Math 120B Rings and Fields: Homework 1

Hand in questions 1, 2, 4, 5, 6, 7 & 11(a,b) at discussion on Tuesday 9th October

1. (a) Let $R = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$. Is this a *ring* (with the usual addition and multiplication)? Is R commutative? Does it have a unity? Is R a field?
   (b) Repeat the question for the set $\{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. What changes?
2. Consider the matrix ring $M_2(\mathbb{Z}_2)$. Find the number of elements in the ring (it's *order*), and describe all the units in the ring. To what elementary group is the set of units isomorphic?
3. Is the function $\phi : A \mapsto \det A$ a ring homomorphism $\phi : M_n(\mathbb{R}) \to \mathbb{R}$. Why/why not?
4. (a) Describe *all* the subrings of $\mathbb{Z}_6$.
   (b) If possible, give an example of a homomorphism $\phi : \mathbb{Z}_6 \to \mathbb{Z}_6$ where $\phi(1) \neq 0$ and $\phi(1) \neq 1$.
   (c) Give an example of a ring with unity $1 \neq 0$ that has a subring with non-zero unity $1' \neq 1$.
   (d) Prove that part (c) cannot happen with subfields: that is, if $G \leq F$ are fields, with $1_G \neq 0_G$, then $1_G = 1_F$.
5. Describe all ring homomorphisms $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.
6. Let $\mathcal{F} = \{f : \mathbb{R} \to \mathbb{R}\}$ be a ring of functions under the usual addition and multiplication. For any $a \in \mathbb{R}$, prove that the *evaluation map* $\phi_a : \mathcal{F} \to \mathbb{R}$ defined by $\phi_a(f) = f(a)$ is a homomorphism of rings.
7. Prove that $a^2 - b^2 = (a - b)(a + b)$ for all a, b in a ring R if and only if R is commutative.
8. Prove that the multiplicative inverse in a ring with unity is unique.
9. Prove that a subset S of a ring R is a subring if and only if:
   - $0 \in S$,
   - $\forall a, b \in S,\ a - b \in S$,
   - $\forall a, b \in S,\ ab \in S$.
10. An element a in a ring R is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$. Show that if R is *commutative* and a, b are nilpotent, so is $a + b$. *Hint: use the binomial theorem...*
11. An element a in a ring R is *idempotent* if $a^2 = a$.
    (a) Prove that the set of idempotents in a commutative ring is closed under multiplication.
    (b) Find all idempotents in the ring $\mathbb{Z}_6 \times \mathbb{Z}_{12}$.
    (c) A ring is *Boolean* if every element is idempotent. Prove that every Boolean ring is commutative. *Hint: consider* $(a + b)^2 \ldots$
    (d) (Hard) Let $\mathcal{U}$ be some set, and $\mathcal{P}(\mathcal{U})$ its power set. For any $A, B \in \mathcal{P}(\mathcal{U})$, define
    $$A + B := (A \cup B) \setminus (A \cap B) \quad \text{and} \quad A \cdot B = A \cap B$$
       i. Give the addition and multiplication tables for $\mathcal{P}(\mathcal{U})$ when $\mathcal{U} = \{a, b\}$ has two elements. (*$\mathcal{P}(\mathcal{U})$ has four elements!*)
       ii. For any set $\mathcal{U}$, prove that $\mathcal{P}(\mathcal{U})$ is a Boolean ring.