Math 206C: Algebra
Final Exam Galois Theory Practice Problems

The goal of this document is to provide you with some practice problems for the Final Exam from past Algebra Qualifying Exams. These problems focus on material that was not covered on Midterm 2, or on Practice Problems– Week 9 document or the Practice Problems– Cyclotomic Fields document.

Finite Fields

1. Algebra Qualifying Exam Spring 2013 #6
   Let E be the splitting field of $x^{35} - 1$ over $\mathbb{F}_2$.
   (a) How many elements does E have?
   (b) How many subfields does E have?
2. Algebra Qualifying Exam Spring 1996 #9
   Let E be a splitting field of $x^8 - 1$ over a field F of 4 elements. Find $|E|$.
3. Algebra Qualifying Exam Fall 2010 #3
   Determine the splitting field of $x^5 - 2$ over $\mathbb{F}_3$. Then determine the Galois group over $\mathbb{F}_3$ of $x^5 - 2$, both as an abstract group and as a set of automorphisms.

Cyclotomic Fields

1. Algebra Qualifying Exam Spring 1997 #9
   Let E be the splitting field of $x^{42} - 1$ over $\mathbb{Q}$. Determine the number of subfields of E.
2. Algebra Qualifying Exam Spring 2008 #5
   Let K be the splitting field of $x^{49} - 1$ over $\mathbb{Q}$.
   Determine the number of fields F such that $\mathbb{Q} \subseteq F \subseteq K$.

Galois Groups: Problems from Lecture (and some closely related ones)

1. Algebra Qualifying Exam Fall 2019 #7
   Calculate the Galois group of $x^4 - 3x^2 + 4$ over $\mathbb{Q}$.
   Note: We discussed this question in Lecture 27.

2. Algebra Qualifying Exam Fall 2014 #5
   Determine the splitting field over $\mathbb{Q}$ of the polynomial $x^4 + x^2 + 1$, and the degree over $\mathbb{Q}$ of the splitting field.
   Note: We completely answered this question in Lecture 27.

3. Algebra Qualifying Exam Spring 2013 #5a
   Let $f(x) \in \mathbb{Q}[x]$ be an irreducible cubic polynomial whose Galois group is denoted by G_f. Prove that if $f(x)$ has exactly one real root, then $G_f \cong S_3$.
   Note: We completely answered this question in Lecture 27.

4. Algebra Qualifying Exam Fall 2001 #4
   Let $f(x)$ be an irreducible degree p polynomial over $\mathbb{Q}$ with exactly $p - 2$ real roots where p is a prime. Regard the Galois group G_f of $f(x)$ as a subgroup of S_p through its action on the roots of f.

   (a) Show that G_f contains a 2-cycle of S_p.

   (b) Show that $G_f \cong S_p$. Hint: Use that the irreducibility of f implies that G_f is a transitive subgroup. Explain why p being a prime now implies that G_f contains a p-cycle.

   (c) Let $f(x) = x^5 - 9x + 2$. Using (a) and (b) show that $G_f = S_p$.

   Note: We discussed this question in Lecture 27.

5. Algebra Qualifying Exam Fall 2002 #8

   (a) Let $g(x)$ be an irreducible degree 5 polynomial over $\mathbb{Q}$ with exactly 3 real roots. Show that the Galois group of $g(x)$ is S_5.

   (b) Consider the polynomial $f(x) = x^5 - 25x + 3$. Show that $f(x)$ is irreducible.

   (c) Using (a) and (b) find the Galois group of $f(x)$.

   Note: This question is similar to the previous two.

6. Algebra Qualifying Exam Fall 2019 #8
   Let $K \subseteq \mathbb{C}$ be a field extension of $\mathbb{Q}$ such that $\mathrm{Gal}(K/\mathbb{Q})$ is cyclic of order 4.
   Prove that $i \notin K$.

Galois Groups: Computations

1. Algebra Qualifying Exam Fall 2010 #5
Find the Galois group over $\mathbb{Q}$ of $x^3 + 4x + 2$, as an abstract group.

2. Algebra Qualifying Exam Fall 1997 #10
Show that the splitting field E of the polynomial
$$f(X) = X^3 + X^2 - 2X - 1$$
over $\mathbb{Q}$ is obtained by adjoining a single root of $f(X)$. Find the Galois group $\mathrm{Gal}(E/\mathbb{Q})$.
Hint: Show first that $f(X)$ divides $f(X^2 - 2)$.

