Math 230b: Algebra

Final Exam

Friday, March 18 2016.

NAME:

- You have two hours for this exam. Pace yourself, and do not spend too much time on any one problem.
- There are 10 total problems on the exam. **You only need to do 8 of them.** On the front page of your exam clearly indicate which two problems you do not want graded by crossing them out.
- **Show your work and justify all of your answers**. The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used. Do not use a calculator.
- If you need more room, use extra pages, and indicate clearly that you have done so.
- You may use results that we proved in lecture or on the homework without proving them here provided you clearly state the result your are using.

Problems	
1	
2	
3	
4	
5	
Total	

Problems	
6	
7	
8	
9	
10	
Total	

1. (a) Let R be a ring with a 1 and M any R-module. Show that $R \otimes_R M \cong M$.

(b) Prove that $\mathbb{Z}[i] \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{C}$.

2. Each of the following examples gives a ring R and an ideal I of R. In each case state (and *justify*)

   - whether or not I is a prime ideal, and
   - whether or not I is a maximal ideal.

   (a) $R = \mathbb{Z} \times \mathbb{Z},\ I$ is the principal ideal generated by $(1, 5)$.

   (b) $R = \mathbb{Z} \times \mathbb{Z},\ I = 7\mathbb{Z} \times 7\mathbb{Z}$.

   (c) $R = \mathbb{Z}[x],\ I$ is the principal ideal $(x^2 - 1)$.

   (d) $R = \mathbb{Z}[x]/(x^2),\ I$ is the principal ideal (x).

   (e) $R = \mathbb{Z}[i],\ I$ is the principal ideal (7).

3. (a) Recall that a commutative ring R with a 1 is *Noetherian* if it satisfies the ascending chain condition on ideals, or equivalently, if every ideal of R is finitely generated.

Prove that if $R[x]$ is Noetherian, then R is Noetherian.

(b) Let R be an arbitrary ring. A free R-module of finite type is an R-module isomorphic to R^n for some integer $n \geq 0$.

Prove that an R-module M is finitely generated if and only if there exists a free R-module P of finite type and a surjective R-module homomorphism $\varphi\colon P \to M$.

4. Let R be a commutative ring with a 1. Recall that an R-module M is *Noetherian* if it satisfies the ascending chain condition on submodules, or equivalently, every submodule of M is finitely generated.

   Let M be a Noetherian R-module and $f\colon M \to M$ a surjective R-module homomorphism. Show that f is an isomorphism.

5. (a) Let F be a field and $R \subset F$ be a subring with the property that for every $x \in F$, either $x \in R$ or $x^{-1} \in R$ (or both). Prove that if I and J are two ideals of R, then either $I \subseteq J$ or $J \subseteq I$.

(b) Construct a proper subring $R \subsetneq \mathbb{Q}$ such that for every $x \in \mathbb{Q}$ either $x \in R$ or $x^{-1} \in R$ (or both).

6. Let $R \subset \mathbb{R}[x]$ be the subring of $\mathbb{R}[x]$ consisting of polynomials whose coefficient of x is 0:

$$R = \left\{ f(x) = a_0 + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n \mid a_i \in \mathbb{R},\ n \in \mathbb{Z}_{\geq 0} \right\}.$$

The ring R is an integral domain (you may use this without proof).

(a) Give an example of an ideal $I \subset R$ that is not principal. You must justify your answer by *proving* that I is not a principal ideal of R.

(b) Prove that R is not a UFD.

7. (a) Let F be a field and $n > 0$. Prove that every ideal in $F[x]/(x^n)$ is principal.

(b) For $n > 1$, prove that $F[x]/(x^n)$ has exactly one maximal ideal, but is not a field.

8. Let $R = \mathbb{Z}[\sqrt{-m}]$ where m is a squarefree odd integer with $m \geq 3$.

   (a) Show that 1 and -1 are the only units of R.

   (b) Show that 2 and $1 + \sqrt{-m}$ are irreducible in R.

   (c) Show that R is not a UFD.

9. Let R be a commutative ring with with a 1 and let J be the intersection of all maximal ideals of R. Then

$$1 + J = \{1 + x \mid x \in J\}$$

is a subgroup of the group of units of R.

10. Let R be a commutative ring with 1 and M a left R-module. The set of torsion elements of M is denoted

$$\text{Tor}(M) = \{m \in M \mid rm = 0 \text{ for some nonzero } r \in R\}.$$

(a) Prove that if R is an integral domain then $\text{Tor}(M)$ is a left submodule of M.

(b) Give an example of a ring R and an R-module M such that $\mathrm{Tor}(M)$ is not a submodule.

(c) If R has zero divisors show that every nonzero R-module has nonzero torsion elements.