# Math 206C: Algebra
# Midterm 1 (Fields): Things to Know

The goal of this document is to give a list of definitions and theorems related to the material we have covered so far on Fields (Section 13.1, most of Section 13.2, and a little bit of Section 13.4 of Dummit and Foote) that will be helpful to know for Midterm 1 on Friday, April 23.

## Fields

### Definitions

1. The characteristic of a field.
2. The prime subfield of a field.
3. Extension of fields.
4. The degree of a field extension.
5. The field generated by $\alpha_1, \ldots, \alpha_k$ over $F$.
   Simple extension.
   Primitive element.
6. Splitting field for $f(x) \in F[x]$.
7. What it means for an element $\alpha$ to be algebraic over $F$.
   What it means for the extension $K/F$ to be algebraic.
8. The minimal polynomial $m_{\alpha,F}(x)$. The degree of $\alpha$ (over $F$).

### Examples

1. $\mathbb{R}[x]/(x^2+1)$, $\mathbb{Q}[x]/(x^2+1)$. (Examples 1,2 page 515 in Section 13.1)
2. $\mathbb{Q}[x]/(x^2-2)$, $\mathbb{Q}[x]/(x^3-2)$. (Examples 3,4 page 515 in Section 13.1)
3. $\mathbb{F}_2[x]/(x^2+x+1)$, $\mathbb{F}_3[x]/(x^2+2)$ and $\mathbb{F}_3[x]/(x^2+2x+2)$.
   (Example 6 page 516 in Section 13.1)
4. $k(t)[x]/(x^2-t)$. (Example 7 page 516 Section 13.1)
5. Examples of minimal polynomials (page 521 Section 13.2)
6. Quadratic extensions of fields of characteristic not equal to 2. (Page 522 Section 13.2)

## Theorems

1. Let $F$ be a field and $p(x) \in F[x]$ be an irreducible. Then there exists a field $K$ containing a subfield isomorphic to $F$ in which $p(x)$ has a root. Identifying $F$ with this isomorphic copy shows that there exists an extension of $F$ in which $p(x)$ has a root.
   (Theorem 3 in Section 13.1)

2. Let $F$ be a field and $f(x) \in F[x]$ be a nonconstant polynomial. Then there exists an extension $K$ of $F$ that is a splitting field for $f(x)$. (Theorem 25 in Section 13.4)

3. Description of the elements of $F[x]/(p(x))$ in terms of $\theta = \overline{x} = x \pmod{(p(x)}$. Description of how to multiply and add elements of this quotient in terms of $\theta$.
   (Theorem 4 and Corollary 5 in Section 13.1)

4. Finding inverses of elements in $F[x]/(p(x))$. We did $\theta^{-1}$ in general, and $\theta^2+1$ in $\mathbb{Q}[x]/(x^3-2)$.
   (Examples 4,5 page 515-6 in Section 13.1).

5. Let $F$ be a field and $p(x) \in F[x]$ be an irreducible polynomial. Suppose $K$ is an extension field of $F$ containing a root $\alpha$ of $p(x)$. Then $F(\alpha) \cong F[x]/(p(x)$. Description of the elements of $F(\alpha)$ as polynomials in $\theta$. (Theorem 6 and Corollary 7 in Section 13.1)

6. Isomorphism $F \to F'$ leads to an isomorphism $F(\alpha) \to F'(\beta)$. (Theorem 8 in Section 13.1)

7. Existence of the minimal polynomial of an element $\alpha$ that is algebraic over $F$.
   (Proposition 9 in Section 13.2)

8. $\alpha$ is algebraic over $F$ if and only if $F(\alpha)/F$ is finite. (Proposition 12 in Section 13.2)

9. Finite extensions are algebraic. (Corollary 13 in Section 13.2)

10. The degrees of field extensions are multiplicative. (Theorem 14 in Section 13.2)

11. $K/F$ is finite if and only if $K$ is generated by a finite number of algebraic elements over $F$.
    (Theorem 17 in Section 13.2)