Math 206C: Algebra
Homework 5

Due Monday, May 17th at 11:59PM.
Please email `nckaplan@math.uci.edu` with questions.

1. Algebra Qualifying Exam Fall 2014 #10
   Suppose p is prime and r and N are positive integers. Consider the map $\sigma\colon \mathbb{F}_{p^r}^* \to \mathbb{F}_{p^r}^*$ defined by $\sigma(x) = x^N$. What is the cardinality of its kernel and image? Fully justify.

2. Exercise 4 in Section 13.5
   In this exercise you prove a result that implies $\mathbb{F}_{p^d} \subseteq \mathbb{F}_{p^n}$ if and only if $d \mid n$.
   We will prove this later in a different way (see Proposition 15 in Section 14.3).

3. Exercise 5 in Section 13.6
   In this exercise you consider the total number of roots of unity in a finite extension of $\mathbb{Q}$.

4. Exercise 6 in Section 13.6
   In this exercise you use what you know about n^{th} roots of unity and $2n^{\text{th}}$ roots of unity to give a relationship between $\Phi_n(x)$ and $\Phi_{2n}(x)$.

5. (a) Let p be prime. Prove that $\Phi_{p^2}(x) = \Phi_p(x^p)$.
   (b) Prove that if n is a positive integer and p is a prime dividing n, then $\Phi_{np}(x) = \Phi_n(x^p)$.

6. Exercises 10, 11, and 12 in Section 13.6
   In these exercises you consider the Frobenius map $\varphi\colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ defined by $\varphi(x) = x^p$. We know from Proposition 35 in Section 13.5 that it is an injective field homomorphism. Here you show that it is an automorphism and determine its rational canonical form and its Jordan canonical form.
   Note: These exercises are closely related to Algebra Qualifying Exam Spring 2020 #9 and also to Algebra Comprehensive Exam Fall 2012 #10.

7. Algebra Qualifying Exam Spring 2011 #7c
   Let F be a field of characteristic p. Define the Frobenius map $\varphi\colon F \to F$.
   Give an example of a field F such that φ is not an automorphism.
   Note: Part (a) of this question is Proposition 35 in Section 13.5 and part (b) is Exercise 11 in Section 13.6.

8. Algebra Qualifying Exam Fall 2011 #2
   Let p be a prime and F be an algebraically closed field of characteristic p. Let $n = p^a m$ where $p \nmid m$. How many n^{th} roots of unity are there in F? Prove your answer.

9. Exercise 1 in Section 14.1
   In this two-part exercise you prove that one can determine how an automorphisms acts on a subfield in terms of how it acts on a generating set for that subfield, and that one can determine how a group of automorphisms acts on a field by how a generating set for that group acts on that field.

10. Exercise 4 in Section 14.1
    In this exercise you show that there is no field isomorphism $\sigma\colon \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{3})$.
    This is very similar to Algebra Qualifying Exam Spring 2012 #9e.