Math 230B: Algebra

Midterm #1

Wednesday, February 1, 2023.

- You have **90 minutes** for this exam. Pace yourself, and do not spend too much time on any one problem.
- **Show your work and justify all of your answers!** The more you explain your thought process, the easier it will be to give partial credit for incomplete solutions.
- This is a closed-book exam. No notes or outside resources can be used.
  Do not use a calculator.
- You may use results that we proved in lecture without proving them here provided you clearly state the result you are using.
  (There is an exception: If a question asks you to prove something that we proved in lecture, you should prove it, don't just state it.)

Problems	
1 (10 Points)	
2 (6 Points)	
3 (4 Points)	
4 (10 Points)	
Total	

Problems	
5 (10 Points)	
6 (12 Points)	
7 (6 Points)	
8 (10 Points)	
Total	

Problems

1. (a) Define Unique Factorization Domain (UFD).
   (b) Define Principal Ideal Domain (PID).
   (c) For the properties "UFD" and "PID" give an example of an integral domain that
      i. satisfies both properties,
      ii. satisfies neither property,
      iii. satisfies one property but not the other.

      Note: You only need to give one example each for (i), (ii), and (iii).
      You do not need to prove that they have these properties.

2. Factor 1300 into a product of irreducible elements in $\mathbb{Z}[i]$.

3. Prove that $x^6 + 30x^5 - 15x^3 + 6x - 120$ is irreducible in $\mathbb{Z}[x]$.

4. Let R be a commutative ring in which every ideal is finitely generated. Prove that if there is an infinite sequence of ideals in R satisfying

$$I_1 \subseteq I_2 \subseteq \cdots$$

then there is some m such that $I_k = I_m$ for all $k \geq m$.

5. Let R be a PID and $\alpha \in R$ be a nonzero nonunit element.
   Prove that α has at least one irreducible factor in R.

   Note: Do not use the fact that every PID is a UFD. The statement you are proving here is one piece of the proof that every PID is a UFD.

6. (a) Determine whether the rings $(\mathbb{Z}/5\mathbb{Z})[x]/(x^2+1)$ and $(\mathbb{Z}/5\mathbb{Z})[x]/(x^2+2)$ are isomorphic.

(b) Prove that $\mathbb{Z}[x]/(3, x^3-1)$ is isomorphic to $(\mathbb{Z}/3\mathbb{Z})[x]/(x^3-1)$.

(c) Give a complete list of the maximal ideals in the ring $(\mathbb{Z}/3\mathbb{Z})[x]/(x^3 - 1)$. Explain how you know your list is complete.

7. Let $R = \mathbb{Z}/n\mathbb{Z}$ where n is a positive integer. Is it necessarily true that a polynomial $f(x) \in R[x]$ with degree d has at most d distinct roots in R?
Explain your answer.

8. Prove that $\mathbb{Z}[\sqrt{10}]$ is not a UFD.