5 Diagonalization

5.1 Eigenvalues and Eigenvectors

Suppose V has a finite basis $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. We've seen that a linear map $\mathrm{T} \in \mathcal{L}(V)$ corresponds to multiplication by a matrix $[\mathrm{T}]_\beta \in M_n(\mathbb{F})$:

$$[\mathrm{T}]_\beta [\mathbf{v}]_\beta = [\mathrm{T}(\mathbf{v})]_\beta$$

The most desirable situation is when this matrix is *diagonal*: otherwise said, $\exists \lambda_i \in \mathbb{F}$ such that

$$[\mathrm{T}]_\beta = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} \quad \text{corresponding to} \quad \forall i,\ \mathrm{T}(\mathbf{v}_i) = \lambda_i \mathbf{v}_i$$

Each vector $\mathbf{v}_i$ is transformed by T in a simple way: without meaningfully changing its direction.

Definition 5.1. Suppose V is a vector space over $\mathbb{F}$ and that $\mathrm{T} \in \mathcal{L}(V)$.

1. A *non-zero* vector $\mathbf{v} \in V$ is an *eigenvector*[a] of T with *eigenvalue* $\lambda \in \mathbb{F}$ if $\mathrm{T}(\mathbf{v}) = \lambda \mathbf{v}$.
2. The eigenvalues/vectors of $A \in M_n(\mathbb{F})$ are those of $\mathrm{L}_A \in \mathcal{L}(\mathbb{F}^n)$: the equation is $A\mathbf{v} = \lambda \mathbf{v}$.
3. If V is finite-dimensional, we say that T is *diagonalizable* if there exists a basis β of eigenvectors: otherwise said, $[\mathrm{T}]_\beta$ is diagonal. We call β an *eigenbasis*.

[a] In German, *eigen* indicates ownership: the term was coined by David Hilbert to indicate how eigenvalues and eigenvectors belong to a linear map. Earlier mathematicians used the word *characteristic* in a similar context.

Example 5.2. If $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ is an eigenbasis for $A = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 4 \end{smallmatrix}\right)$, then for $\mathbf{v}_j = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ and $\lambda_j = \lambda$, we have

$$\begin{aligned} &\begin{cases} 2x + y = \lambda x \\ 3x + 4y = \lambda y \end{cases} \iff \begin{cases} y = (\lambda - 2)x \\ 3x = (\lambda - 4)y \end{cases} \iff \begin{cases} y = (\lambda - 2)x \\ 3x = (\lambda - 4)(\lambda - 2)x \end{cases} \\ &\iff (\lambda - 4)(\lambda - 2) = 3 \end{aligned} \tag{$*$}$$

since $x = y = 0$ does not produce a basis vector. The polynomial has solutions $\lambda_1 = 5$, $\lambda_2 = 1$ which, upon substitution into the original equations, result in the eigenvectors[a]

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Plainly L_A is diagonalizable since $[\mathrm{L}_A]_\beta = \left(\begin{smallmatrix} 5 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ is diagonal, and we conclude that $\{\mathbf{v}_1, \mathbf{v}_2\}$ really is an eigenbasis. Moreover, if $\epsilon = \{\mathbf{i}, \mathbf{j}\}$ is the standard basis, then

$$A = [\mathrm{L}_A]_\epsilon = Q_\beta^\epsilon [\mathrm{L}_A]_\beta Q_\epsilon^\beta = \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 1 \end{pmatrix}^{-1}$$

where Q_β^ϵ is the change of co-ordinate matrix: thus $A = QDQ^{-1}$ where D is diagonal.

[a] There is some freedom here: any non-zero scalar multiples of $\mathbf{v}_1, \mathbf{v}_2$ are also eigenvectors; $[\mathrm{L}_A]_\beta$ is unchanged.

Warnings! The definition and example should remind you of the following critical facts:

- $\mathbf{0}$ is *never* an eigenvector! It is completely uninteresting to observe that $\mathbf{v} = \mathbf{0}$ solves *every* equation of the form $\mathrm{T}(\mathbf{v}) = \lambda\mathbf{v}$.

