Math 206B: Algebra
Midterm 1 Practice Problems

The goal of this document is to provide you with some practice problems for Midterm 1 from past Algebra Comprehensive and Qualifying Exams. I have made an attempt to divide up the problems by topic and also to indicate which ones we have already proven in lecture.

In these problems you should use the definitions for rings given in Dummit and Foote. For example, a ring R does not necessarily have an identity, and a ring homomorphism between two rings with identity does not necessarily take the identity of the first ring to the identity of the second ring.

Quadratic Rings

Quadratic rings provide interesting examples of many of the properties that are discussed in Chapter 8 of Dummit and Foote, so they often come up on Algebra Comprehensive and Qualifying Exams.

1. Algebra Comprehensive Exam Spring 2019: 4
   Let $\mathfrak{p}$ be a nonzero prime ideal in $\mathbb{Z}[\sqrt{3}]$. Prove that $\mathfrak{p}$ contains a unique prime integer.
   Note: We spent quite a while talking about this problem in Office Hours on Monday, January 25th. I have written a comment about this on the Discussion Board topic about Midterm 1.

2. Algebra Qualifying Exam Spring 2007: 3
   Let p be an odd positive integer. Show that if n is an integer such that p divides $n^2 + 1$ then $p \equiv 1 \pmod 4$.
   Note: This is quite close to something we proved in the lecture about factoring in $\mathbb{Z}[i]$.

3. Advisory Advisory Exam Fall 2010: R4
   Show that the quotient ring $\mathbb{Z}[i]/(3)$ is a field with 9 elements while $\mathbb{Z}[i]/(2)$ is not a field.
   Note: In some sense we solved this problem in the lecture about factoring in $\mathbb{Z}[i]$.

4. Algebra Qualifying Exam Spring 2019: 5
   Recall that $\mathbb{Z}[i]$ has a Euclidean norm.

   (a) Prove that for every nonzero ideal I in $\mathbb{Z}[i]$, $\mathbb{Z}[i]/I$ is finite.

   (b) What ring is $\mathbb{Z}[i]/(1 + i)$ isomorphic to?

5. Algebra Qualifying Exam Fall 2015: 10c
   Show that $\mathbb{Z}[\sqrt{10}]$ is not a UFD.

6. Algebra Qualifying Exam Fall 2004: 11

   (a) Show that (13) is not a prime ideal of $\mathbb{Z}[\sqrt{10}]$.

   (b) Show that (17) is a prime ideal of $\mathbb{Z}[\sqrt{10}]$.

7. Algebra Comprehensive Exam Spring 2020: 6
   Let $I \subseteq \mathbb{Z}[\sqrt{-3}]$ be a non-zero ideal. Prove that the quotient ring $\mathbb{Z}[\sqrt{-3}]/I$ is finite.

8. Algebra Qualifying Exam Spring 2006: 3
   For which primes p can one find a nonzero homomorphism from $\mathbb{Z}[i]$ to $\mathbb{Z}/p\mathbb{Z}$?
   Note: A version of this came up as Algebra Qualifying Exam Fall 2017: 9b.

9. Algebra Qualifying Exam Spring 2020: 4
   Show that $\mathbb{Z}[\sqrt{-2}]$ is a Euclidean domain. Find an irreducible element of this ring that is not in $\mathbb{Z}$.
   Note: This also came up as Algebra Qualifying Exam Spring 2007 #5.

10. Algebra Qualifying Exam Fall 2019: 4

    (a) Is $3+\sqrt{2}$ a prime in $\mathbb{Z}[\sqrt{2}]$?

    (b) Identify the quotient $\mathbb{Z}[\sqrt{2}]/(3+\sqrt{2})$. Which ring is it?
    Note: You may use the fact that $\mathbb{Z}[\sqrt{2}]$ has a Euclidean norm.

11. Algebra Qualifying Exam Spring 2011: 2
    Prove that the ideal $(2, 3-\sqrt{-5})$ is a maximal ideal in $\mathbb{Z}[\sqrt{-5}]$.

12. Algebra Qualifying Exam Spring 2010: 3

    (a) Show that $\mathbb{Z}[\sqrt{-5}]$ is not a UFD.

    (b) Factor the principal ideal (6) into a product of prime ideals in $\mathbb{Z}[\sqrt{-5}]$.

    Note: This is basically the last problem on HW3, the one you do not have to hand in. This also came up as Algebra Qualifying Exam Spring 2005 #4, as Algebra Qualifying Exam Winter 2003 #8, and as Algebra Qualifying Exam Winter 2000 #9.

Rings of Fractions and the Chinese Remainder Theorem

Not all of these problems directly use the material from Sections 7.5 and 7.6 of Dummit and Foote. Some of them are about examples of rings that are like ones that arise from taking rings of fractions.

1. Algebra Advisory Exam Fall 2008: G2
   Prove that $13^{20} - 1$ is divisible by 33.
   Note: This is listed as a 'Group Theory' problem on the exam, and that is correct, but how can we use the Chinese Remainder Theorem here?

