Math 206B: Algebra
Midterm 1
Friday, January 29, 2021.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (6 Points)	
2 (13 Points)	
3 (10 Points)	
4 (12 Points)	
Total	

Problems	
5 (6 Points)	
6 (10 Points)	
7 (6 Points)	
8 (10 Points)	
Total	

Problems

1. Describe all maximal ideals in $\mathbb{Z}/n\mathbb{Z}$ where n is a positive integer.

2. (a) True or False: If R is an integral domain and $I \cap J = \{0\}$ where I and J are ideals in R, then $I = \{0\}$ or $J = \{0\}$.

   (b) Let R be a ring with identity $1 \neq 0$. Define the *characteristic* of R.

   (c) True or False: If K and L are fields and $\varphi\colon K \to L$ is a ring homomorphism that takes the identity of K to the identity of L, then K and L must have the same characteristic.

3. A ring R is called Noetherian if every strictly increasing chain of ideals $I_1 \subsetneq I_2 \subsetneq \cdots$ must be finite in length. Prove that if R is Noetherian, then every ideal of R is finitely generated. Prove that $\mathbb{Z}$ is Noetherian.

4. (a) If R is an integral domain, show that any prime element is irreducible.

   (b) If R is a UFD show that any irreducible element is prime.

5. (a) Find a decomposition of 11 into a product of irreducible elements in $\mathbb{Z}[i]$.

   (b) Find a decomposition of 13 into a product of irreducible elements in $\mathbb{Z}[i]$.

6. Suppose that I is an ideal of $R = \mathbb{Z}[x]$ and suppose that $p \in I$ for some prime number p. Prove that I can be generated by 2 elements.

7. Does there exist a non-principal ideal in $\mathbb{Z}[\sqrt{-13}]$?
   Either give an example, or prove that no such example exists.

8. Let $\mathbb{Q}(x)$ be the field of fractions of the integral domain $\mathbb{Q}[x]$. For the subring

$$A = \left\{ \frac{f(x)}{g(x)} \in \mathbb{Q}(x)\colon g(0) \neq 0 \right\},$$

   of $\mathbb{Q}(x)$, prove the following:

   (a) A is a PID.

   (b) A has a unique irreducible element up to associates.