

4 Determinants

To a square matrix A , it turns out that we can attach a single number, its *determinant*, which encapsulates the extent to which the linear map L_A enlarges or contracts space.

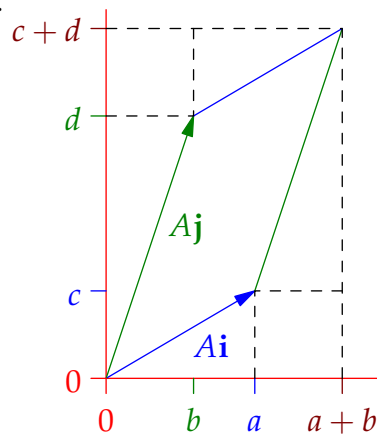
For instance, consider a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Its columns are the result of multiplying the standard basis vectors \mathbf{i}, \mathbf{j} by A :

$$A\mathbf{i} = \begin{pmatrix} a \\ c \end{pmatrix} \quad A\mathbf{j} = \begin{pmatrix} b \\ d \end{pmatrix}$$

For simplicity, suppose $a, b, c, d > 0$ and that the columns are oriented as in the picture. The *unit square* spanned by \mathbf{i}, \mathbf{j} is transformed by A to a *parallelogram*, whose area is

$$(a+b)(c+d) - 2bc - 2 \cdot \frac{1}{2}bd - 2 \cdot \frac{1}{2}ac = ad - bc$$

This one number neatly summarizes how left-multiplication by A changes the *area* of a shape.



4.1 Determinants of Order 2

Definition 4.1. The *determinant* $\det A = |A|$ of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the scalar

$$\det A = ad - bc$$

Example 4.2. If $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 1 & -2 \end{pmatrix}$, then

$$\det A = 1 \cdot 3 - 2 \cdot 4 = -5, \quad \det B = 5 \cdot (-2) - 0 \cdot 1 = -10$$

Note that $\det : M_2(\mathbb{F}) \rightarrow \mathbb{F}$ is a *non-linear* function; for instance

$$\det(A+B) = \begin{vmatrix} 6 & 2 \\ 5 & 1 \end{vmatrix} = 6 - 10 = -4 \neq \det A + \det B$$

However, determinant does play nicely with matrix multiplication:

$$\det AB = \begin{vmatrix} 7 & -4 \\ 23 & -6 \end{vmatrix} = -42 + 92 = 50 = \det A \det B$$

In the following results, we summarize the key properties of order 2 determinants: with the exception of the explicit inverse formula, these will eventually be seen to hold in higher dimensions.

Theorem 4.3 (Basic properties of order-two determinants). 1. $\det A^T = \det A$

2. $\det A = 0$ if and only if the columns (rows) of A are parallel (linearly dependent).

3. Determinant is a bilinear function of the columns (rows) of A .

4. $\det AB = \det A \det B$

These are easily verified directly: write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, etc. The third property benefits from a little expansion: writing a matrix in terms of its columns, determinant can be thought of as a function

$$\det : \mathbb{F}^2 \times \mathbb{F}^2 \rightarrow \mathbb{F} : (\mathbf{a}_1 \ \mathbf{a}_2) \mapsto \det(\mathbf{a}_1 \ \mathbf{a}_2)$$

and property 3 claims that

$$\det(\lambda \mathbf{u} + \mathbf{v}, \mathbf{w}) = \lambda \det(\mathbf{u}, \mathbf{w}) + \det(\mathbf{v}, \mathbf{w}) \quad \text{and} \quad \det(\mathbf{u}, \lambda \mathbf{v} + \mathbf{w}) = \lambda \det(\mathbf{u}, \mathbf{v}) + \det(\mathbf{u}, \mathbf{w})$$

and similarly with regard to the rows of a matrix (a pair of row vectors).

Example 4.4. The fact that the first rows are identical means we can combine

$$-24 = -5 - 19 = \det \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} + \det \begin{pmatrix} 1 & 2 \\ 5 & -9 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 \\ 9 & -6 \end{pmatrix} = -6 - 18 = -24$$

Properties 2 and 3 partly overlap with the effect of row/column operations on determinant.

Corollary 4.5 (Row/column operations).

Type I: *Swapping rows (or columns) changes the sign of $\det A$*

Type II: *Multiplying a row (or column) by λ multiplies $\det A$ by λ*

Type III: *Adding a multiple of one row (or column) to another leaves $\det A$ unchanged*

Proof. Simply combine the product formula with the list of all elementary matrices

	Type	Matrices	Determinant
$\det EA = \det AE = \det E \det A$	I	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\det E = -1$
	II	$\begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$	$\det E = \lambda$
	III	$\begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$	$\det E = 1$

Alternatively you can compute all possibilities directly. ■

Corollary 4.6 (Inverses). A is invertible (non-singular) if and only if $\det A \neq 0$. In such a case, $\det A^{-1} = \frac{1}{\det A}$ and

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof. (\Rightarrow) If A is invertible, the product formula tells us that

$$1 = \det I_2 = \det(AA^{-1}) = \det A \det A^{-1} \implies \det A \neq 0 \text{ and } \det A^{-1} = \frac{1}{\det A}$$

(\Leftarrow) Observe that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (\det A) I_2$$

If $\det A \neq 0$, we divide by $\det A$ to see that A has an inverse given by the desired expression. ■

Oriented Area and the Determinant

In the introduction, we considered a basic example of how determinant relates to area. We now proceed more formally.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\mathbf{u} \ \mathbf{v})$ be written in terms of its columns: we consider the parallelogram \mathcal{P} obtained by applying A to the unit square, i.e. \mathcal{P} is spanned by the vectors $\mathbf{u} = A\mathbf{i}$ and $\mathbf{v} = A\mathbf{j}$.