- If $\mathbf{v}$ is an eigenvector of T with eigenvalue λ, then so is any *non-zero* scalar multiple:

  $$\mathrm{T}(k\mathbf{v}) = k\mathrm{T}(\mathbf{v}) = k\lambda\mathbf{v} = \lambda(k\mathbf{v})$$

  Indeed in Example 5.2, the (infinitely many) eigenvectors of A have the form

  $$a\mathbf{v}_1 = \begin{pmatrix} a \\ 3a \end{pmatrix}, \quad b\mathbf{v}_2 = \begin{pmatrix} -b \\ b \end{pmatrix} \quad \text{where } a, b \neq 0$$

  What we really care about are *linearly independent* eigenvectors, of which A has only *two*: $\mathbf{v}_1, \mathbf{v}_2$. While strictly nonsense, it is common and acceptable to state that "A has two eigenvectors," rather than the more precise "A has two *linearly independent* eigenvectors."

Is every linear map diagonalizable? Does every linear map have eigenvectors?

These are the most obvious questions arising from the definition: the answers to both are a resounding *no*! To illustrate, here are several examples where we obtain many eigenvectors or very few.

Examples 5.3. 1. Let $A = \left(\begin{smallmatrix} 1 & 4 \\ 0 & 1 \end{smallmatrix}\right)$. If $\mathbf{v} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is an eigenvector with eigenvector λ, then

$$\begin{cases} x + 4y = \lambda x \\ y = \lambda y \end{cases} \implies xy + 4y^2 = \lambda xy = xy \implies y = 0 \implies \lambda = 1$$

A is non-diagonalizable: it has one independent eigenvector $\mathbf{v} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ with eigenvalue $\lambda = 1$.

2. The matrix $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{R})$ acts by rotation counter-clockwise by $90°$ in $\mathbb{R}^2$. Since $A\mathbf{v}$ is perpendicular to $\mathbf{v}$, we see that A has no eigenvectors! In particular, A is not diagonalizable.

   However, see Example 5.7.3 for what happens when A is viewed as a *complex* matrix.

3. Let $\mathrm{T} = \frac{\mathrm{d}}{\mathrm{d}x}$ be defined by differentiation on some vector space of functions V.

   A non-zero function $f \in V$ is an eigenvector (*eigenfunction*) of T with eigenvalue λ if and only if it satisfies the *natural growth equation* $f' = \lambda f$. As seen in calculus/ODEs, all solutions have the form $f(x) = ce^{\lambda x}$ where c is constant. Here are three specific cases:

   (a) If V is the space of all differentiable functions, then T has *infinitely many* linearly independent[a] eigenvectors $f(x) = e^{\lambda x}$. In this context diagonalizability is meaningless since V is infinite-dimensional.

   (b) If $V = P(\mathbb{R})$ is the space of polynomials, then T has exactly one independent eigenvector $f(x) = 1$ with eigenvalue $\lambda = 0$.

   (c) Let $\beta = \{\sin x, \cos x\}$ and $V = \operatorname{Span}_{\mathbb{R}}\{\sin x, \cos x\}$, then $[\mathrm{T}]_\beta = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ is the matrix above, and so T has no eigenvectors.

   (d) If $\beta = \{e^x, e^{2x}, e^{5x}\}$ and $V = \operatorname{Span}_{\mathbb{R}} \beta$, then T is diagonalizable; indeed β is an eigenbasis.

[a] That these functions are linearly independent is a little tricky and was discussed in the first chapter.

Finding Eigenvalues and Eigenvectors in Finite-Dimensions

As Example 5.3.3 shows, linear operators on infinite-dimensional vector spaces can have eigenvectors, though the computation of such is usually case-specific. In the finite-dimensional situation, we can approach matters systematically. First we observe that we need only consider *matrices.*

Lemma 5.4. *Let* $\mathrm{T} \in \mathcal{L}(V)$ *where* $\dim V = n$ *and suppose* β *is a basis of* V*. Then*

$$\mathrm{T}(\mathbf{v}) = \lambda\mathbf{v} \iff [\mathrm{T}]_\beta[\mathbf{v}]_\beta = \lambda[\mathbf{v}]_\beta$$

Otherwise said:

- T *has the same eigenvalues as any matrix of* T *with respect to any basis.*
- *The co-ordinate isomorphism* $\phi_\beta : V \to \mathbb{F}^n : \mathbf{v} \mapsto [\mathbf{v}]_\beta$ *maps eigenvectors of* T *to those of* $[\mathrm{T}]_\beta$*.*

The lemma says that to compute the eigenvalues and eigenvectors of T, we simply compute those of its matrix $[\mathrm{T}]_\beta$ with respect to any basis β and then translate.