2. Algebra Qualifying Exam Fall 2009: 4
   Let A be a commutative ring with identity. Let S be a non-empty multiplicative subset of A such that $0 \notin S$. Let P be an ideal of A, which is maximal in the set of all ideals that do not intersect S. Prove that P is a prime ideal.

3. Algebra Qualifying Exam Spring 2001: 6
   Let $\varphi(\cdot)$ denote the 'Euler phi' function. That is, $\varphi(n)$ is the number of positive integers $x \leq n$ for which $\gcd(x, n) = 1$.

   (a) Calculate $\varphi(p^r)$ where p is prime.

   (b) If R_1 and R_2 are rings, show that $(R_1 \times R_2)^* \cong R_1^* \times R_2^*$.

   (c) If the prime factorization of n is given by

   $$n = p_1^{r_1} \cdots p_k^{r_k},$$

   give a formula for $\varphi(n)$ in terms of $p_1, \ldots, p_k, r_1, \ldots, r_k$. Justify your answer.

4. Algebra Qualifying Exam Fall 2018: 4
   Let $R \subset \mathbb{Q}$ be the subring consisting of fraction $\frac{a}{b}$ with $a \in \mathbb{Z}$ and $b = 2^k 3^l$ with $k, l \geq 0$. Describe the ideals of R. Is R a PID?

5. Algebra Qualifying Exam Spring 2011: 10e
   True or False: If R is a commutative ring with identity and R has a unique prime ideal then R is a field.

Statements from Lecture

It is not uncommon for problems on the Comprehensive and Qualifying Exams to ask you to prove a statement directly from lecture. Here are some examples.

1. Algebra Advisory Exam Fall 2005: R6
   Prove that a PID R satisfies the Ascending Chain Condition on ideals.
   Note: We proved a more general version of this statement in our lecture on UFDs. Dummit and Foote give an argument for this more specific case on page 288.

2. Algebra Comprehensive Exam Spring 2017: 3
   (a) Define Euclidean domain.
   (b) Prove that every Euclidean domain is a PID.

   Note: We proved this statement in our lecture on Euclidean domains. This also came up as Algebra Qualifying Exam Spring 2006 9a.

3. Algebra Comprehensive Exam Spring 2005: R2
   (a) If R is an integral domain, show that any prime element is irreducible.
   (b) If R is a UFD show that any irreducible element is prime.

   Note: The first statement is Proposition 10 in Section 8.3. It also came up as Algebra Comprehensive Exam Spring 2008 R5. The second statement is Proposition 12 in Section 8.3. Both parts of this problem came up again as Algebra Qualifying Exam Spring 2017 #3.

4. Algebra Qualifying Exam Fall 2015: 6
   Let F be a field. Prove that every ideal of $F[x]$ is principal.
   Note: This follows directly from Theorem 3 in Section 9.2.

5. Algebra Qualifying Exam Spring 2015: 2
   (a) Define UFD.
   (b) Define PID.
   (c) For the properties "UFD" and "PID" give an example of an integral domain that
      i. satisfies both properties,
      ii. satisfies neither property,
      iii. satisfies one property but not the other.

6. Algebra Qualifying Exam Fall 2014: 3
   True or False: If R is a PID and P is a nonzero prime ideal of R, then P is a maximal ideal of R.
   Note: This is Proposition 7 in Section 8.2.

7. Algebra Qualifying Exam Fall 2014: 7
   (a) Let R be an integral domain. Prove that $R[x]^x = R^x$, that is, the units in $R[x]$ are the constant polynomials whose constant term is a unit in R.
   (b) Find an example of a commutative ring R and nonconstant polynomials $f(x), g(x) \in R[x]$ such that $f(x)g(x) = 1$.

   Note: Part(a) is Proposition 4(2) in Section 7.2.

Additional Problems

Here we include some additional problems that did not fit so nicely into one of the groups described above.

1. Algebra Comprehensive Exam Spring 2017: 10b
   True or False: Let R be a commutative ring with unity $1 \neq 0$. There exists a field k and a surjective ring homomorphism $f\colon R \to k$.

2. Algebra Comprehensive Exam Fall 2011: 1
   Let R be an integral domain. A nonzero non-unit element $s \in R$ is said to be *special* if, for every element $a \in R$, there exist $q, r \in R$ with $a = qs + r$ such that r is either 0 or a unit in R.

   (a) If $s \in R$ is special, prove that the principal ideal (s) generated by s is maximal in R.

   (b) Show that every polynomial in $\mathbb{Q}[x]$ of degree 1 is special in $\mathbb{Q}[x]$.

3. Algebra Qualifying Exam Spring 2018: 4
   Let R be a UFD and assume that any ideal I in R is finitely generated. Show that for every nonzero $a, b \in R$ and any greatest common divisor d of a and b, $d = ar + sb$ for some $r, s \in R$. Prove that R is a PID.
   Note: This is closely related to Exercise 7ab of Section 8.2.

4. Algebra Qualifying Exam Fall 2011: 3
   A commutative ring R with identity $1 \neq 0$ is boolean if $x^2 = x$ for every $x \in R$. Find all boolean integral domains. Prove that every prime ideal in a boolean ring is maximal.