First observe (Theorem 4.3, part 2) that

$$\det A = \det(\mathbf{u}, \mathbf{v}) = 0 \iff \mathbf{u}, \mathbf{v} \text{ are parallel} \iff \mathcal{P} \text{ has zero area}$$

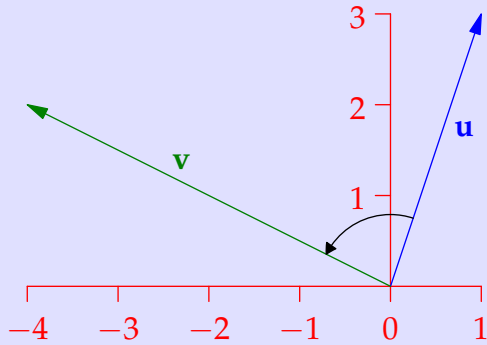
This gives a pictorial way to understand zero determinant and lack of invertibility: if \mathbf{u}, \mathbf{v} are parallel then any purported inverse would map these to *parallel* vectors $A^{-1}\mathbf{u}, A^{-1}\mathbf{v}$ which couldn't span the original unit square! A linear map can *scale* area, but it cannot create area out of nothing.

Now consider the case when $\det A \neq 0$. In particular, this requires at least one of $a, c \neq 0$. It is straightforward to check that the matrix $R = \frac{1}{\sqrt{a^2+c^2}} \begin{pmatrix} a & c \\ -c & a \end{pmatrix}$ acts by *counter-clockwise rotation* and therefore *preserves area*. Now compute,

$$RA = \frac{1}{\sqrt{a^2+c^2}} \begin{pmatrix} a & c \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{a^2+c^2}} \begin{pmatrix} a^2+c^2 & ab+cd \\ 0 & \det A \end{pmatrix} \quad (*)$$

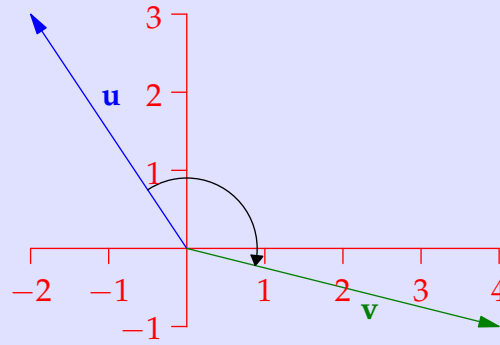
Since $a^2 + c^2 > 0$, we see that the sign of $\det A$ determines whether we rotate counter-clockwise ($\det A > 0$) or clockwise ($\det A < 0$) to get from the first to the second column of A .

Definition 4.7. An ordered pair of vectors (\mathbf{u}, \mathbf{v}) in \mathbb{R}^2 is *positively-oriented* if $\det(\mathbf{u}, \mathbf{v}) > 0$ and *negatively-oriented* if $\det(\mathbf{u}, \mathbf{v}) < 0$.



Positively-oriented: counter-clockwise rotation

$$\det(\mathbf{u}, \mathbf{v}) = \begin{vmatrix} 1 & -4 \\ 3 & 2 \end{vmatrix} = 14 > 0$$



Negatively-oriented: clockwise rotation

$$\det(\mathbf{u}, \mathbf{v}) = \begin{vmatrix} -2 & 4 \\ 3 & -1 \end{vmatrix} = -10 < 0$$

A short exercise following from $(*)$ results in a complete proof of the following:

Theorem 4.8. The area of the parallelogram spanned by the columns of A is

- $\det A$ if the columns are positively-oriented;
- $-\det A$ if the columns are negatively-oriented.

As such, $\det A$ is often known as the *oriented area* of a parallelogram.

Exercises 4.1 1. Compute the determinants of the following matrices and, if possible, their inverses:

- (a) $\begin{pmatrix} 1 & 4 \\ 7 & -3 \end{pmatrix} \in M_2(\mathbb{R})$ (b) $\begin{pmatrix} i & 1+i \\ 1 & i \end{pmatrix} \in M_2(\mathbb{C})$ (c) $\begin{pmatrix} 3 & 2 \\ 1 & 4 \end{pmatrix} \in M_2(\mathbb{Z}_5)$ (d) $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \in M_2(\mathbb{Z}_5)$
 (Recall that $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ is the field of remainders modulo 5)

2. Explicitly prove all parts of Theorem 4.3.

3. Find the area of the triangles (*a triangle is half a parallelogram...*):

- (a) With vertices $(0, 0)$, $(2, 4)$ and $(1, -2)$.
 (b) With vertices $(2, 1)$, $(3, -2)$ and $(-5, 4)$.