With this identification out of the way, let $A \in M_n(\mathbb{F})$ have eigenvector $\mathbf{v}$ with eigenvalue λ. Observe:

$$A\mathbf{v} = \lambda\mathbf{v} \iff (A - \lambda I)\mathbf{v} = \mathbf{0} \qquad (\dagger)$$

where I is the identity matrix. Since $\mathbf{v} \neq \mathbf{0}$, the nullspace $\mathcal{N}(A - \lambda I)$ is non-trivial. Indeed

$$\begin{aligned} \lambda \text{ is an eigenvalue of } A &\iff \operatorname{null}(A - \lambda I) > 0 \iff \operatorname{rank}(A - \lambda I) < n \\ &\iff \det(A - \lambda I) = 0 \end{aligned}$$

where we used the Rank–Nullity Theorem and a standard property of the determinant.

Definition 5.5. The *characteristic polynomial* of a matrix A is $p(t) := \det(A - tI)$.
When $\dim V = n$, the *characteristic polynomial* of $\mathrm{T} \in \mathcal{L}(V)$ may be computed with respect to any basis β of V

$$p(t) = \det(\mathrm{T} - t\mathrm{I}) := \det[\mathrm{T} - t\mathrm{I}]_\beta = \det([\mathrm{T}]_\beta - tI_n)$$

In either case, the *characteristic equation* is $p(t) = 0$.

Plainly λ is an eigenvalue if and only if $p(\lambda) = 0$. Once we have an eigenvalue, (†) says that the corresponding eigenvectors lie in the nullspace $\mathcal{N}(A - \lambda I)$. To summarize:

Theorem 5.6. *Let* $A \in M_n(\mathbb{F})$*.*

1. *The characteristic polynomial* $p(t)$ *is a degree* n *polynomial in* t *with leading term* $(-1)^n t^n$*.*
2. A *has at most* n *eigenvalues, precisely the solutions to the characteristic equation* $p(t) = 0$*.*
3. *An eigenvector with eigenvalue* λ *is any non-zero element of the* eigenspace $E_\lambda := \mathcal{N}(A - \lambda I)$*.*

Once part 1 is proved, the rest follows immediately from our above discussion and the fact that a degree n polynomial has at most n solutions. Before seeing this, we revisit our past examples in this language and see another.

Examples 5.7. 1. (Example 5.2) $A = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 4 \end{smallmatrix}\right)$ has characteristic polynomial

$$p(t) = \det \begin{pmatrix} 2-t & 1 \\ 3 & 4-t \end{pmatrix} = (2-t)(4-t) - 3 = t^2 - 6t + 5 = (t-5)(t-1)$$

recovering the eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 1$. We can now find the nullspaces:

- $\lambda_1 = 5$: $\mathcal{N}(A - \lambda_1 I) = \mathcal{N} \begin{pmatrix} -3 & 1 \\ 3 & -1 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$
- $\lambda_2 = 1$: $\mathcal{N}(A - \lambda_2 I) = \mathcal{N} \begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix} = \text{Span} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

We may therefore choose two independent eigenvectors $\mathbf{v}_1 = \left(\begin{smallmatrix} 1 \\ 3 \end{smallmatrix}\right)$, $\mathbf{v}_2 = \left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$: these form an eigenbasis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

2. (Example 5.3.2) $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ has characteristic equation $p(t) = \det \left(\begin{smallmatrix} -t & -1 \\ 1 & -t \end{smallmatrix}\right) = t^2 + 1 = 0$. Since this has no solutions (in $\mathbb{R}$), we see that A has no eigenvalues. However, if we consider $A \in M_2(\mathbb{C})$ as a *complex* matrix, then there are two eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$: indeed

$$\mathcal{N}(A - \lambda_1 I) = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} = \text{Span} \begin{pmatrix} i \\ 1 \end{pmatrix} \quad \text{and} \quad \mathcal{N}(A - \lambda_2 I) = \text{Span} \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

so we may choose two independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{C}^2$. These form a basis and so A is diagonalizable as a complex matrix.

