Math 206C: Algebra
Final Exam
Thursday, June 10, 2021.

- You have **2 hours** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (10 Points)	
2 (10 Points)	
3 (10 Points)	
4 (8 Points)	
5 (6 Points)	
Total	

Problems	
6 (8 Points)	
7 (10 Points)	
8 (10 Points)	
9 (10 Points)	
10 (10 Points)	
11 (8 Points)	
Total	

Problems

1. Let V be a vector space over $\mathbb{Q}$ of dimension at most $p-2$ where p is prime. Let T be a linear transformation on V such that $T^p = I$ (where I denotes the identity linear transformation). Show that $T = I$.

2. Determine up to similarity all 3×3 matrices in $\mathrm{GL}_3(\mathbb{Q})$ of order **exactly** 6.
   That is, give a list of matrices in $\mathrm{GL}_3(\mathbb{Q})$ of order exactly 6 with such that any matrix in $\mathrm{GL}_3(\mathbb{Q})$ of order exactly 6 is similar to a unique matrix in your list.

3. Let $F \subseteq K \subseteq L$ be fields and suppose that L/F is finite. Prove that $[L\colon F] = [L\colon K] \cdot [K\colon F]$.

4. Let K/F be a field extension and $\sigma \in \mathrm{Aut}(K/F)$ be an automorphism of K fixing F.
   Suppose $f(x) \in F[x]$ and $\alpha \in K$.
   (a) Prove that $\sigma(f(\alpha)) = f(\sigma(\alpha))$.
   (b) Prove that σ permutes the set of roots of $f(x)$ in K.

5. (a) State the Primitive Element Theorem.
   (b) Define what if means for a field F of characteristic p to be perfect.
   (c) Let F be a field. Define what it means for a field to be an algebraic closure of F.

6. Let F be any field. Prove that if K/F is a finite extension, then it is an algebraic extension.

7. Determine the Galois group of $(x^3 - x + 1)(x^2 - 2)$ over $\mathbb{Q}$ as an abstract group.

8. Let K be the splitting field over $\mathbb{Q}$ of $x^8 - 1$.
   (a) Find $[K\colon \mathbb{Q}]$.
   (b) Describe the Galois group $G = \mathrm{Gal}(K/\mathbb{Q})$ both as an abstract group and as a set of automorphisms.
   (c) Find explicitly all subgroups of G and the corresponding subfields of K under the Galois correspondence.

9. Determine the Galois group of the splitting field of $x^3 + 2$ over $\mathbb{F}_3$, over $\mathbb{F}_7$, and over $\mathbb{F}_{11}$.

10. Fix a prime p. For all positive integers m and n, let $f(m, n)$ be the number of nonzero ring homomorphisms from $\mathbb{F}_{p^m}$ to $\mathbb{F}_{p^n}$.
    Note: For this question you should assume that a ring homomorphism must take 1 to 1.
    (a) What is $f(m, 6)$?
    (b) What is $f(6, n)$?

11. Prove that $\mathbb{Q}(\sqrt[3]{5})$ is not a subfield of any cyclotomic field over $\mathbb{Q}$.