4. (a) Show that the matrix $R = \frac{1}{\sqrt{a^2+c^2}} \begin{pmatrix} a & c \\ -c & a \end{pmatrix}$ acts by counter-clockwise rotation.
 (Hint: the columns of R are related by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ -c \end{pmatrix} = \begin{pmatrix} c \\ a \end{pmatrix}$)
 (b) Complete the proof of Theorem 4.8.

5. Let \mathbb{F} be a field and $\Delta : \mathbb{F}^2 \times \mathbb{F}^2 \rightarrow \mathbb{F}$ be a bilinear function: $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{F}^2, \lambda \in \mathbb{F}$,

$$\Delta(\lambda \mathbf{u} + \mathbf{v}, \mathbf{w}) = \lambda \Delta(\mathbf{u}, \mathbf{w}) + \Delta(\mathbf{v}, \mathbf{w}) \quad \text{and} \quad \Delta(\mathbf{u}, \lambda \mathbf{v} + \mathbf{w}) = \lambda \Delta(\mathbf{u}, \mathbf{v}) + \Delta(\mathbf{u}, \mathbf{w})$$

- (a) We say that Δ is *alternating* if $\Delta(\mathbf{u}, \mathbf{u}) = 0$ for all $\mathbf{u} \in \mathbb{F}^2$.
 i. Prove that Δ alternating $\implies \Delta(\mathbf{v}, \mathbf{u}) = -\Delta(\mathbf{u}, \mathbf{v})$ for all \mathbf{u}, \mathbf{v} .
 ii. Prove the converse to part (a) (*provided $2 \neq 0$ in \mathbb{F} !*).
 (b) Prove that if Δ is an alternating bilinear function satisfying $\Delta(\mathbf{i}, \mathbf{j}) = 1$, then $\Delta = \det$.

6. (*A link to multivariable calculus*) Let $D, E \subseteq \mathbb{R}^2$ and $T : D \rightarrow E$ be a change of co-ordinates

$$T(u, v) = (x(u, v), y(u, v))$$

Assume T is bijective and that the partial derivatives of x, y with respect to u, v exist and are continuous. Note that T does not have to be linear!

Let $P = (u_0, v_0) \in D$, and consider small positive quantities $\Delta u, \Delta v$ to define points

$$Q = (u_0 + \Delta u, v_0), \quad R = (u_0, v_0 + \Delta v)$$

The area of the parallelogram spanned by $\overrightarrow{PQ} = \mathbf{i}\Delta u$ and $\overrightarrow{PR} = \mathbf{j}\Delta v$ is therefore $\Delta u \Delta v$.

Prove that the parallelogram spanned by $\overrightarrow{T(P)T(Q)}$ and $\overrightarrow{T(P)T(R)}$ has

$$\text{Area} \approx \left| \det \begin{pmatrix} x_u(P) & x_v(P) \\ y_u(P) & y_v(P) \end{pmatrix} \right| \Delta u \Delta v$$

where $x_u = \frac{\partial x}{\partial u}$, etc., denote partial derivatives.

This determinant is the Jacobian $J(T) = \frac{\partial(x, y)}{\partial(u, v)}$. The above is essentially the justification for the change of variables formula for double integrals:

$$\iint_E f(x, y) \, dx \, dy = \iint_D f(x(u, v), y(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| \, du \, dv$$

4.2 Higher-Order Determinants

We now extend the definition of determinant to any size of square matrix. The goal is to establish all the basic properties seen for order 2 determinants. This will take a little time...

Definition 4.9. Let $A \in M_n(\mathbb{F})$. For each i, j we defined the ij^{th} minor of A to be the matrix \tilde{A}_{ij} obtained by deleting the i^{th} row and j^{th} column of A . The *determinant* of A to be the sum

$$\det A = \sum_{j=1}^n (-1)^{1+j} a_{1j} \det \tilde{A}_{1j} = a_{11} \det \tilde{A}_{11} - a_{12} \det \tilde{A}_{12} + \cdots + (-1)^{1+n} a_{1n} \det \tilde{A}_{1n}$$

This is known as the *cofactor expansion* of $\det A$ along the first row of A .

The difficulty should be immediately obvious: the definition is *inductive*! For instance the determinant of $A \in M_4(\mathbb{F})$ is defined in terms of the *four* 3×3 determinants, each of which is computed using *three* 2×2 determinants: in total we need *twelve* order 2 determinants!