3. Let $\mathrm{T} \in \mathcal{L}(P_2(\mathbb{R}))$ be defined by

$$\mathrm{T}(f)(x) = \int_0^2 f(x)\mathrm{d}x + (x-3)f'(x)$$

With respect to the standard basis, we have the matrix $A = [\mathrm{T}]_\epsilon = \left(\begin{smallmatrix} 2 & -1 & \frac{8}{3} \\ 0 & 1 & -6 \\ 0 & 0 & 2 \end{smallmatrix}\right)$ whose eigenvalues are the solutions of the characteristic equation

$$0 = p(t) = \det(A - tI) = (2-t)^2(1-t) \iff t = 1, 2$$

Now compute the nullspaces:

- $\lambda_1 = 1$: $\mathcal{N}(A - \lambda_1 I) = \mathcal{N} \begin{pmatrix} 1 & -1 & \frac{8}{3} \\ 0 & 0 & -6 \\ 0 & 0 & 1 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
- $\lambda_2 = 2$: $\mathcal{N}(A - \lambda_2 I) = \mathcal{N} \begin{pmatrix} 0 & -1 & \frac{8}{3} \\ 0 & -1 & -6 \\ 0 & 0 & 0 \end{pmatrix} = \text{Span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

We may therefore choose two independent eigenvectors of A; $\mathbf{v}_1 = \left(\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right)$ and $\mathbf{v}_2 = \left(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right)$. These correspond to polynomials $f_1, f_2 \in P_2(\mathbb{R})$ or *eigenfunctions* of T:

$$\begin{aligned} \mathbf{v}_1 = [f_1]_\epsilon &\implies f_1(x) = 1 + x \\ \mathbf{v}_2 = [f_2]_\epsilon &\implies f_2(x) = 1 \end{aligned}$$

It is easily checked directly that $\mathrm{T}(f_1) = f_1$ and $\mathrm{T}(f_2) = 2f_2$. Since T has insufficient independent eigenvectors, we see that it is *not* diagonalizable.

We now give an induction argument to complete the proof of Theorem 5.6: that $p(t)$ is a degree n polynomial. First observe the following obvious fact: in any cofactor expansion of the determinant, one never multiplies an entry of a matrix by itself...

Lemma 5.8. *If $B(t)$ is a square matrix, k of whose entries are linear functions of t with the rest constant, then* $\det B(t)$ *is a polynomial in t with degree $\leq k$.*

The main argument is a little difficult to follow, so first consider an example where we expand the characteristic polynomial of a 3×3 matrix along the first row.

$$B = \begin{pmatrix} 1 & 3 & 2 \\ -1 & 0 & 4 \\ 1 & -2 & 5 \end{pmatrix} \implies p(t) = \begin{vmatrix} 1-t & 3 & 2 \\ -1 & -t & 4 \\ 1 & -2 & 5-t \end{vmatrix}$$
$$= (1-t)\begin{vmatrix} -t & 4 \\ -2 & 5-t \end{vmatrix} - 3\begin{vmatrix} -1 & 4 \\ 1 & 5-t \end{vmatrix} + 2\begin{vmatrix} -1 & -t \\ 1 & -2 \end{vmatrix}$$
$$= (b_{11}-t)\det \tilde{B}_{11}(t) - b_{12}\det \tilde{B}_{12}(t) + b_{13}\det \tilde{B}_{13}(t)$$

In each case $\tilde{B}_{1j}(t)$ is the $1j^{\text{th}}$ minor of the matrix $B - tI$. Observe that

$$\deg(\det \tilde{B}_{1j}(t)) = \begin{cases} 2 & \text{if } j = 1 \\ 1 & \text{otherwise} \end{cases} \implies \deg(p(t)) = 3$$

This is essentially the induction step in the following proof with $n = 2$.

