product is constructed as a quotient of the free module on the elements of $M \times n$).
2. Not only are elements of $M \otimes_R N$ finite R-linear combinations of elementary tensors $m \otimes n$, but every element of $M \otimes_R N$ is a finite sum of elementary tensors.
3. Spanning sets of tensor products. (Theorem 3.3 of Conrad's notes.)
4. $m \otimes 0 = 0 \otimes n = 0$. (Theorem 3.5 of Conrad's notes.)

5. $\mathbb{Z}/a\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/b\mathbb{Z} \cong \mathbb{Z}/\gcd(a,b)\mathbb{Z}$. (Theorem 4.1 of Conrad's notes.)

6. A basis for the tensor product of two free modules. (Theorem 4.9 of Conrad's notes.)

7. $R/I \otimes_R R/J \cong R/(I+J)$. (Theorem 4.3 of Conrad's notes– we stated the map giving the isomorphism, but did not do the proof.)

8. $R/I \otimes_R M \cong M/IM$. (Theorem 4.5 of Conrad's notes– we stated the map giving the isomorphism, but did not do the proof.)

9. Let R be an integral domain with field of fractions K. Let V be a vector space over K. Then $K \otimes_R V \cong V$. (Theorem 4.21 of Conrad's notes– we stated the map giving the isomorphism, but did not do the proof.)

10. $M \otimes_R N \cong N \otimes_R M$. (Theorem 5.1 of Conrad's notes– we stated the map giving the isomorphism, but did not do the full proof.)

11. $(M \otimes_R N) \otimes_R P \cong M \otimes_R (N \otimes_R P)$. (Theorem 5.2 of Conrad's notes– we stated the map giving the isomorphism, but did not do the full proof.)

12. $M \otimes_R (N \oplus P) \cong (M \otimes_R N) \oplus (N \otimes_R P)$. (Theorem 5.3 of Conrad's notes– we stated the map giving the isomorphism, but did not do the full proof.)

13. $\mathrm{Bil}_R(M, N; P) \cong \mathrm{Hom}_R(M \otimes_R N; P)$ as R-modules. (First part of Theorem 5.7 of Conrad's notes– we stated the map giving the isomorphism, but did not do the full proof.)

Vector Spaces

Definitions

1. Vector space.
2. Linear Transformation.
3. Dimension of a vector space.
4. Null space and nullity of a linear transformation.
5. Rank of a linear transformation.
6. What it means for a linear transformation to be nonsingular.
7. What it means for a linear transformation to represent a linear transformation φ with respect to bases $\mathcal{B}$ and $\mathcal{E}$.
8. What it means for a matrix to be nonsingular.
9. What it means for two $n \times n$ matrices to be similar. What is means for two linear transformations from V to V to be similar.
10. The transition matrix/change of basis matrix from the basis $\mathcal{B}$ to the basis $\mathcal{E}$.

Examples

1. $F[x]/(f(x))$ is a vector space. $\overline{1}, \overline{x}, \ldots, \overline{x^{n-1}}$ is a basis for it.

Theorems

1. Any finite minimal spanning set for a vector space is a basis. (Proposition 1 Sec.11.1.)
2. Any finite spanning set of vector space contains a basis for it. (Corollary 2 Sec.11.1.)
3. The Replacement Theorem for bases and linearly independent vectors. (Theorem 3 in Sec.11.1.) Also, its corollary showing that if V is a finite dimensional vector space then every basis has exactly n elements.
4. 'Building Up Lemma'– Any set of linearly independent vectors is contained in a basis. (Corollary 5 in Sec.11.1.)
5. An n-dimensional vector space V over F satisfies $V \cong F^n$. (Theorem 6 in Sec. 11.1.)
6. The number of ordered bases for an n-dimensional vector space over a finite field of size q. The number of k-dimensional subspaces of such a vector space. (Theorem from the start of Lecture 23– Example in Sec. 11.1.)

7. The dimension of the quotient of two vector spaces. (Theorem 7 Sec. 11.1.) The 'Rank-Nullity' theorem for linear transformations between vector spaces. (Corollary 8 Sec. 11.1.)

8. Let V be an n-dimensional vector space over a finite field of size q. The size of the general linear group $\mathrm{GL}(V)$. (Lecture 23– Example in Sec. 11.1.) $|\mathrm{GL}_n(\mathbb{F}_q)|$ (Corollary 14 in Sec. 11.2.)

9. $\mathrm{Hom}_F(V, W) \cong M_{m\times n}(F)$, an isomorphism of vector spaces. (Theorem 10 in Sec. 11.2.)

10. Composing linear transformations corresponds to multiplying matrices. (Theorem 12 in Sec. 11.2.) The corollary of this: Matrix multiplication is associative and distributive. An $n \times n$ matrix is nonsingular if and only if it is invertible.

11. $\mathrm{Hom}_F(V, C) \cong M_{n\times n}(F)$, an isomorphism of vector spaces and an isomorphism of rings. (Corollary 14 in Sec. 11.2.)

12. Two matrices associated to the same linear transformation with respect to different bases are similar. Two similar matrices represent the same linear transformation with respect to different choices of a basis for V. (Sec. 11.2.)

Modules over a PID

Definitions

1. Invariant factors of a finitely generated R-module over a PID. Elementary divisors of a finitely generated R-module over a PID.

2. Noetherian R-module.

3. The Rank of an R-module.

Theorems

1. If R is a PID, every cyclic R-module C satisfies $C \cong R/(a)$ for some principal ideal $(a) \subseteq R$. (Sec. 12.1.)

2. Classification of Modules over a PID: Existence: Invariant Factor Form, Existence: Elementary Divisor Form, The Primary Decomposition Theorem, Uniqueness. (Sec. 12.1.)

3. Equivalent conditions for an R-module to be Noetherian (Theorem 1 in Sec. 12.1.)