Example 4.10. Let $A = \begin{pmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & 2 & -1 \\ -2 & 3 & 2 & 6 \\ 1 & 2 & 1 & 1 \end{pmatrix}$, then

$$\begin{aligned} \det A &= 1 \begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & 6 \\ 2 & 1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 2 & -1 \\ -2 & 2 & 6 \\ 1 & 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 & -1 \\ -2 & 3 & 6 \\ 1 & 2 & 1 \end{vmatrix} - (-3) \begin{vmatrix} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 2 & -1 \\ 3 & 2 & 6 \\ 2 & 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 1 & -1 \\ -2 & 3 & 6 \\ 1 & 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 & 2 \\ -2 & 3 & 2 \\ 1 & 2 & 1 \end{vmatrix} \end{aligned}$$

Next we compute the remaining 3×3 determinants using the cofactor expansion:

$$\begin{aligned} \det A &= \left(\begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \right) \\ &\quad + 2 \left(0 \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 6 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} \right) \\ &\quad + 3 \left(0 \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} + 2 \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} \right) \\ &= \begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 6 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 6 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} + 6 \begin{vmatrix} -2 & 3 \\ 1 & 2 \end{vmatrix} \end{aligned}$$

Each of the remaining 2×2 determinants can now be evaluated:

$$\det A = -4 - 2(-9) - (-1) - 2(-8) - 2(-7) - 3(-4) + 6(-7) = 15$$

While the above calculation was assisted by the fact that we only needed to compute three 3×3 determinants, it is still very slow-going. As the order n gets larger, things become ugly very quickly. In order to facilitate more rapid calculations it is useful to develop some of the properties we saw in the previous section. We begin by computing the determinant of the $n \times n$ identity matrix I_n . The

cofactor expansion along the first row yields

$$\det I_n = 1 \cdot \det I_{n-1} - 0 \det (\tilde{I}_n)_{12} + 0 \det (\tilde{I}_n)_{13} - \cdots + (-1)^{1+n} \cdot 0 \det (\tilde{I}_n)_{1n} = \det I_{n-1}$$

By induction, since $\det I_2 = 1$, we conclude:

Lemma 4.11. $\det I_n = 1$.

The next result combines several basic properties, all of which we have checked for order 2 determinants. Several of the arguments rely on lengthy inductions.

Theorem 4.12. 1. $\det : M_n(\mathbb{F}) \rightarrow \mathbb{F}$ is a linear function of each row when the other rows are fixed.

2. If A has a row of zeros, then $\det A = 0$.

3. The determinant can be evaluated using the cofactor expansion along any row:

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det \tilde{A}_{ij}$$

4. If A has two identical rows then $\det A = 0$.

Sketch Proof. 1. This is by induction. We know the statement is true for all 2×2 matrices. Fix n and suppose the claim is true for all $n \times n$ matrices. Let A be order $n+1$, with i^{th} row a linear combination

$$\mathbf{a}_i^T = \lambda \mathbf{b}_i^T + \mathbf{c}_i^T \quad (*)$$

Also let B, C be the matrices obtained by replacing row i of A with \mathbf{b}^T and \mathbf{c}^T respectively. There are two cases.

(a) If $i = 1$, then the cofactor expansion along the first row is plainly linear since B, C have the same minors as A :

$$a_{1j} = \lambda b_j + c_j \implies \det A = \sum_{j=1}^{n+1} (-1)^{1+j} (\lambda b_j + c_j) |\tilde{A}_{1j}| = \lambda \det B + \det C$$

(b) If $i > 1$, then each of the $n \times n$ minors \tilde{A}_{1j} has its $(i-1)^{\text{th}}$ row observing the same linear combination (*). By the induction hypothesis,

$$\det A = \sum_{j=1}^{n+1} (-1)^{1+j} a_{1j} |\tilde{A}_{1j}| = \sum_{j=1}^{n+1} (-1)^{1+j} a_{1j} (\lambda |\tilde{B}_{1j}| + |\tilde{C}_{1j}|) = \lambda \det B + \det C$$

2. Re-run the proof of part 1 without C and taking $\lambda = 0$: i.e. suppose (*) is simply $\mathbf{a}^T = 0\mathbf{b}^T$ where \mathbf{b}^T is any row vector.

3. We omit the argument since it requires a *long* induction based on parts 1 and 2.

4. This is Exercise 4.2.5. ■

Examples 4.13. 1. Look for rows with many zeros when computing the determinant! Compare, for instance, the cofactor expansions of the following along the first and second rows:

$$\text{First row: } \det \begin{pmatrix} 5 & 3 & 2 \\ 0 & 0 & 2 \\ 7 & 2 & 1 \end{pmatrix} = 5 \begin{vmatrix} 0 & 2 \\ 7 & 1 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 7 & 1 \end{vmatrix} + 2 \begin{vmatrix} 0 & 0 \\ 7 & 2 \end{vmatrix} = 5 \cdot (-4) - 3 \cdot (-14) + 2 \cdot 0 = 22$$

$$\text{Second row: } \det \begin{pmatrix} 5 & 3 & 2 \\ 0 & 0 & 2 \\ 7 & 2 & 1 \end{pmatrix} = -0 + 0 - 2 \begin{vmatrix} 5 & 3 \\ 7 & 2 \end{vmatrix} = 22$$

2. Here we take advantage of linearity and the fact that the first two rows are nearly identical before expanding along the second row of the resulting 3×3 determinant:

$$\det \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 5 \end{pmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 5 \end{vmatrix} - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 0 & -1 \\ 1 & 0 & 2 & 5 \end{vmatrix} = 0 - 4 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix} = -4 \left(- \begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} \right) = 24$$

Elementary matrices and the determinant

We now consider the effect of elementary row operations on the determinant.