3. Algebra Qualifying Exam Fall 2018 #8
Find the Galois group of $x^6 - 2$ over $\mathbb{Q}$ and over $\mathbb{F}_5$.

4. Algebra Qualifying Exam Spring 2017 #7
    (a) Find a polynomial $f(x) \in \mathbb{Q}[x]$ whose splitting field L_f has Galois group $\mathrm{Gal}(L_f/\mathbb{Q})$ isomorphic to $\mathbb{Z}/2\mathbb{Z}$.
    (b) Find a polynomial $g(x) \in \mathbb{Q}[x]$ whose splitting field L_g has Galois group $\mathrm{Gal}(L_g/\mathbb{Q})$ isomorphic to S_3.
    (c) Find a polynomial $h(x) \in \mathbb{Q}[x]$ whose splitting field L_h has Galois group $\mathrm{Gal}(L_h/\mathbb{Q})$ isomorphic to $\mathbb{Z}/2\mathbb{Z} \times S_3$.

5. Algebra Qualifying Exam Winter 2003 #11
Let K be the splitting field over $\mathbb{Q}$ of the polynomial
$$f(x) = (x^2 - 2x - 1)(x^4 - 1).$$
Determine the Galois group G of $f(x)$ and determine all the intermediate fields explicitly.

6. Algebra Qualifying Exam Fall 2007 #3b
Find a field F such that $\mathrm{Gal}(F/\mathbb{Q}) \cong \mathbb{Z}/3\mathbb{Z}$. Prove your answer.

7. Algebra Qualifying Exam Spring 2011 #4
Let G be the Galois group of the polynomial $x^6 - 27$ over $\mathbb{Q}$.
Determine all elements of G be describing their actions on the generators of the splitting field.
Also describe G as an abstract group.

8. Algebra Qualifying Exam Fall 1997 #8
Determine the degree $[E\colon \mathbb{Q}]$ of the splitting field E of $X^{10} - 5$ over $\mathbb{Q}$.

9. Algebra Qualifying Exam Fall 2009 #10
Compute the Galois group of the polynomial $f(x) = x^5 - 4x + 2$ over $\mathbb{Q}$.

10. Algebra Qualifying Exam Spring 1996 #8
Find the Galois group of the polynomial $3x^3 - 9x^2 + 9x - 5$ over $\mathbb{Q}$.

1. Algebra Qualifying Exam Fall 2017 #5
   Suppose that K is a field of characteristic 0, and L is the splitting field of the irreducible polynomial $f(x) \in K[x]$. Prove that if $\mathrm{Gal}(L/K)$ is abelian, and if $a \in L$ is a root of f, then $L = K(a)$.

2. Algebra Qualifying Exam Fall 2011 #1
   Show that $\sqrt[4]{2}$ is not contained in any field L that is Galois over $\mathbb{Q}$ with $\mathrm{Gal}(L/\mathbb{Q}) \cong S_n$, for any positive integer n. You may use without proof the fact that the Galois group of the polynomial $x^4 - 2$ over $\mathbb{Q}$ is the dihedral group of order 8.

3. Algebra Comprehensive Exam Spring 2005 #F3
   Let E/F be a finite Galois extension with Galois group G. Suppose that $|G| = 2p^n$ where p is an odd prime and $n \geq 1$. Prove that there is an intermediate field $F < K < E$ with $K \neq E$ such that $F < K$ is normal.

4. Algebra Qualifying Exam Spring 2013 #9
   Suppose that K is a Galois extension of F of degree pq, where $p < q$ are distinct primes. Show that K has a subfield L that is Galois over F with $[L\colon F] = p$.

5. Algebra Qualifying Exam Fall 2004 #8
   Suppose $f(x) \in \mathbb{Q}[x]$ is irreducible and let K denote its splitting field.

   (a) Suppose $\mathrm{Gal}(K/\mathbb{Q}) = Q_8$ (the quaternion group of order 8).
   What are the possibilities for the degree of f?

