Math 230a: Algebra
In-Class Exam 1: Solutions

1. (a) Let G be a group and X a set. Explain what it means to have a left group action of G on X.

   Solution: A left action of the group G on the set X is the choice, for each $g \in G$ a permutation $\pi_g : X \to X$ such that the following two conditions hold:

   - π_e is the identity: $\pi_e(x) = x$ for each $x \in X$,
   - for every $g_1, g_2 \in G$, $\pi_{g_1} \circ \pi_{g_2} = \pi_{g_1 g_2}$.

   (b) True/False (Circle One): Let the set X be equal to G. The map that takes a pair (g, x) to $g^{-1}x$ defines a left group action.

   Solution: This is false because the action does not generally satisfy the second property above.

2. State the Second (Diamond) Isomorphism Theorem.

   Solution: Let G be a group and A, B be subgroups of G satisfying $A \leq N_G(B)$. Then AB is a subgroup of G, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ and $AB/B \cong A/(A \cap B)$.

3. (a) Define what it means for a group G to be solvable.

   Solution: A group G is solvable if there is a chain of subgroups

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq G_2 \trianglelefteq \cdots \trianglelefteq G_s = G$$

   such that G_{i+1}/G_i is abelian for $i = 0, \ldots, s-1$.

   (b) For which $n \geq 3$ is the dihedral group of order $2n$, D_{2n}, solvable? Justify your answer.

   Solution: The group

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1,\ rs = sr^{-1} \rangle.$$

   is solvable for all $n \geq 3$ since

$$1 \trianglelefteq \langle r \rangle \trianglelefteq D_{2n}.$$

   We see that $\langle r \rangle$ is a subgroup of index 2, so it is normal. Moreover, $D_{2n}/\langle r \rangle \cong \mathbb{Z}/2\mathbb{Z}$ is abelian, as is $\langle r \rangle$, since it is cyclic.

4. Suppose that H and K are subgroups of a group G. Prove that if $H \cup K$ is a subgroup, then either $H \subseteq K$ or $K \subseteq H$.

   Solution: Suppose H is not a subset of K and K is not a subset of H. Then there exists some $h \in H$ with $h \notin K$ and some $k \in K$ with $k \notin H$. Since $H \cup K$ is a subgroup of G by assumption, $hk \in H \cup K$. If $hk \in H$, then $h^{-1}hk = k \in H$, which is a contradiction. If $hk \in K$, then $hkk^{-1} = h \in K$, which is also a contradiction. Therefore, $H \subseteq K$ or $K \subseteq H$.

5. Determine all finite groups that have at most three conjugacy classes.

   Solution: We note that an element is in a conjugacy class of size 1 if and only if it is in the center of G. In particular, the identity element is always in its own conjugacy class. This shows that the trivial group is the only group with exactly one conjugacy class.

Let $x_1, \ldots, x_r$ be representatives of the conjugacy classes of G that have size greater than 1. By the class equation

$$|G| = |Z(G)| + \sum_{i=1}^{r} |\operatorname{Orb}_{x_i}|,$$

and $|\operatorname{Orb}_{x_i}|$ divides $|G|$ for each i since the size of the orbit is equal to the index of the centralizer $C_G(x_i)$.

Suppose we have a finite group with exactly two conjugacy classes. One of these classes has size 1, the class of the identity element, and the other has size a for some positive integer a. So $|G| = 1 + a$. Since $a \mid 1 + a$, we must have $a = 1$. A group is abelian if and only if $Z(G) = G$, which is the case here, and we see that $G \cong \mathbb{Z}/2\mathbb{Z}$.

Suppose we now have three conjugacy classes. They have sizes $1, a, b$ where without loss of generality $a \leq b$. We have $|G| = 1 + a + b$ and $a \mid (1 + a + b)$, which implies $a \mid 1 + b$, and similarly $b \mid 1 + a$. Therefore we see that $b \leq 1 + a$, so we must have either $a = b$ or $a + 1 = b$. If $a = b$ then $a \mid a + 1$ and again we see that $a = 1$. In this case we have conjugacy classes of sizes $1, 1$ and 1, giving an abelian group of order 3, which must be $\mathbb{Z}/3\mathbb{Z}$. If $a + 1 = b$, then $a \mid a + 2$ implies that either $a = 1$ or $a = 2$. In the first case we get a group with conjugacy classes of sizes $1, 1$, and 2, a non-abelian group of order 4, which we know cannot exist. If $a = 2$ then $b = 3$ and we get a non-abelian group of order 6, which we saw in class must be isomorphic to S_3.

Therefore, the complete list of groups we are looking for is the abelian groups of order at most 3 and S_3.

6. (a) Prove that conjugate elements of a group G have the same order.

   Solution: Suppose that x has order n. Let $y = gxg^{-1}$ be a conjugate of x. We see that for any positive integer m,

$$y^m = \left(gxg^{-1}\right)\left(gxg^{-1}\right)\cdots\left(gxg^{-1}\right) = gx^m g^{-1}.$$

   Therefore $y^n = gx^n g^{-1} = e$, so y has order at most n. If $y^m = e$ for some $m < n$ then $gx^m g^{-1} = e$, and multiplying both sides on the right by g and cancelling gives $x^m = e$, contradicting the fact that x has order n.

   (b) Is the converse true? Justify your answer.

   Solution: The converse is false and any abelian group of order at least 3 gives an example. Both non-identity elements of $\mathbb{Z}/3\mathbb{Z}$ have order 3, but since this group is abelian they each have their own conjugacy class.