Corollary 4.14. Let A be an $n \times n$ matrix and let E an elementary matrix. For each type:

Type I: $\det EA = -\det A$, and $\det E = -1$;

Type II: If E multiplies a row by λ , then $\det EA = \lambda \det A$, and $\det E = \lambda$;

Type III: $\det EA = \det A$ and $\det E = 1$.

Warning! Multiplying every row by λ yields $\det(\lambda A) = \lambda^n \det A$

It is worth taking stock for a moment:

- For order two determinants (Corollary 4.5) these results followed from the multiplicative property $\det AB = \det A \det B$.
- For higher order, we need to prove directly (using Theorem 4.12). Indeed the Corollary establishes the limited multiplicative property $\det EA = \det E \det A$ whenever E is elementary. We will use this fact in the next section to prove the full multiplication formula.

Proof. 1. See Exercise 4.2.5

2. Suppose $E = E_i^{(\lambda)}$ multiplies row i by λ . Since \det is linear in each row, we immediately see that $\det EA = \lambda \det A$.

3. Suppose $E_{ik}^{(\lambda)}$ adds λ times row k to row i . Since determinant is linear in row i , we see that

$$\det EA = \det A + \lambda \det B$$

where B is the matrix obtained from A by replacing row i with row k . Since B has two identical rows, we conclude that $\det EA = \det A$.

Letting $A = I_n$ in each case gives the advertised values for $\det E$. ■

Examples 4.15. 1. In case the argument for part 3 of the theorem is unclear, here is an example: if

$$E = E_{21}^{(5)} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \text{ and } A = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix}, \text{ then}$$

$$B = \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix} \text{ and } \det EA = \det \begin{pmatrix} 4 & 2 \\ 3+20 & -1+10 \end{pmatrix} = \det \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} + 5 \det \begin{pmatrix} 4 & 2 \\ 4 & 2 \end{pmatrix}$$

2. We can use all the above results to assist us in computing determinants. For instance,

$$\begin{aligned} \det \begin{pmatrix} 1 & 1 & 2 & -3 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 2 & 7 & 4 & -6 \end{pmatrix} &= 0 + 3 \begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & -1 \\ 2 & 4 & -6 \end{vmatrix} - \begin{vmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 2 & 7 & -6 \end{vmatrix} + 0 \quad (\text{cofactor expansion 2nd row}) \\ &= 3 \begin{vmatrix} 1 & 2 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{vmatrix} - \begin{vmatrix} 1 & 1 & -3 \\ 0 & 1 & -1 \\ 0 & 5 & 0 \end{vmatrix} \quad (\text{type III row operations simplify third rows}) \\ &= 0 - \left(0 - 5 \begin{vmatrix} 1 & -3 \\ 0 & -1 \end{vmatrix} + 0 \right) \quad (\text{equal rows/cofactor expansion along third row}) \\ &= 5(-1 - 0) = -5 \end{aligned}$$

We are now in a position to establish the relationships between determinant, products and inverses.

Theorem 4.16. Let $A, B \in M_n(\mathbb{F})$. Then:

1. $\det AB = \det A \det B$
2. A is invertible $\iff \det A \neq 0$, in which case $\det A^{-1} = \frac{1}{\det A}$

Proof. We first establish both results when A is non-invertible (singular). Since the row space has dimension $\text{rank } A < n$, at least one row (row i say) is a linear combination of the others:

$$\mathbf{a}_i^T = \sum_{k \neq i} c_k \mathbf{a}_k^T$$

Applying row operations of type III, namely multiplication by $E_{ik}^{(-c_k)}$, we see that

$$\det A = \det \left(\prod_{k \neq i} E_{ik}^{(-c_k)} A \right) = 0$$

since the right hand matrix has row i identically zero. Moreover, $\text{rank } AB \leq \text{rank } A < n$ so that AB is also singular. We conclude that $\det AB = 0 = \det A \det B$.

Now suppose A is invertible. Then $A = E_k \cdots E_1$ is a product of elementary matrices. The multiplicative property for elementary matrices (Corollary 4.14) now proves the general result!

$$\det AB = \det E_k \cdots \det E_1 \det B = \det(E_k \cdots E_1) \det B = \det A \det B$$

Let $B = I$ to see that $\det A = \det E_k \cdots \det E_1 \neq 0$. Finally, let $B = A^{-1}$ to see that

$$1 = \det I_n = \det AA^{-1} = \det A \det A^{-1} \implies \det A^{-1} = \frac{1}{\det A}$$

■

Only one basic property of determinant now remains.

Theorem 4.17. $\det A^T = \det A$.

Proof. First observe that $\det A = 0 \iff \det A^T = 0$, since $\text{rank } A = \text{rank } A^T$.

Suppose now that A is invertible. Then $A = E_k \cdots E_1$ is the product of elementary matrices. But then

$$A^T = E_1^T \cdots E_k^T$$

is also the product of elementary matrices. Moreover, E_i^T is an elementary matrix of the same type and determinant as E_i . It follows that

$$\det A^T = \det E_1^T \cdots \det E_k^T = \det E_1 \cdots \det E_k = \det A$$

Any statements regarding rows or row operations now apply equally to columns and vice versa. In particular, we may compute $\det A$ using the cofactor expansion along any row or *down any column*.

