Math 120B Rings and Fields: Homework 5

Hand in questions 3, 4, 6, 7 & 10 at discussion on Tuesday 27th November

1. Find all prime ideals and all maximal ideals of the following rings.

   (a) $\mathbb{Z}_6$ (b) $\mathbb{Z}_2 \times \mathbb{Z}_2$ (c) $\mathbb{Z}_2 \times \mathbb{Z}_4$
   (*Hint: for parts (b), (c), you might find it helpful to consider each additive subgroup.*)

2. Find all elements $c \in \mathbb{Z}_3$ such that each of the following is a field.

   (a) $\mathbb{Z}_3[x] / \langle x^2 + c \rangle$ (b) $\mathbb{Z}_3[x] / \langle x^3 + x^2 + c \rangle$ (c) $\mathbb{Z}_3[x] / \langle x^3 + cx^2 + 1 \rangle$

3. Repeat the previous question for the following quotient rings where $c \in \mathbb{Z}_5$.

   (a) $\mathbb{Z}_5[x] / \langle x^2 + x + c \rangle$ (b) $\mathbb{Z}_5[x] / \langle x^2 + cx + 1 \rangle$

4. Find a maximal ideal of $\mathbb{Z} \times \mathbb{Z}$: justify your answer.

5. $x^2 - 5x + 6 \in \mathbb{Q}[x]$ is reducible over $\mathbb{Q}$. Prove that the quotient ring $\mathbb{Q}[x] / \langle x^2 - 5x + 6 \rangle$ is not a field by finding a non-zero non-unit element.

6. Let R be a finite commutative ring with unity. Show that every prime ideal in R is a maximal ideal.

7. We've seen that every ring with unity contains a subring isomorphic to either $\mathbb{Z}$ or some $\mathbb{Z}_n$.

   (a) Is it possible that a ring with unity may contain *two* subrings isomorphic to $\mathbb{Z}_m$ and $\mathbb{Z}_n$ with $m \neq n$? If yes, give an example. If no, prove it.
   (b) Is part (a) possible for subrings $\mathbb{Z}_p, \mathbb{Z}_q$ where $p \neq q$ are both *prime*? Explain.
   (c) Repeat part (b) for an *integral domain.*

   These last three questions are more of a challenge.

8. Prove that N is a maximal ideal in a ring R if and only if R/N is non-trivial and has no proper nontrivial ideals (R/N is called a *simple ring*).
   (*Hint: let* $\gamma : R \to R/N$ *be the canonical homomorphism, and appeal to question 7 of the last homework. Note that* γ *is surjective...*)

9. Prove that $M_2(\mathbb{Z}_2)$ is a simple ring (this is the set of 2×2 matrices over $\mathbb{Z}_2$).
   (*Hint: Any proper ideal cannot contain any units, now rule out the remaining non-trivial elements...*)

10. Let $\mathbb{F}$ be a field and let $f, g \in \mathbb{F}[x]$. Show that

    $$N = \{rf + sg : r, s \in \mathbb{F}[x]\}$$

    is an ideal of $\mathbb{F}[x]$. Show that if $\deg(f) \neq \deg(g)$ and that if N is a proper ideal, then f and g cannot both be irreducible over $\mathbb{F}$.
    (*Hint: Recall that every ideal of* $\mathbb{F}[x]$ *is principal...*
    This question is related to the Euclidean Algorithm, and the notion of a greatest common divisor of f, g.)