Math 120B Rings and Fields: Homework 4

Hand in questions 2, 3, 5, 6, 9 & 10 at discussion on Tuesday 13th November

1. Describe all ring homomorphisms of $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z} \times \mathbb{Z}$.

2. Find all positive integers n such that $\mathbb{Z}_n$ contains a subring isomorphic to $\mathbb{Z}_2$.
(*Hint: find the subrings of $\mathbb{Z}_4$ and $\mathbb{Z}_6$ which have* two *elements. Are they* both *isomorphic to $\mathbb{Z}_2$?*)

3. (a) Prove that if an ideal $N \leq R$ contains a unit in R, then $N = R$.
   (b) Prove that a field only has precisely two ideals.
   (c) Find a subring of $\mathbb{Z} \times \mathbb{Z}$ which is *not* an ideal of $\mathbb{Z} \times \mathbb{Z}$.

4. A student is asked to prove that a factor ring R/N is commutative if and only if

   $$\forall r, s \in R,\ rs - sr \in N$$

   The student starts by writing,

   Assume R/N is commutative. Then $rs = sr$ for all $r, s \in R/N \ldots$

   (a) Why does the reader expect nonsense from here on?
   (b) Prove the assertion properly.

5. Prove that each homomorphism of from a field to a ring is either injective or maps everything to zero.

6. Let R be a commutative ring with unity with prime characteristic p. Prove that $\phi : R \to R$ defined by $\phi(x) = x^p$ is a homomorphism.

7. Let $\phi : R \to S$ be a ring homomorphism and let N be an ideal of R.
   (a) Prove that $\phi(N)$ is an ideal of $\phi(R)$.
   (b) Show that $\phi(N)$ need not be an ideal of S.
   (c) Let M be an ideal of $\phi(R)$ or S. Prove that $\phi^{-1}(M) = \{x \in R : \phi(x) \in M\}$ is an ideal of R.

8. Let R be a commutative ring and let $a \in R$. Prove that $I_a = \{x \in R : ax = 0\}$ is an ideal of R.

9. Prove that the intersection of ideals of a ring R is an ideal of R. What ideal is the intersection $4\mathbb{Z} \cap 10\mathbb{Z}$ of $\mathbb{Z}$? In general, what is $m\mathbb{Z} \cap n\mathbb{Z}$?

10. Recall that an element a of a ring R is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$.
    (a) Prove that the set of nilpotent elements in a commutative ring R forms an ideal (this is the *nilradical* of R).
    (b) Find the nilradicals of the rings $\mathbb{Z}_{12}, \mathbb{Z}$ and $\mathbb{Z}_{32}$. Can you describe the nilradical in $\mathbb{Z}_n$ in general?
    (c) If N is the nilradical of R, prove that R/N has nilradial $\{0 + N\}$.