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det \tilde{A}_{ij} \quad (\text{expansion along } i^{\text{th}} \text{ row})$$

$$= \sum_{i=1}^n (-1)^{i+j} a_{ij} \det \tilde{A}_{ij} \quad (\text{expansion along } j^{\text{th}} \text{ column})$$

Example 4.18. It is sensible to evaluate the determinant of the following matrix using the cofactor expansion down the 3rd column:

$$\begin{aligned} \det \begin{pmatrix} 1 & 2 & 0 & 4 \\ -2 & 3 & 1 & 5 \\ 2 & 4 & 2 & 8 \\ -1 & -3 & 0 & 2 \end{pmatrix} &= - \begin{vmatrix} 1 & 2 & 4 \\ -2 & 3 & 5 \\ 2 & 4 & 8 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 & 4 \\ -2 & 3 & 5 \\ -1 & -3 & 2 \end{vmatrix} = 0 + 2 \begin{vmatrix} 1 & 2 & 4 \\ 0 & 7 & 13 \\ 0 & -1 & 6 \end{vmatrix} \quad (\text{rank} = 0 \text{ and row operations}) \\ &= 2 \begin{vmatrix} 7 & 13 \\ -1 & 6 \end{vmatrix} = 110 \quad (\text{cofactor first column}) \end{aligned}$$

Order 3 Determinants and Volume

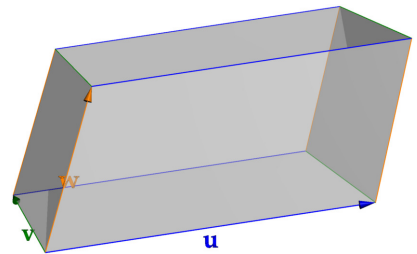
As in two dimensions, we have a geometric interpretation.

Theorem 4.19. Let $A = (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in M_3(\mathbb{R})$ have columns $\mathbf{u}, \mathbf{v}, \mathbf{w}$.

1. $\det A = [\mathbf{u}, \mathbf{v}, \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ is the scalar triple product of its columns
2. The volume of the parallelepiped spanned by $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is given by $|\det A|$.

The ordered triple $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is said to be *positively-oriented* if $\det A > 0$. By the theorem, this is if and only if \mathbf{u} lies on the same side of the \mathbf{v}, \mathbf{w} -plane as the normal vector $\mathbf{v} \times \mathbf{w}$: this is the familiar *right-hand rule*. The picture represents a positively oriented triple.

One can also extend this interpretation of oriented volume to higher dimensions.



Example 4.20. The parallelepiped spanned by the vectors $\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 6 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}\right\}$ has volume

$$\left| \det \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 6 & -2 \end{pmatrix} \right| = |-12| = 12$$

Since the determinant is negative, the three vectors are *negatively-oriented*.

Cramer's Rule

Finally, we present an application to the solution of $n \times n$ systems of linear equations $A\mathbf{x} = \mathbf{b}$:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{n1}x_1 + \cdots + a_{nn}x_n = b_n \end{cases} \quad \text{where } A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

Theorem 4.21. Suppose $A\mathbf{x} = \mathbf{b}$ where A is invertible. Then the unique solution \mathbf{x} has k^{th} entry

$$x_k = \frac{1}{\det A} \det M_k$$

where M_k is the matrix obtained by replacing column k of A by \mathbf{b} .

Proof. For each k , define $X_k = A^{-1}M_k$. Since the columns of A and M_k , except the k^{th} , are identical

$$X_k \mathbf{e}_i = A^{-1}M_k \mathbf{e}_i = \begin{cases} A^{-1}\mathbf{b} = \mathbf{x} & \text{if } i = k \\ A^{-1}A\mathbf{e}_i = \mathbf{e}_i & \text{if } i \neq k \end{cases}$$

Therefore X_k is the identity matrix except that the k^{th} column is the solution \mathbf{x} . Use the k^{th} row cofactor expansion of X_k to see that

$$x_k = (-1)^{k+k} x_k \det I_{n-1} = \det X_k = \det A^{-1} \det M_k$$

Cramer's rule is particularly useful if you only want to find a small number of the solution values x_1, \dots, x_n . If you want to find *all* the values, you are usually better off solving the system using an augmented matrix approach, or even computing the inverse A^{-1} .

Example 4.22. We find y by applying Cramer's rule to the system
$$\begin{cases} 2x + 3y + 7z = 1 \\ x - y + 3z = 2 \\ 3x + 5y + z = 8 \end{cases}$$

$$y = \frac{\begin{vmatrix} 2 & 1 & 7 \\ 1 & 2 & 3 \\ 3 & 8 & 1 \end{vmatrix}}{\begin{vmatrix} 2 & 3 & 7 \\ 1 & -1 & 3 \\ 3 & 5 & 1 \end{vmatrix}} = \frac{2 \begin{vmatrix} 2 & 3 \\ 8 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 7 \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix}}{2 \begin{vmatrix} -1 & 3 \\ 5 & 1 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} + 7 \begin{vmatrix} 1 & -1 \\ 3 & 5 \end{vmatrix}} = \frac{-44 + 8 + 14}{-32 + 24 + 56} = \frac{-22}{48} = -\frac{11}{24}$$

If you want to find x and z using this method, you need to do a lot of calculating!

