Math 206B: Algebra
Homework 4

Due Wednesday, February 10th at 12:00PM.
Please email `nckaplan@math.uci.edu` with questions.

1. Exercise 3 of Section 9.1
   This is an exercise that I mentioned in lecture, which gives a more general version of the observation that $\mathbb{Z}[x][y] \cong \mathbb{Z}[y][x]$.

2. Exercise 5 of Section 9.1
   This exercise is similar to Exercise 4 of Section 9.1, which was the last example we talked about in Lecture 8.

3. Exercises 1 and 2 of Section 9.2
   First, note that Exercise 3 of Section 9.2 is part of Proposition 15 of Section 9.5, which we proved in Lecture 11.

   Let p be a prime and k be a positive integer. These three exercises taken together show that in order to construct a finite field of order p^k all we need to do is find an irreducible polynomial of degree k in $(\mathbb{Z}/p\mathbb{Z})[x]$. In 206C we will give a formula for the **number** of irreducible polynomials in $(\mathbb{Z}/p\mathbb{Z})[x]$ of degree k (it is on page 588 in Section 14.3). In particular, we will see that there is always at least one such polynomial. Combining this with Exercise 3 of Section 7.5, which we solved on HW1, shows that a finite field of order q exists if and only if q is a power of a prime.

4. Exercise 4 of Section 9.2
   One idea for this problem is to imitate Euclid's proof that there are infinitely many primes in $\mathbb{Z}$: Suppose that there are only finitely many primes $\{p_1, \ldots, p_N\}$. What can we say about $p_1 \cdots p_N + 1$?

5. Exercise 11 of Section 9.2
   This is an exercise I discussed in lecture. It is the analogue in $\mathbb{Q}[x]$ of Exercise 4 in Section 8.1, which you solved on an earlier HW.

6. Exercises 5 of Section 9.2
   In this exercise you determine the structure of the set of ideals of $F[x]/(p(x))$. This ends up being very useful in solving several previous Algebra Comprehensive and Qualifying Exam problems.

7. Exercise 1 of Section 9.3
   This exercise asks about the relationship between factorizations in $R[x]$ and in $F[x]$, and uses these ideas to show that a certain rings is not a UFD.

8. Exercise 2 of Section 9.3
   This exercise gives a nice fact about products of polynomials with rational coefficients.

9. Exercise 3 of Section 9.3.
   This exercise is basically the same as Spring 2020 Algebra Comprehensive Exam #4. It is a nice example showing that sometimes you can find a ring that is not a UFD inside of a ring that is a UFD.

10. Exercise 3 of Section 9.4
    This exercise gives a nice example of different technique for showing that a polynomial is irreducible.