Math 120A Homework 2

Hand in questions 1(a), 2(e), 3(b), 4, 5(b), 8, 9(a,b,c) & 11 at the discussion on Tuesday 13th October

1. Prove directly that the following are structural properties: i.e. if \( \phi \) is an isomorphism of binary structures, then each property is preserved by \( \phi \).

   (a) \( \exists \) element \( x \) such that \( x * x = x \).
   (b) \( * \) is associative.
   (c) The cardinality of the set of units\(^1\)

2. Either prove that the following are groups, or if not say why they are not.

   (a) \( (\mathbb{Q}, \cdot) \),
   (b) \( (\mathbb{R}^+, \cdot) \),
   (c) \( (3\mathbb{Z}, +) \),
   (d) \( (\mathbb{R}, a * b = a - b) \),
   (e) \( N_2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \neq 0 \) with matrix multiplication.

3. \( \phi(n) = 2 - n \) is a bijection of \( \mathbb{Z} \) with itself. For each of the following, define a binary relation \( * \) on \( \mathbb{Z} \) such that \( \phi \) is an isomorphism of binary relations.

   (a) \( \phi : (\mathbb{Z}, *) \cong (\mathbb{Z}, +) \).
   (b) \( \phi : (\mathbb{Z}, *) \cong (\mathbb{Z}, \cdot) \).
   (c) \( \phi : (\mathbb{Z}, *) \cong (\mathbb{Z}, \max(a, b)) \).

4. (a) Prove that the set of matrices \( S := \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : M_2(\mathbb{R}) \right\} \) forms a group under matrix addition.
   (b) Prove that \( T = S \setminus \{0\} \) (\( S \) without the zero matrix), forms a group under matrix multiplication.
   (c) \( \phi : S \to \mathbb{C} \) given by \( \phi \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a + ib \), is certainly a bijection. Indeed \( \phi|_T : T \to \mathbb{C}^\times \) is also a bijection. Prove that \( \phi \) and \( \phi|_T \) are both isomorphisms\(^2\)

\[ \phi : (S, +) \cong (\mathbb{C}, +), \quad \phi|_T : (T, \cdot) \cong (\mathbb{C}^\times, \cdot) \]

5. Let \( G \) be a group. Prove or disprove the following conjectures:

   (a) \( \forall g, h \in G, (ghg^{-1})^2 = gh^2g^{-1} \).
   (b) \( \forall g, h \in G, (gh)^{-1} = g^{-1}h^{-1} \iff G \) is Abelian.
   (c) \( \forall g \in G, (g^{-1})^{-1} = g \).

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\(^1\)If \( e \) is an identity for \( (X, *) \), then a unit is an element \( x \in X \) which has an inverse.

\(^2\)In 120B this will be described as an isomorphism of rings, and, in this case, fields.
6. Define a binary relation on the half-open interval \([0,1)\) by

\[ x \ast y = x + y - \lfloor x + y \rfloor \]

where \(\lfloor x \rfloor\) is the floor (integer part) function. Thus \(x \ast y\) is the fractional part of \(x + y\).

Let \(S^1 = \{z \in \mathbb{C} : |z| = 1\}\) be the unit circle in the complex plane. Find an isomorphism

\[ \phi([0,1), \ast) \cong (S^1, \cdot) \]

and prove that it is such. Is this an isomorphism of groups, or merely of binary structures?

7. Let \(([0,1), \ast)\) be the binary structure in the previous question. Prove or disprove:

\[ ([0,1), \ast) \cong (\mathbb{R}, +) \]

*Hint: how many solutions are there to the equation \(x \ast x = 0\)?*

8. Let \(\mathcal{U}\) be some universal set and let \(\mathcal{P}(\mathcal{U})\) be its power set (the set of subsets of \(\mathcal{U}\)).

(a) Which of the group axioms does the union operator \(\cup\) satisfy on \(\mathcal{P}(\mathcal{U})\)? That is, how close does the structure \((\mathcal{P}(\mathcal{U}), \cup)\) come to being a group?

(b) Repeat part (a) for the intersection operator.

(c) The symmetric difference of two sets \(A, B\) is the set

\[ A \triangle B := (A \cup B) \setminus (A \cap B) \]

i. Use Venn diagrams to give a sketch argument that \(\triangle\) is associative on \(\mathcal{P}(\mathcal{U})\).

ii. Does \(\triangle\) have an identity element? What is it?

iii. Is \((\mathcal{P}(\mathcal{U}), \triangle)\) a group?

9. (a) Show that the set \(\{1, 2, 3, 4\}\) is a group under multiplication modulo 5.

(b) What about \(\{1, 2, 3, 4, 5\}\) under multiplication modulo 6?

(c) Hypothesise for which \(n \in \mathbb{Z}^+\) the set \(\{1, 2, 3, \ldots, n - 1\}\) is a group under multiplication modulo \(n\).

(d) (Challenge) Prove your assertion (you will need to recall the Euclidean algorithm…).

10. Consider the sets of functions:

- \(C[0,1] = \{f : [0,1] \rightarrow \mathbb{R}, \text{ continuous}\}\)
- \(C^1[0,1] = \{f : [0,1] \rightarrow \mathbb{R}, \text{ differentiable with continuous first derivative}\}\)

(a) Prove or disprove: \((C[0,1], +)\) is a group, where \(f + g\) is the function defined by

\[ (f + g)(x) = f(x) + g(x) \]

Is the same claim true for \((C^1[0,1], +)\)?

(b) Find a homomorphism \(\phi : (C^1[0,1], +) \rightarrow (C[0,1], +)\).

(c) Is your homomorphism in part (b) an isomorphism? Why/why not? If not, can you restrict either of the sets \(C^1[0,1]\) or \(C[0,1]\) so that \(\phi\) becomes an isomorphism?
11. Let $G$ be a group and let $h \in G$ be some fixed element. *Conjugation by $h$* is the function

$$\phi_h : G \rightarrow G : g \mapsto hgh^{-1}$$

(a) Prove that $\phi_h$ is an isomorphism of $G$ with itself (called an *automorphism*).

(b) Suppose that $G$ is Abelian. Describe the function $\phi_h$ in a simpler way.

(c) (Harder) Look up the multiplication table for $S_3$ (or $D_3$). Compute the function $\phi_{\rho_1}$. Can you describe in words what this automorphism is doing to the symmetries of the triangle?

(d) Repeat part (c) for the automorphism $\phi_{\mu_3}$. 