Exercises 4.2 1. Compute the determinants of the following matrices using any method you like:

$$A = \begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix} \quad B = \begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$$

2. Prove that the determinant of an upper triangular matrix is the product of the terms on its diagonal.
3. We saw that a general 4×4 determinant requires the computation of $4 \cdot 3 = 12$ determinants of order 2. How many order 2 determinants does an order n determinant require?
4. Establish the ‘diagonal’ method for computing a 3×3 determinant: as the sum of the products of the falling diagonals minus the products of the rising diagonals:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}$$

Now use this to quickly compute the determinant of $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

(Warning! This method is special to order 3 determinants: if $n \geq 4$ you have to calculate the slow way!)

5. (a) Prove part 4 of Theorem 4.12: a matrix with two identical rows has determinant zero.
(b) Prove the type I case of Corollary 4.14: switching two rows changes the sign of the determinant.

(Hint: You can prove these in either order. You’ll need an induction for whichever you do first, then the other should follow quite easily...)

$$6. \text{ Find the determinant of } A = \begin{pmatrix} 1 & 4 & 1 & 1 \\ 2 & 3 & 0 & 1 \\ 2 & 2 & 0 & 1 \\ 2 & 1 & -1 & 1 \end{pmatrix}$$

$$7. \text{ Use Cramer’s rule to solve the system of equations } \begin{cases} 2x + y - 3z = 1 \\ x - 2y + z = 0 \\ 3x + 4y - 2z = -5 \end{cases}$$

8. Suppose that n is odd and that A is skew-symmetric ($A^T = -A$). Prove that A is singular. Can we say anything if n is even?
9. Suppose $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \subseteq \mathbb{F}^n$. Prove that β is a basis if and only if $\det(\mathbf{v}_1 \cdots \mathbf{v}_n) \neq 0$.
10. Let $\dim V = n$, let $T \in \mathcal{L}(V)$ and suppose β is a basis of V . Define $\det T := \det[T]_\beta$. Explain why this definition is independent of the choice of basis β .

11. (a) Suppose $X, Y \in M_n(\mathbb{F})$ can be written in block form

$$X = \left(\begin{array}{c|c} A & B \\ \hline O & I \end{array} \right) \quad Y = \left(\begin{array}{c|c} I & C \\ \hline O & D \end{array} \right)$$

where O is a zero matrix and I an identity. Prove that $\det X = \det A$ and $\det Y = \det D$.

- (b) Use part (a) to prove that $\det \left(\begin{array}{c|c} A & B \\ \hline O & C \end{array} \right) = \det A \det C$

12. Consider the system of equations

$$\begin{cases} 2x + 3y = 1 \\ x + 4y = 2 \end{cases} \iff \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

- (a) Compute the inverse of the square matrix and, supposing $x, y \in \mathbb{R}$, find the solution.
(b) Suppose the above are now equations in the field \mathbb{Z}_7 of remainders modulo 7:

$$\begin{cases} 2x + 3y \equiv 1 \pmod{7} \\ x + 4y \equiv 2 \pmod{7} \end{cases}$$

Compute the inverse of the matrix in \mathbb{Z}_7 and use it to find x, y .

- (c) What happens if you try to solve the system in \mathbb{Z}_5 ? Instead find the solutions to the system

$$\begin{cases} 2x + 3y \equiv 4 \pmod{5} \\ x + 4y \equiv 2 \pmod{5} \end{cases}$$

13. (a) Use Cramer's rule to prove the *classical adjoint* formula for the inverse:

$$(A^{-1})_{ij} = \frac{(-1)^{i+j}}{\det A} \det \tilde{A}_{ji}$$

Moreover, conclude that $\sum_{j=1}^n (-1)^{i+j} a_{ij} |\tilde{A}_{kj}| = \begin{cases} \det A & \text{if } i = k \\ 0 & \text{otherwise} \end{cases}$

- (b) Use part (a) to compute the inverse of $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{pmatrix}$
(c) Suppose that $A \in M_n(\mathbb{Z})$ is an invertible square matrix, all of whose entries are integers. Prove that the entries of A^{-1} are all integers if and only if $\det A = \pm 1$.

4.3 A Characterization of Determinant: non-examinable

In Exercise 4.1.5 we can give an alternative interpretation of the determinant. This construction can be done in general, though it involves a more advanced type of vector space.

Definition 4.23. Let V be a vector space over \mathbb{F} and let $f : V \times \cdots \times V \rightarrow \mathbb{F}$ be a function from k copies of V to the field. We say that f is k -multilinear if it is linear in each entry: for each j ,

$$f(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{x} + \lambda \mathbf{y}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k) = f(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{x}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k) + \lambda f(\mathbf{v}_1, \dots, \mathbf{v}_{j-1}, \mathbf{y}, \mathbf{v}_{j+1}, \dots, \mathbf{v}_k)$$

An *alternating k -form* on V is a k -multilinear function f which evaluates to zero whenever two entries in the domain are equal:

$$f(\dots, \mathbf{v}, \dots, \mathbf{v}, \dots) = 0$$

Since the codomain of an alternating k -form is the (one-dimensional) vector space \mathbb{F} , the set of all alternating k -forms is a vector space over \mathbb{F} in its own right, denoted $\wedge^k V^*$.

