Math 230B: Algebra

Final Exam

Thursday, March 23, 2023.

NAME:

- You have **two hours** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (10 Points)	
2 (10 Points)	
3 (10 Points)	
4 (10 Points)	
5 (10 Points)	
Total	

Problems	
6 (10 Points)	
7 (10 Points)	
8 (10 Points)	
9 (10 Points)	
10 (10 Points)	
Total	

Problems

1. Let K be a field and let A be an $n \times n$ matrix with entries in K.
   Suppose that $f \in K[x]$ is an irreducible polynomial such that $f(A) = 0$.
   Prove that $\deg(f) \mid n$.

2. Let A be a finite abelian group of order n. What is the cardinality of $\mathbb{Q} \otimes_{\mathbb{Z}} A$? Prove that your answer is correct.

3. Is $\mathbb{Q}$ is a free $\mathbb{Z}$-module? Prove that your answer is correct.

4. Let R be a UFD and let α be an irreducible element of R. Prove that α is prime.

5. Is every Euclidean domain a PID? Either prove that the answer is yes or give an example to show that the answer is no (and explain why your example works).

6. Let V be a vector space of dimension n over a field F. Let W be a subspace of V of dimension m. Let s be the dimension of the vector space V/W. Prove that $m + s = n$.

   Note: Do not use the Rank-Nullity theorem for linear transformations to prove this statement. (We used this statement to prove the Rank-Nullity theorem.)

7. (a) Let R be a commutative ring with $1 \neq 0$ and let M be an R-module. Define $\mathrm{Ann}(M)$, the annihilator of M.

   (b) Prove that $\mathrm{Ann}(M)$ is an ideal of R.

   (c) Let R be a PID, let B be a torsion R-module and let p be a prime in R. Prove that if $pb = 0$ for some nonzero $b \in B$, then $\mathrm{Ann}(B) \subseteq (p)$.

8. (a) How many similarity classes of 4×4 matrices A with entries in $\mathbb{R}$ satisfy $A^3 = I$? Explain how you know this number is correct.

   (b) Give an example of one matrix in each of these similarity classes.

9. For each of the following rings, list all of its maximal ideals.
   Prove that your list is complete.

   (a) $\mathbb{Q}[x]/(x^2+1)$
   (b) $\mathbb{C}[x]/(x^2+1)$
   (c) $\mathbb{Q}[x]/(x^3+x^2)$.

10. Let F be a field and let G be a finite subgroup of F^*. Prove that G is cyclic.