Math 120B Rings and Fields: Homework 6

Hand in questions 1(b,c,d,e), 2, 3, 4(b), & 5 at discussion on Tuesday 4th December

1. For each of the following complex numbers α: identify whether α is algebraic or transcendental over the base field $\mathbb{F}$, justify your answer and find the minimal polynomial $m_{\alpha,\mathbb{F}}$ if algebraic.

	α	$\mathbb{F}$		α	$\mathbb{F}$
(a)	$1+\sqrt{2}$	$\mathbb{Q}$	(e)	$\sqrt{\pi+1}$	$\mathbb{Q}$
(b)	$1+i$	$\mathbb{Q}$	(f)	$\sqrt{i+\sqrt{2}}$	$\mathbb{Q}(i)$
(c)	$\sqrt{1+\sqrt[3]{2}}$	$\mathbb{Q}$	(g)	$\sqrt{i+\sqrt{2}}$	$\mathbb{Q}$
(d)	$\sqrt{1+\sqrt[3]{2}}$	$\mathbb{Q}(\sqrt[3]{2})$	(h)	$\sqrt{e^2+4}$	$\mathbb{Q}(e)$

2. (a) Find a subfield $\mathbb{F}$ of $\mathbb{R}$ such that π is algebraic of degree 3 over $\mathbb{F}$.
   (b) Find a subfield $\mathbb{F}$ of $\mathbb{R}$ such that e^2 is algebraic of degree 5 over $\mathbb{F}$.

3. Prove that $3^{1/3} \notin \mathbb{Q}(2^{1/2})$ and thus find a basis for the field $\mathbb{Q}(2^{1/2}, 3^{1/3})$ over $\mathbb{Q}$.

4. (a) Prove that $\mathbb{Q}(\sqrt{3}+\sqrt{5}) = \mathbb{Q}(\sqrt{3},\sqrt{5})$. Find all the intermediate fields of the extension $\mathbb{Q}(\sqrt{3}+\sqrt{5}) : \mathbb{Q}$. (*Look at the calculation for* $\mathbb{Q}(\sqrt{2}+i)$ *in the notes...*)
   (b) Find an element $\alpha \in \mathbb{C}$ so that $\mathbb{Q}(\alpha) = \mathbb{Q}(\sqrt{2},\sqrt[3]{2})$. Find all the intermediate fields of this extension.

5. Show that the polynomial $x^2+1 \in \mathbb{Z}_3[x]$ is irreducible. Let α be a zero of this polynomial and construct the simple extension

$$\mathbb{F}_9 := \mathbb{Z}_3(\alpha) = \{0,\ 1,\ 2,\ \alpha,\ 1+\alpha,\ 2+\alpha,\ 2\alpha,\ 1+2\alpha,\ 2+2\alpha\} \cong \mathbb{Z}_3[x] \big/ \langle x^2+1 \rangle$$

   Find the addition and multiplication tables of this field. To what elementary groups are $(\mathbb{F}_9,+)$ and $(\mathbb{F}_9^\times,\cdot)$ isomorphic?
   (*Hint: consider the cyclic group* $\langle 1+\alpha\rangle \leq (\mathbb{F}_9^\times,\cdot)$)

6. Repeat the previous question with the polynomial $x^3+x^2+1 \in \mathbb{Z}_2[x]$. You should construct a field $\mathbb{F}_8$ with eight elements.

7. Let $\mathbb{F}$ be a finite field with q elements.
   (a) Explain why $\mathbb{F}$ must have *prime* characteristic.
   (b) If $\mathbb{E}$ is a finite extension of $\mathbb{F}$ such that $[\mathbb{E}:\mathbb{F}] = n$, prove that $|\mathbb{E}| = q^n$.
   (c) Let $\mathbb{K}$ be a finite field of characteristic p. Prove that $\mathbb{K}$ contains a subfield (isomorphic to) $\mathbb{Z}_p$. Thus explain why $\mathbb{K}$ has order p^n for some $n \in \mathbb{N}$ and why $\mathbb{K} : \mathbb{Z}_p$ is an algebraic extension.
   (d) If $|\mathbb{K}| = p^n$, describe the group $(\mathbb{K},+)$ in accordance with the Fundamental Theorem of Finitely Generate Abelian Groups.
   Show that every element of $\alpha \in \mathbb{K}^\times$ satisfies the equation $\alpha^{p^n-1} = 1$. Explain why $\exists k$ a divisor of p^n-1 for which all elements $\alpha \in \mathbb{K}^\times$ satisfy $\alpha^k = 1$. How may distinct zeros can the polynomial x^k-1 have over a field? Hence identify the group $(\mathbb{K}^\times,\cdot)$.