Proof of Theorem 5.6, part 1. Since only n entries of the matrix $A - tI$ contain t, the Lemma tells us that the maximum degree of $p(t) = \det(A - tI)$ is n.

It remains to prove that the leading term of $p(t)$ is $(-1)^n t^n$: we prove by induction on n.

(*Base Case*) If $n = 1$, then $A = (a)$ and so $p(t) = -t + a$ as required.

(*Induction Step*) Fix $n \in \mathbb{N}$ and assume for every matrix $A \in M_n(\mathbb{F})$ that

$$p(t) = (-1)^n t^n + \cdots$$

Let $B \in M_{n+1}(\mathbb{F})$ and compute using the cofactor expansion along the first row:

$$\det(B - tI) = (b_{11} - t)\det \tilde{B}_{11}(t) - b_{12}\det \tilde{B}_{12}(t) + \cdots$$

where $\tilde{B}_{1j}(t)$ is the $n \times n$ minor obtained by deleting the 1^{st} row and j^{th} column of $B - tI$. There are two cases:

If $j = 1$: $\tilde{B}_{11}(t) = B_{11} - tI \in M_n(\mathbb{F})$. By the induction hypothesis its determinant is a degree n polynomial with leading term $(-1)^n t^n$. It follows that

$$(b_{11} - t)\det \tilde{B}_{11}(t) = (-1)^{n+1} t^{n+1} + \text{lower order terms}$$

If $j \geq 2$: $\tilde{B}_{1j}(t) \in M_n(\mathbb{F})$ where $n-1$ of the entries contain a t: we have deleted the first row and j^{th} column and thus removed *two* of the diagonal t-terms from $B - tI$. By the Lemma, $\det \tilde{B}_{1j}(t)$ is a polynomial of degree *at most* $n - 1$.

Summing these polynomials completes the proof. ∎

Exercises 5.1 1. For each matrix $A \in M_n(\mathbb{F})$, find the eigenvalues and a set of linearly independent eigenvectors. If an eigenbasis exists, state an invertible matrix Q and a diagonal matrix D such that $A = QDQ^{-1}$.

(a) $A = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \in M_2(\mathbb{R})$ (b) $A = \begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix} \in M_3(\mathbb{R})$

(c) $A = \begin{pmatrix} i & 1 \\ 2 & -i \end{pmatrix} \in M_2(\mathbb{C})$ (d) $A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 1 & -4 \\ 2 & 0 & -1 \end{pmatrix} \in M_3(\mathbb{R})$

2. For each linear operator T on a vector space V, find an ordered basis β such that $[\mathrm{T}]_\beta$ is diagonal.

   (a) $V = P_2(\mathbb{R})$ and $\mathrm{T}\big(f(x)\big) = xf'(x) + f(2)x + f(3)$

   (b) $V = P_3(\mathbb{R})$ and $\mathrm{T}\big(f(x)\big) = xf'(x) + f''(x) - f(2)$

3. If A and B are similar matrices ($B = QAQ^{-1}$ for some Q), prove that $\mathbf{v}$ is an eigenvector of A if and only if $Q\mathbf{v}$ is an eigenvector of B with the same eigenvalue.

4. Prove that the characteristic polynomial $p(t) = \det(\mathrm{T} - t\mathrm{I}) = \det([\mathrm{T}]_\beta - tI)$ of a linear map $\mathrm{T} \in \mathcal{L}(V)$ is independent of the choice of basis β used in its computation.

5. Suppose $A \in M_n(\mathbb{C})$ is a *real* matrix with eigenvalue $\lambda \in \mathbb{C}$ and eigenvector $\mathbf{v} \in \mathbb{C}^n$.

   (a) Prove that the complex conjugate $\overline{\mathbf{v}}$ is also an eigenvector. What is its eigenvalue?

   (b) Prove that if $\mathbf{v} = a\mathbf{w}$ for some (complex) scalar a and *real vector* $\mathbf{w} \in \mathbb{R}^n$, then $\lambda \in \mathbb{R}$.

