Math 206C: Algebra
Homework 2

Due Wednesday, April 21st at 11:59PM.
Please email `nckaplan@math.uci.edu` with questions.

Let R be a ring and $\mathrm{Mat}_n(R)$ be the ring of $n \times n$ matrices with entries in R. For $A \in \mathrm{Mat}_n(R)$, let $\mathrm{tr}(A)$ denote the trace of A, the sum of the diagonal entries of A.

1. Algebra Comprehensive Exam Spring 2015 #7
   Suppose $A \in \mathrm{Mat}_n(\mathbb{C})$ satisfies $A^k = I$ for some k. Show that $|\mathrm{tr}(A)| \leq n$.

2. Algebra Qualifying Exam Spring 2017 #9
   Suppose p is a prime.
   (a) Show that all matrices $A \in \mathrm{GL}_2(\mathbb{F}_p)$ of order exactly p have the same characteristic polynomial and find that polynomial.
   (b) Show that all matrices $A \in \mathrm{GL}_2(\mathbb{F}_p)$ of order exactly p have the same minimal polynomial and find that polynomial.

3. Exercise 2 of Section 12.2
   This is the exercise about the minimal polynomial of a block diagonal matrix that we used in Lecture 5 to prove Corollary 25 from Section 12.3.

4. Exercise 4 of Section 12.3
   In Lecture 6 we showed one way to diagonalize $xI - A$ where A is the matrix in this exercise, and saw how this let us write down the Jordan canonical form C of A. Using this information together with the algorithm described in Section 12.3 allows you to write down a matrix P such that $C = P^{-1}AP$.

5. Exercise 18 of Section 12.3
   In this exercise you determine all possible Jordan canonical forms for a linear transformation with a particular characteristic polynomial. It is similar to an example from lecture.

6. Exercise 19 of Section 12.3
   In this exercise you determine conditions on $f(x)$ so that all $n \times n$ matrices with characteristic polynomial $f(x)$ are similar.

7. Exercise 23 of Section 12.3
   In lecture we solved an exercise about the possible invariant factors of a 2×2 matrix. In this exercise you compute the rational canonical form and Jordan canonical form of a 2×2 matrix satisfying a certain equation.

   Note: In an earlier version of this document there was a typo– I wrote 'Exercise 22' instead of Exercise 23. Maybe some of you have solved Exercise 22 already. That is totally fine. You only need to hand in **either** Exercise 22 or Exercise 23.

8. Exercise 24 of Section 12.3
   In this exercise you show that there are no 3×3 matrices with entries in $\mathbb{Q}$ satisfying $A^8 = I$ but $A^4 \neq I$. There are such matrices over $\mathbb{C}$, for example, consider a diagonal 3×3 matrix where each diagonal entry is an 8th root of unity that is not a 4th root of unity.

9. Exercise 31 of Section 12.3
   In this exercise you consider the Jordan canonical form of a nilpotent matrix.

10. Exercise 33 of Section 12.3
    In this exercise you show that a certain kind of matrix is nilpotent.