Math 120B Rings and Fields: Extra Questions for Midterm

Submit nothing!

1. Given $f, g \in \mathbb{F}[x]$, find the polynomials $q, r \in \mathbb{F}[x]$ satisfying the division algorithm.
   (a) $f(x) = x^6 + 3x^5 + 4x^2 - 3x + 2$ and $g(x) = x^2 + 2x - 3$ in $\mathbb{Z}_7[x]$.
   (b) $f(x) = x^5 - 2x^4 + 3x - 5$ and $g(x) = 2x + 1$ in $\mathbb{Z}_{11}[x]$.
2. Express the polynomial as a product of linear factors in the given polynomial ring.
   (a) $x^4 + 4$ in $\mathbb{Z}_5[x]$.
   (b) $x^3 + 2x^2 + 2x + 1$ in $\mathbb{Z}_7[x]$.
   (c) $2x^3 + 3x^2 - 7x - 5$ in $\mathbb{Z}_{11}[x]$.
3. Express each of the following polynomials $\mathbb{Z}_5[x]$ as a product of irreducible factors.
   (a) $x^3 + 2x + 3$
   (b) $2x^3 + x^2 + 2x + 2$
4. Prove that each of the following polynomials is irreducible over $\mathbb{Q}$. Are they irreducible over $\mathbb{R}$ or $\mathbb{C}$?
   (a) $x^2 + 8x - 2$
   (b) $x^2 + 6x + 12$
   (c) $x^3 + 3x^2 - 8$
   (d) $x^4 - 22x^2 + 1$
5. Determine whether each polynomial in $\mathbb{Z}[x]$ satisfies an Eisenstein criterion for irreducibility over $\mathbb{Q}$.
   (a) $x^2 - 12$
   (b) $8x^3 + 6x^2 - 9x + 24$
   (c) $4x^{10} - 9x^3 + 24x - 18$
   (d) $2x^{10} - 25x^3 + 10x^2 - 30$
6. Find all the zeros of $6x^4 + 17x^3 + 7x^2 + x - 10$ in $\mathbb{Q}$.
7. Find all prime numbers p such that $x + 2$ is a factor of $x^4 + x^3 - x^2 - x + 1 \in \mathbb{Z}_p[x]$.
8. Find the number of irreducible degree 3 polynomials in $\mathbb{Z}_3[x]$.
9. Find the number of irreducible quadratic polynomials in $\mathbb{Z}_p[x]$ where p is a prime. (*Hint: Find the number of reducible quadratics first!*)
10. If p is a prime $a \in \mathbb{Z}_p$, prove that the polynomial $x^p + a$ is reducible in $\mathbb{Z}_p[x]$.
    (*Hint: Consider $p = 2$ separately from when p is odd*)

11. If $\mathbb{F}$ is a field and $a \neq 0$ is a zero of $f(x) = a_n x^n + \cdots + a_1 x + a_0$, prove that a^{-1} is a zero of $g(x) = a_0 x^n + \cdots + a_{n-1} x + a_n$.

12. (a) Prove that if $\sigma : \mathbb{Z} \to \mathbb{Z}_m$ is the homomorphism $\sigma(a) = a \mod m$, then the function

$$\widetilde{\sigma} : \mathbb{Z}[x] \to \mathbb{Z}_m[x] : \sum a_k x^k \mapsto \sum \sigma(a_k) x^k$$

is a *surjective* homomorphism.

(b) Hence properly justify the argument seen in class: if p is prime and $f(x) \in \mathbb{Z}[x]$ is a reducible *monic* polynomial, so is $\widetilde{\sigma}(f(x)) \in \mathbb{Z}_p[x]$.

(c) Prove that $x^3 + 17x + 36$ is irreducible over $\mathbb{Q}$.
(*Hint: find a suitable prime...*)