Math 120B Rings and Fields: Homework 3

Hand in questions 3(b), 5, 6, 8, 11 & 12 at discussion on Tuesday 23rd October

1. Describe the field of fractions $\text{Frac}(D)$ of the *Gaussian integers*

   $$D = \{x + iy : x, y \in \mathbb{Z}\}$$

2. Let R be a commutative ring, and let T be a non-empty subset of R which is closed under multiplication and contains neither zero nor any zero-divisors. Define a relation on $R \times T$ by

   $$(a, b) \sim (c, d) \iff ad = bc$$

   and define the operations $+, \cdot$ as in the construction of the field of fractions of an integral domain. Convince yourself that we can follow the exact procedure to create a *partial ring of quotients* $Q(R, T)$. In particular, prove the following:

   (a) $Q(R, T)$ has a unity, even if R does not.

   (b) Every element of T is a unit in $Q(R, T)$.

3. Consider the previous exercise.

   (a) How many elements are there in the ring $Q(\mathbb{Z}_4, \{1, 3\})$? Is this ring a field? Why/why not?

   (b) Describe the ring $Q(\mathbb{Z}, \{2^n : n \in \mathbb{N}\})$ by describing a subring of $\mathbb{R}$ to which it is isomorphic. Find the units in this ring.

   (c) Repeat part (b) for the ring $Q(3\mathbb{Z}, \{6^n : n \in \mathbb{N}\})$.

4. Compute the sum and the product of the following polynomials in $\mathbb{Z}_8[x]$:

   $$f(x) = 4x - 5 \qquad g(x) = 2x^2 - 4x + 2$$

5. (a) List all polynomials of degree ≤ 3 in $\mathbb{Z}_2[x]$ (include 0).

   (b) *How many* polynomials of degree ≤ 2 are there in $\mathbb{Z}_5[x]$? Explain.

6. Compute the evaluation homomorphism $\phi_\alpha : \mathbb{Z}_7[x] \to \mathbb{Z}_7$ for each of the following:

   (a) $\phi_3[(x^4 + 2x)(x^3 - 3x^2 + 3)]$

   (b) $\phi_5[(x^3 + 2)(4x^2 + 3)(x^7 + 3x^2 + 1)]$

   (c) $\phi_4[3x^{106} + 5x^{99} + 2x^{53}]$

7. Find all the zeros in the indicated finite field of the given polynomial with coefficients in that field.

   (a) $x^3 + 2x + 2$ in $\mathbb{Z}_7$

   (b) $x^5 + 3x^3 + x^2 + 2x$ in $\mathbb{Z}_5$

   (c) $f(x)g(x)$ where $f(x) = x^3 + 2x^2 + 5$ and $g(x) = 3x^2 + 2x$ in $\mathbb{Z}_7$

8. Use Fermat's Little Theorem to find all the zeros in $\mathbb{Z}_5$ of $2x^{219} + 3x^{74} + 2x^{57} + 3x^{44}$.

9. Consider the evaluation homomorphism $\phi_5 : \mathbb{Q}[x] \to \mathbb{R}$. Find six elements in the kernel of the homomorphism ϕ_5.

10. Find a polynomial of degree > 0 in $\mathbb{Z}_4[x]$ that is a unit.

11. (a) Let D be an integral domain. Describe the units in $D[x]$.
    (b) Find the units in $\mathbb{Z}[x]$.
    (c) Find the units in $\mathbb{Z}_7[x]$.

12. Let F be a field of characteristic zero and let D be the formal polynomial differentiation map:

$$D(a_n x^n + \cdots + a_2 x + a_1 x + a_0) = n \cdot a_n x^{n-1} + \cdots + 2 \cdot a_2 x + a_1$$

    (a) Prove that $D : (F[x], +) \to (F[x], +)$ is a group homomorphism.
    (b) Find the kernel of D.
    (c) Find the image of $F[x]$ under D. Explain your answer.
    (d) Explain why D is *not* a ring homomorphism. Hypothesize an expression for $D(f(x)g(x))$ and *prove* your assertion.
    (*Hint: For parts (c) and (d), think about why you only have to prove for* monomials...)

13. Suppose that $f(x), g(x) \in R[x]$, that $\deg(f(x)) = m$ and $\deg(g(x)) = n$.

    (a) Prove that $\deg(f(x) + g(x)) = N = \max\{m, n\}$ *unless* $m = n$ in which case we can only claim that $\deg(f(x) + g(x)) \leq N$.
    (b) If $k > m + n$, prove that $a_i b_{k-i} = 0$ for all $i \in \{0, \ldots, k\}$, so that $\deg(f(x)g(x)) \leq \deg(f(x)) + \deg(g(x))$.
    (c) In each case, give an explicit example which shows that the inequalities can be strict.