Math 120B Rings and Fields: Extra Questions for Final

1. Find a basis for the given field extension: there is no need to do so, but you should be able to justify your assertions if necessary.

   (a) $\mathbb{Q}(\sqrt{2}, \sqrt{5}) : \mathbb{Q}$
   (b) $\mathbb{Q}(\sqrt{2}\sqrt{3}) : \mathbb{Q}$
   (c) $\mathbb{Q}(\sqrt{2} + \sqrt{5}) : \mathbb{Q}$
   (d) $\mathbb{Q}(\sqrt{2}, \sqrt[3]{5}) : \mathbb{Q}$
   (e) $\mathbb{Q}(\sqrt[3]{2}, \sqrt[3]{6}, \sqrt[3]{24}) : \mathbb{Q}$
   (f) $\mathbb{Q}(\sqrt{2}, \sqrt{6}) : \mathbb{Q}(\sqrt{3})$
   (g) $\mathbb{Q}(\sqrt{2} + \sqrt{3}) : \mathbb{Q}(\sqrt{3})$
   (h) $\mathbb{Q}(\sqrt{2}, \sqrt{6} + \sqrt{10}) : \mathbb{Q}(\sqrt{3} + \sqrt{5})$

2. Let $\alpha = a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $b \neq 0$. Prove that $\mathbb{C} = \mathbb{R}(\alpha)$. Hence or otherwise, prove that every *finite* extension field of $\mathbb{R}$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$.

3. Let α be a zero of the irreducible polynomial $x^2 + x + 1$ over $\mathbb{Z}_2$. The field extension $\mathbb{Z}_2(\alpha)$ is therefore algebraic and so the element $1 + \alpha \in \mathbb{Z}_2(\alpha)$ is also algebraic over $\mathbb{Z}_2$. Find the minimal polynomial of $1 + \alpha$ over $\mathbb{Z}_2$.

4. Prove that $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt[3]{2})$.

5. Let $a, b \in \mathbb{Q}$. Prove that $\mathbb{Q}(\sqrt{a} + \sqrt{b}) = \mathbb{Q}(\sqrt{a}, \sqrt{b})$.
   (*You may assume that* $\sqrt{a} + \sqrt{b} \neq 0$)

6. Let $\mathbb{E}$ be a finite extension of $\mathbb{F}$ and let $f \in \mathbb{F}[x]$ be irreducible and have a degree which is not a divisor of $[\mathbb{E} : \mathbb{F}]$. Prove that f has no zeros in $\mathbb{E}$. By giving a counterexample, explain why we need the *irreducibility* condition?

7. Let $\alpha \in \mathbb{E}$ be algebraic of odd degree over $\mathbb{F}$. Prove that α^2 is algebraic of odd degree over $\mathbb{F}$ and that $\mathbb{F}(\alpha^2) = \mathbb{F}(\alpha)$.

8. Prove that no finite field $\mathbb{F}$ can be algebraically closed.
   (*Hint: If* $\mathbb{F} = \{a_1, \ldots, a_n\}$*, can you construct a polynomial with no zeros in* $\mathbb{F}$*?*)

9. Prove that every finite extension field of $\mathbb{R}$ is isomorphic to either $\mathbb{R}$ or $\mathbb{C}$.
   (*Be very careful with this!*)

10. For each $n \in \mathbb{N}_0$, define the field $\mathbb{F}_n$ inductively:

$$\mathbb{F}_1 = \mathbb{Q}, \qquad \mathbb{F}_{n+1} = \mathbb{F}_n(n^{1/n})$$

   (a) What, explicitly are the fields $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5$ and what are their degrees over $\mathbb{Q}$?

   (b) Prove explicitly that the set $\mathbb{E} := \bigcup_{n=1}^{\infty} \mathbb{F}_n$ is an infinite algebraic field extension of $\mathbb{Q}$.