   (b) Suppose $\mathrm{Gal}(K/\mathbb{Q}) = D_8$ (the dihedral group of order 8).
   What are the possibilities for the degree of f?

Additional Problems

1. Algebra Qualifying Exam Fall 2000 #8
   Suppose α is a root of a monic irreducible polynomial $f \in \mathbb{Q}[x]$ of degree 9. In this case, $K = \mathbb{Q}[x]/(f(x))$ is a degree 9 extension of $\mathbb{Q}$ isomorphic to $\mathbb{Q}(\alpha)$.
   (a) Suppose α is in $\mathbb{R}$, but none of the other roots of f are real.
   Explain why K has no (non-trivial) field automorphisms.
   (b) Suppose there is a field M properly between K and $\mathbb{Q}$.
   What are the possible degrees of $M/\mathbb{Q}$.
   (c) Suppose the Galois closure of $K/\mathbb{Q}$ is L and $\mathrm{Gal}(L/\mathbb{Q}) \cong S_9$.
   Explain why there is no field properly between K and $\mathbb{Q}$.

2. Algebra Qualifying Exam Spring 2020 #8
   Let $f(x) \in \mathbb{Q}[x]$ be a monic cubic polynomial with distinct roots r, s, t. Let $g(x)$ be the monic cubic polynomial with roots $r^2 + s + t, s^2 + t + r$ and $t^2 + r + s$.
   (a) Prove that $g(x) \in \mathbb{Q}[x]$, that is, the coefficients of $g(x)$ are in $\mathbb{Q}$.
   (b) Prove that if $\mathrm{Gal}(f) \cong S_3$ then $\mathrm{Gal}(g) \cong S_3$.

3. Algebra Qualifying Exam Spring 2016 #6
   Let $L/\mathbb{Q}$ be a Galois extension with Galois group isomorphic to A_4.
   (a) Does there exist a quadratic extension $K/\mathbb{Q}$ contained in L? Prove your answer.
   (b) Does there exist a degree 4 polynomial in $\mathbb{Q}[x]$ with splitting field equal to L?
   Prove your answer.

4. Algebra Qualifying Exam Fall 2012 #5
   Suppose F is a Galois extension of $\mathbb{Q}$ and $\mathrm{Gal}(F/\mathbb{Q}) \cong S_4$. Show that there is an irreducible polynomial $g(x) \in \mathbb{Q}[x]$ of degree 4 such that the splitting field of $g(x)$ is F.

5. Algebra Qualifying Exam Fall 1997 #9
   Let F be a field and let $f(X) \in F[X]$ be a separable irreducible polynomial of degree 4. Determine, as explicitly as possible, the Galois group G, of the splitting field of $f(X))$ over F, when G has order 8.

6. Algebra Qualifying Exam Spring 2012 #10c
   Does there exist an extension F of $\mathbb{R}$ with $[F : \mathbb{R}] = 4$?
   Either give an example or explain briefly why no such example exists.

7. Algebra Qualifying Exam Fall 2020 #7
   Let $E < F$ be a field extension of degree 5 and K the smallest subfield in the algebraic closure of E such that K is Galois over E and contains F.
   Show that the degree of K over E is at most 120.

8. Algebra Qualifying Exam Fall 2016 #9b
   Is the following statement True or False? (Give a brief explanation.)
   A finite extension of $\mathbb{Q}$ cannot have infinitely many distinct subfields.

9. Algebra Qualifying Exam Spring 2018 #8
   Let F be a field and let $f(x) \in F[x]$ be an irreducible polynomial. Suppose E is a splitting field for $f(x)$ over F and assume that there exists an element $\alpha \in E$ such that both α and $\alpha + 1$ are roots of $f(x)$.

(a) Show that the characteristic of F is not zero.

(b) Prove that there exists a field L between F and E such that the degree $[E\colon L]$ is equal to the characteristic of F.

Note: Part (a) of this problem came up as Algebra Qualifying Exam Fall 2011 #7.

10. Algebra Qualifying Exam Spring 2016 #9a
Either give an example or state that none exists. In either case, give a **brief** explanation.
An element $\alpha \in \mathbb{Q}(\sqrt{2}, i)$ such that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2}, i)$.