Subgroups of $A_4$

The alternating group $A_4$ is written in cycle notation as follows:

$$A_4 = \{ e, (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243) \}.$$

We find its subgroups.

By Lagrange’s Theorem, the proper subgroups of $A_4$ can only have orders 1, 2, 3, 4 or 6. Looking first for cyclic subgroups, we see that $A_4$ has one element of order 1 ($e$), three of order 2 (the pairs of disjoint transpositions), and eight of order 3. There are thus no cyclic subgroups of order 4 or 6. Indeed the only cyclic subgroups of $A_4$ are the following:

$$C_1 = \{ e \},$$
$$C_2 \cong \{ e, (12)(34) \} \cong \{ e, (13)(24) \} \cong \{ e, (14)(23) \},$$
$$C_3 \cong \{ e, (123), (132) \} \cong \{ e, (124), (142) \} \cong \{ e, (134), (143) \} \cong \{ e, (234), (243) \}.$$

We are left with the possibility of non-cyclic subgroups of orders 4 or 6, i.e. either the Klein 4-group $V$ or the symmetric group $S_3$. $V$ consists of the identity plus three elements of order 2. There are only three elements of order 2 in $A_4$ and indeed you can check by multiplying out that the identity together with the three pairs of disjoint 2-cycles forms a group:

$$V \cong \{ e, (12)(34), (13)(24), (14)(23) \}.$$

Now consider the possibility of $S_3$ being a subgroup of $A_4$. $S_3$ consists of the identity, three elements of order 2, and two elements of order 3. However, there are only three elements of order 2 in $A_4$, so if $S_3$ was a subgroup of $A_4$ then these elements must be in $S_3$. However this means that the subgroup $V$ given above is in $S_3$; a contradiction of Lagrange, since $4 = |V|$ is not a divisor of $6 = |S_3|$.

The full subgroup diagram of $A_4$ is as follows:

In particular there is no subgroup of order 6, showing that the converse to Lagrange’s Theorem is false:

$$6 \mid |A_4| \not \Rightarrow \exists H \leq A_4 \text{ with } |H| = 6$$