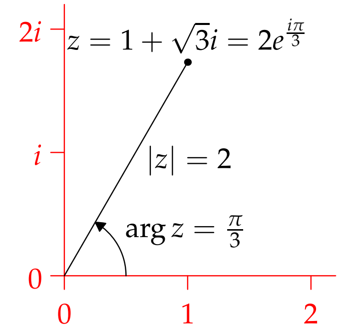
A Brief Review of Complex Number Notation

The complex numbers $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ are the real vector space $\mathbb{R}^2$ spanned by a basis $\{1, i\}$ where $i^2 = -1$ is a 'number' satisfying $i^2 = -1$. To multiply complex numbers, simply expand out and replace i^2 with -1: for example

$$(2 + 3i)(1 - i) = 2 - i + 3i - 3i^2 = 2 + 2i + 3 = 5 + 2i$$



***Complex conjugate*:** Given a complex number $z = x + iy \in \mathbb{C}$, its *conjugate* $\bar{z} = x - iy$ is the reflection of z in the real axis.

***Modulus (length)*:** $r = |z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$ is the distance of z from the origin.

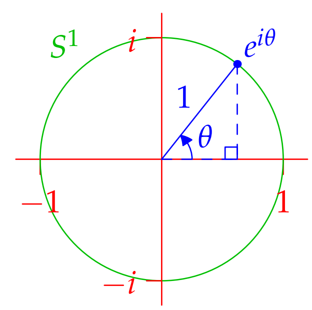
***Argument (angle)*:** If $z \neq 0$, then $\theta = \arg z$ is the angle measured counter-clockwise from the positive real axis to the ray $\overrightarrow{0z}$.

***Polar form*:** $z = re^{i\theta} = r\cos\theta + ir\sin\theta$. The complex exponential obeys the usual exponential laws and is $2\pi i$-periodic: for instance

- $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)}$
- $e^{i\theta} = 1 \iff \theta = 2\pi k$ for some integer k

The modulus and argument are the usual polar co-ordinates of a point in $\mathbb{R}^2$. The exponential laws show that the polar form behaves nicely with respect to complex multiplication:

$$|zw| = |z|\,|w| \quad \text{and} \quad \arg(zw) \equiv \arg z + \arg w \pmod{2\pi}$$



***Euler's Formula & the Unit Circle*:** When $r = 1$ we have *Euler's formula*:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

the source of the famous identity $e^{i\pi} = -1$. These complex numbers comprise the unit circle

$$S^1 = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\theta} : \theta \in [0, 2\pi)\}$$

***Rotations*:** Let θ be a fixed real number. The complex function $\text{rot}_\theta(z) = e^{i\theta}z$ has the effect of *rotating z about the origin by θ radians.* To see why, write $z = re^{i\psi}$ in polar form and observe that

$$\text{rot}_\theta(z) = e^{i\theta}re^{i\psi} = re^{i(\theta+\psi)}$$

has the *same modulus r*, but has had *θ radians added to its modulus.* For this reason, the unit circle S^1 can be thought of as the set of *rotations around the origin.*