Math 120A: Homework 1

Definitions  Complete the following sentences.

1. A binary operation \( \ast \) on a set \( X \) is **associative** if and only if

2. A binary operation \( \ast \) on a set \( X \) is **commutative** if and only if

3. The table of a commutative binary operation has ____________ (type of symmetry)

4. An element \( e \in X \) is an **identity** for the binary operation \( \ast \) if

Questions  Hand in questions 2, 4, 5 & 7(a,b) at discussion on Tuesday 9th October

1. Given the following binary operation table, calculate

   \[
   \begin{array}{cccc}
   \ast & a & b & c & d \\
   a & c & d & a & b \\
   b & d & c & b & a \\
   c & a & b & c & d \\
   d & b & a & d & c \\
   \end{array}
   \]

   (a) \( c \ast d \)

   (b) \( a \ast (c \ast b) \)

   (c) \( (c \ast b) \ast a \)

   (d) \( (d \ast c) \ast (b \ast a) \)

2. Does it make sense to write \( a \ast b \ast c \) for the following binary operation \( \ast \)? Explain why/why not.

   \[
   \begin{array}{ccc}
   \ast & a & b & c \\
   a & b & c & b \\
   b & c & a & a \\
   c & b & a & c \\
   \end{array}
   \]

3. Are the binary operations in 1,2 commutative? Explain your answer.

4. (a) Exhibit a binary operation on the two-element set \( \{a, b\} \) which is commutative.

   (b) Write down the multiplication tables of all non-commutative binary operations \( \ast \) on \( \{a, b\} \).

   (c) How many binary operations are there in total on \( \{a, b\} \)?

   (d) (Harder) How many distinct binary operations are there on a set of \( n \) letters? How many are commutative?

5. For which of the following pairs is \( \ast \) a binary operation on the set? For those that are operations, which are commutative and which are associative?

   (a) \( (\mathbb{Z}, \ast) \), \( a \ast b = a - b \)

   (b) \( (\mathbb{R}, \ast) \), \( a \ast b = 2a + b \)

   (c) \( (\mathbb{Q}^+, \ast) \), \( a \ast b = a^b \)

   (d) \( (\mathbb{N}, \ast) \), \( a \ast b = a^b \)

   (e) \( (\mathbb{N}, \ast) \), \( a \ast b = \text{product of all distinct prime factors of } ab \) (e.g. for \( 360 = 2^3 \cdot 3^2 \cdot 5 \) and \( 10 = 2 \cdot 5 \) we have \( 360 \ast 10 = 2 \cdot 3 \cdot 5 \)). Also define \( 1 \ast 1 = 1 \).
6. (a) Suppose that \( V \) is a set containing at least two elements. Prove that there exist functions \( f, g : V \to V \) such that \( f \circ g \neq g \circ f \) and thus show that the set of functions \( \{ f : V \to V \} \) is non-commutative.

   \( \text{(Hint: suppose that } \{ a, b \} \subseteq V \ldots) \)

(b) Show that matrix multiplication is non-commutative.

7. This question deals with sets of functions. The notation \( C \) means continuous and \( C^1 \) means differentiable functions with continuous first derivative. The domain is also indicated. Thus \( C^1[0, 1] \) is the set of differentiable functions with continuous first derivatives, and domain \( [0, 1] \).

   (a) Define \( * \) on \( C^1[0, 1] \) by

   \[
   (f * g)(x) = \int_0^x f'(t)g'(t)\,dt + f(0) + g(0)
   \]


   \( \text{(You will need to use the Fundamental Theorem of Calculus...)} \)

(b) Suppose that \( e \in C^1[0, 1] \) is an identity for \( * \). Show that \( e \) would have to satisfy the integral equation

   \[
   \int_0^x f'(t)e'(t)\,dt + f(0) + e(0) = f(x)
   \]

   for all functions \( f \in C^1[0, 1] \). Does such an \( e \) exist? Is it unique?

(c) (If you’ve done 140A...) Define \( * \) by

   \[
   f * g = \begin{cases} f & \text{if } \max f \geq \max g, \\ g & \text{if } \max g > \max f. \end{cases}
   \]

   Which result guarantees that this is a binary operation on \( C[0, 1] \)? Why isn’t it a binary operation on \( C(0, 1) \)? Is the binary operation commutative? Does it have an identity? Explain your answer.

8. The Lie bracket \([,]\) of two \( n \times n \) matrices is defined by \([A, B] = AB - BA\).

   (a) Show that \([,]\) is an anti-commutative binary operation on \( M_n(\mathbb{R}) \), that is,

   \[
   \forall A, B \in M_n(\mathbb{R}), \quad [A, B] = -[B, A]
   \]

   (b) Give a counter-example with \( 2 \times 2 \) matrices which shows that the Lie bracket is non-associative.

   (c) Let \( \mathfrak{s}o(n) = \{ A \in M_n(\mathbb{R}) : A^T = -A \} \) be the set of skew-symmetric \( n \times n \) matrices.

   i. Is matrix multiplication a binary operation on \( \mathfrak{s}o(n) \)?

   ii. Is the Lie bracket a binary operation on \( \mathfrak{s}o(n) \)?

   Justify your answers. \( \text{If you like, try proving this first with } 2 \times 2 \text{ matrices, then see if you can generalize your arguments. You will only need the abstract definitions, not any explicit matrices.} \)