Math 206C: Algebra
Midterm 2 (Fields): Things to Know

This document will build on the earlier 'Midterm 1 (Fields): Things to Know' document. We will give some definitions, examples, and theorems that will be helpful to know for Midterm 2. This document will focus on material that was not covered on Midterm 1 that we learned before the start of our unit on Galois theory: Lectures 11-15.

Fields

Definitions

1. The field of algebraic numbers $\overline{\mathbb{Q}}$.
2. What it means for $\alpha \in \mathbb{C}$ to be transcendental over $\mathbb{Q}$.
3. The composite field of K_1 and K_2. The composite of a collection of subfields.
4. The n^{th} cyclotomic field $\mathbb{Q}(\zeta_n)$.
5. Primitive n^{th} roots of unity.
6. An algebraic closure of a field F.
7. What it means for a field to be algebraically closed.
8. What it means for $f(x) \in F[x]$ to be separable/inseparable.
9. What it means for an element α that is algebraic over F to be separable over F.
10. What is means for an algebraic extension K/F to be separable/inseparable.
11. What it means for a field to be perfect.
12. The derivative of $f(x) \in F[x]$.
13. Let F be a field of characteristic p. The Frobenius endomorphism of F.
14. The n^{th} cyclotomic polynomial $\Phi_n(x)$.

Examples

1. Expressing the elements of $F(\alpha, \beta)$ as polynomials in α and β. (p. 525-526, Section 13.2)
2. $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$ and related examples. (Exercise 7 in Section 13.2)
3. $\overline{\mathbb{Q}}$ is a proper subfield of $\mathbb{C}$ that is an infinite extension of $\mathbb{Q}$.
4. Examples of the composite of two fields. (p. 528-529 in Section 13.2)

5. Examples of splitting fields: $x^2 - 2, (x^2 - 2)(x^2 - 3), x^3 - 2, x^p - 2, x^4 - 2, x^4 + 2$.

6. Splitting field of $x^n - 1$: $\mathbb{Q}(\zeta_n)$.

7. Examples of separable/inseparable polynomials.

8. How to compute cyclotomic polynomials: $x^n - 1 = \prod_{d|n} \Phi_d(x)$.

Theorems

1. Suppose α and β are algebraic over F. Then $\alpha \pm \beta, \alpha\beta, \alpha/\beta$ (for $\beta \neq 0$), are all algebraic. Let L/F be an arbitrary extension. The collection of elements of L that are algebraic over F forms a subfield of F. (Corollaries 18 and 19 in Section 13.2)

2. If K is algebraic over F and L is algebraic over K, then L is algebraic over F.
(Theorem 20 in Section 13.2)

3. Results on the degree of a composite of two fields.
(Proposition 21 and Corollary 22 of Section 13.2)

4. Uniqueness of splitting fields up to isomorphism.
(Theorem 27 and Corollary 28 in Section 13.4)

5. If $f(x) \in F[x]$ has degree n, then a splitting field for $f(x)$ over F has degree at most $n!$. In fact, more is true, the degree divides $n!$.
(Proposition 26 in Section 13.4 and Algebra Comprehensive Exam Fall 2011 #8)

6. $K = \overline{K}$ if and only if K is algebraically closed.

7. Let $\overline{F}$ be an algebraic closure of F. Then $\overline{F}$ is algebraically closed.
(Proposition 29 in Section 13.4)

8. Let F be a field. There exists an algebraic closure of F.

9. Let K be an algebraically closed field and let F be a subfield of K. The set of elements $\overline{F}$ of K that are algebraic over F is an algebraic closure of F. (Proposition 31 in Section 13.4)

10. All algebraic closures of F are isomorphic.
(We did not prove this in lecture but it is a good fact to know.)

11. The Fundamental Theorem of Algebra: $\mathbb{C}$ is algebraically closed.
(This is stated at the end of Section 13.5. We have not proven it yet, but will come back to it in Section 14.6. It is good to know this fact and you may find it useful to apply it even before we prove it.)

12. Every finite extension of a perfect field is separable. (Corollary 39 in Section 13.5)

13. $f(x)$ has α as a multiple root if and only if α is also a root of $D_x(f(x))$.

14. Let $f(x) \in F[x]$ be irreducible. Then $f(x)$ is separable if and only if $D_x(f(x))$ is not 0 in $F[x]$. In particular, when $\text{char}(F) = 0$, every irreducible polynomial is separable. When $\text{char}(F) = p$, an irreducible polynomial is separable if and only if it is not a polynomial in x^p.

15. Finite fields are perfect. (Corollary 36 in Section 13.5)

16. For every prime p and every positive integer n there exists a finite field of order p^n. This field is unique up to isomorphism.

17. $[\mathbb{Q}(\zeta_n)\colon \mathbb{Q}] = \varphi(n)$. (Corollary 42 in Section 13.6)

18. The n^{th} cyclotomic polynomial has integer coefficients and is irreducible in $\mathbb{Z}[x]$. (Lemma 40 and Theorem 41 in Section 13.6)