Math 206C: Algebra

Midterm 2

Friday, May 21, 2021.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (8 Points)	
2 (8 Points)	
3 (12 Points)	
4 (10 Points)	
5 (8 Points)	
Total	

Problems	
6 (4 Points)	
7 (6 Points)	
8 (6 Points)	
9 (4 Points)	
Total	

Problems

1. Prove that the polynomial $f(x) = 1 + \frac{x}{1} + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} \in \mathbb{Q}[x]$ has no multiple roots in $\mathbb{C}$.

2. Suppose that V is a finite dimensional vector space and $T \colon V \to V$ is a linear transformation that has characteristic polynomial which is irreducible over $\mathbb{Q}$.
Show that the matrix of T (in any basis of V) can be diagonalized **over the field** $\mathbb{C}$.

3. Factor $x^4 + 1 \in F[x]$ and find the splitting field over F if the ground field F is:

$$(a)\ \mathbb{Q}, \qquad (b)\ \mathbb{F}_2, \qquad (c)\ \mathbb{R}.$$

4. Let p be prime and $\mathbb{F}_p \subset \mathbb{F}_{p^n}$ be a degree $n > 1$ extension of finite fields. Consider the Frobenius automorphism $\Phi \colon \mathbb{F}_{p^n} \to \mathbb{F}_{p^n}$ sending α to α^p. Show that Φ is $\mathbb{F}_p$-linear, that its minimal polynomial $m_\Phi(x)$ has degree n, and then compute the minimal polynomial.

5. Let n be a positive integer. Prove that the n^{th} cyclotomic polynomial $\Phi_n(x)$ has integer coefficients.

6. Let p be an odd prime. How many subfields of $\mathbb{F}_{p^{12}}$ are there?
Note: You only need to write down a number. No explanation is necessary.

7. Does there exist a field F and an extension K/F with $[K \colon F] = 2$ that is **not** a Galois extension? Either give an example and explain why it has this property, or prove that no example exists.

8. Let $K = \mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$ and $F = \mathbb{Q}(\sqrt{-3})$. Is K/F a Galois extension? Justify your answer.

9. Let K be a field and H be a subgroup of $\mathrm{Aut}(K)$.
Recall that K^H denotes the subfield of K consisting of elements fixed by every $\sigma \in H$.
Is it true that $H \subseteq \mathrm{Aut}(K/K^H)$?
Either prove this statement or give a counterexample.