Math 206B: Algebra
Homework 2

Due Friday, January 22nd at 12:00PM.
Please email `nckaplan@math.uci.edu` with questions.

All exercises are from Dummit and Foote.

1. Exercise 3 of Section 8.1.
   We have seen that in the quadratic number rings of Section 7.1, we can identify the units by finding the elements of norm ± 1. In this exercise you study elements of minimum norm in a general Euclidean domain.

2. Exercise 4 of Section 8.1.
   I mentioned part (b) of this exercise in lecture. You apply what we know about the Euclidean algorithm to the special case of $\mathbb{Z}$ to give a complete description of the integer solutions to the linear Diophantine equation $ax + by = N$.

3. Exercise 6 of Section 8.1.
   This exercise is also about a linear Diophantine equation in the integers, but now you are taking linear combinations of two integers with *nonnegative* coefficients. This question is related to the theory of *numerical semigroups*, which was the subject of an REU that I participated in as an undergraduate.

4. Exercise 7 of Section 8.1.
   We know that $\mathbb{Z}[i]$ is a PID, so the ideal $(85, 1 + 13i)$ is principal. In this exercise you find a generator for this ideal (and for one other example) by applying the Euclidean algorithm.

5. Exercise 8a of Section 8.1 (but only for $D = -3$).
   We proved that $\mathbb{Z}[i]$ is a Euclidean domain with respect to the field norm N in $\mathbb{Q}(i)$. In this exercise you prove the analogous fact for $\mathbb{Z}\left[\frac{1+\sqrt{-3}}{2}\right]$. The main idea is the same, but since $-3 \equiv 1 \pmod 4$, the details are a little different. (This exercise has many parts, but they are all similar and I do not think it's worth the effort to repeat all the details.)

6. Exercise 9 of Section 8.1.
   In this exercise you prove that $\mathbb{Z}[\sqrt{2}]$ is a Euclidean domain. The strategy is similar to the strategy in the previous problem, but since D is positive the details are different.

7. Exercise 11 of Section 8.1 and Exercise 2 in Section 8.2.
   This week we talked quite a bit about greatest common divisors. For integers, gcds are often paired with lcms. In the first exercise you develop the basics of the least common multiple of two elements in a commutative ring. In the second exercise you show that in a PID any two nonzero elements do have a least common multiple.

8. Exercise 1 of Section 8.2.
   In Section 7.6 when we discussed The Chinese Remainder Theorem, we talked about comaximal ideals. In this exercise you relate this concept to the greatest common divisor.

9. Exercises 3 and 8 of Section 8.2.
   In these two exercises you show that the property of being a PID is preserved under certain operations.

10. Exercise 5 of Section 8.2.
In lecture, of the ways that we saw that $\mathbb{Z}[\sqrt{-5}]$ is not a PID was by producing an ideal generated by two elements that we proved was not principal. In this exercise you prove that certain other ideals are not principal. You then see that products of these non-principal ideals can give principal ideals.