This may seem very abstract, but you've already seen an example: the determinant! Viewed as a function

$$\det : \mathbb{F}^n \times \cdots \times \mathbb{F}^n \rightarrow \mathbb{F}$$

Theorem 4.12 (part 1) and Corollary 4.14 say that $\det \in \wedge^n (\mathbb{F}^n)^*$ is an alternating n -form. We now observe something very special about this vector space.

Theorem 4.24. If $\dim V = n$, then $\dim \wedge^n V^* = 1$.

Proof. Let $f \in \wedge^n V^*$. Since f multilinear, it is determined completely by its values when applied to a basis $\beta = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ of V . In particular, it is determined by the n^n scalars

$$f(\mathbf{w}_1, \dots, \mathbf{w}_n) \quad \text{where each } \mathbf{w}_i \in \beta \tag{*}$$

Since f is alternating, this is non-zero only if $\{\mathbf{w}_1, \dots, \mathbf{w}_n\} = \beta$ is the whole basis. Moreover, for any $\mathbf{x}, \mathbf{y} \in V$

$$\begin{aligned} 0 &= f(\dots, \mathbf{x} + \mathbf{y}, \dots, \mathbf{x} + \mathbf{y}, \dots) \\ &= f(\dots, \mathbf{x}, \dots, \mathbf{x}, \dots) + f(\dots, \mathbf{x}, \dots, \mathbf{y}, \dots) + f(\dots, \mathbf{y}, \dots, \mathbf{x}, \dots) + f(\dots, \mathbf{y}, \dots, \mathbf{y}, \dots) \\ &= f(\dots, \mathbf{x}, \dots, \mathbf{y}, \dots) + f(\dots, \mathbf{y}, \dots, \mathbf{x}, \dots) \quad (\text{since } f \text{ is alternating}) \\ \implies f(\dots, \mathbf{y}, \dots, \mathbf{x}, \dots) &= -f(\dots, \mathbf{x}, \dots, \mathbf{y}, \dots) \end{aligned}$$

Applying the 'entry swapping' to the values (*), we see that we may rearrange the order of the basis vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ so that they are in exactly the same order as they appear in β , at the cost of a \pm -sign. Otherwise said, f is *completely determined by the single value* $f(\mathbf{v}_1, \dots, \mathbf{v}_n)$.

A specific alternating n -form f may be defined by choosing $f(\mathbf{v}_1, \dots, \mathbf{v}_n) = 1$. If $g \in \wedge^n V^*$ is another alternating n -form, then

$$g = af \quad \text{where } a = g(\mathbf{v}_1, \dots, \mathbf{v}_n)$$

All alternating n -forms are a scalar multiple of f , and so $\dim \wedge^n V^* = 1$. ■

This immediately leads to an alternative *non-inductive* definition of determinant.

Definition 4.25. The order n determinant is the unique alternating n -form $\det \in \wedge^n(\mathbb{F}^n)^*$ for which $\det(\mathbf{e}_1 \cdots \mathbf{e}_n) = 1$.

This is very sneaky. Rather than defining determinant inductively in terms of smaller determinants, we demand only that it satisfy certain properties. The problem with this definition is that it is very difficult to compute with explicitly, so it is good that we have the more elementary discussion to rely on. The alternative formulation is much more useful in the abstract.

The *exterior algebra* of alternating forms is widely applied in modern Geometry and Physics with a purpose analogous to how determinants measure (hyper-)volume.

Exercises 4.3 1. Let V be a vector space over \mathbb{F} with dual space $V^* = \mathcal{L}(V, \mathbb{F})$. For any $f, g \in V^*$, define the *wedge product* $f \wedge g$ by

$$\forall \mathbf{v}, \mathbf{w} \in V, \quad f \wedge g(\mathbf{v}, \mathbf{w}) := \det \begin{pmatrix} f(\mathbf{v}) & f(\mathbf{w}) \\ g(\mathbf{v}) & g(\mathbf{w}) \end{pmatrix}$$

Prove that $f \wedge g \in \wedge^2 V^*$.

2. Let $\{f_1, f_2, f_3\}$ be the dual basis to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ in \mathbb{R}^3 (that is $f_i(\mathbf{e}_j) = \delta_{ij}$). Prove that the cross-product of vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ is given by

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} f_2 \wedge f_3(\mathbf{v}, \mathbf{w}) \\ f_3 \wedge f_1(\mathbf{v}, \mathbf{w}) \\ f_1 \wedge f_2(\mathbf{v}, \mathbf{w}) \end{pmatrix}$$

A similar approach can be used to construct an analogue of the cross product of three vectors in \mathbb{R}^4 : if you want a challenge, try to figure out how to create a vector orthogonal to $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^4$ using three-forms.

3. If $k \leq n = \dim V$, prove that $\dim \wedge^k V^* = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, namely the number of size- k subsets of a size n set.

(Hint: given a basis β of V , what values are needed to defined f ?)