   (c) Is the converse of part (b) true? Explain. In particular, if $\lambda \in \mathbb{R}$, consider the real and imaginary parts of $\mathbf{v}$

$$\mathbf{x} := \frac{1}{2}(\mathbf{v} + \overline{\mathbf{v}}) \qquad \mathbf{y} := \frac{1}{2i}(\mathbf{v} - \overline{\mathbf{v}})$$

   and prove that $\dim_{\mathbb{R}} \operatorname{Span}\{\mathbf{x}, \mathbf{y}\} = \dim_{\mathbb{C}} \operatorname{Span}\{\mathbf{v}, \overline{\mathbf{v}}\}$. What does this mean for the eigenvectors of A?

6. Let $p(t) = (-1)^n t^n + c_{n-1}t^{n-1} \cdots + c_0$ be the characteristic polynomial of a matrix A.

   (a) Prove that $c_0 = \det A$ and hence conclude that A is invertible if and only if $c_0 \neq 0$.

   (b) Prove that $p(t) = (a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t) + q(t)$ where $\deg q(t) \leq n - 2$. Hence argue that $c_{n-1} = (-1)^{n-1} \operatorname{tr} A$.
   (*Hint: try an induction proof*)

5.2 Diagonalizability

We have now seen how to compute eigenvectors in finite dimensions, and observed that diagonalizability is equivalent to the existence of an eigenbasis. In this section we consider the question of when an eigenbasis might exist.

Theorem 5.9. *Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are eigenvectors of $\mathrm{T} \in \mathcal{L}(V)$ corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.*

Proof. We prove by induction on k. The base case $k = 1$ is trivial.

Fix $k \in \mathbb{N}$: for the induction hypothesis, suppose *every* set of k eigenvectors corresponding to k distinct eigenvalues is linearly independent. To obtain a contradiction, suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is a *linearly dependent* set of $k+1$ eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_{k+1}$. WLOG, we may assume

$$\exists a_1, \ldots, a_k \in \mathbb{F} \quad \text{such that} \quad a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k + \mathbf{v}_{k+1} = \mathbf{0} \tag{$*$}$$

Apply T to this linear dependence and substitute for $\mathbf{v}_{k+1}$ using $(*)$:

$$\begin{aligned} \sum_{j=1}^{k} a_j\lambda_j\mathbf{v}_j + \lambda_{k+1}\mathbf{v}_{k+1} = \mathbf{0} &\implies \sum_{j=1}^{k} a_j(\lambda_j - \lambda_{k+1})\mathbf{v}_j = \mathbf{0} \\ &\implies a_j(\lambda_j - \lambda_{k+1}) = 0 \implies a_j = 0 \end{aligned}$$

where we used the linear independence of $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and the distinctness of the $\lambda_1, \ldots, \lambda_{k+1}$. But this shows that $\mathbf{v}_{k+1} = \mathbf{0}$ is not an eignevector: contradiction. We conclude that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{k+1}\}$ is linearly independent.

By induction, the result is proved. ■

Suppose $\dim V = n$ and that the degree n characteristic polynomial of $\mathrm{T} \in \mathcal{L}(V)$ has distinct roots;[1]

$$p(t) = (-1)^n(t - \lambda_1)\cdots(t - \lambda_n) = (\lambda_1 - t)\cdots(\lambda_n - t)$$

where $\lambda_1, \ldots, \lambda_n$ are the distinct eigenvalues of T. Since each λ_j implies the existence of at least one eigenvector $\mathbf{v}_j$, the Theorem says that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent and thus a basis of V (an *eigenbasis* for T). We therefore have a simple sufficient condition for the diagonalizability of T.

Corollary 5.10. *Suppose $\dim_{\mathbb{F}} V = n$ and $\mathrm{T} \in \mathcal{L}(V)$. If T has n distinct eigenvalues (equivalently $p(t)$ has n distinct roots in the field $\mathbb{F}$), then T is diagonalizable.*

To orient ourselves, recall Examples 5.7.

1. $A = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 4 \end{smallmatrix}\right) \in M_2(\mathbb{R})$ has distinct eigenvalues $\lambda = 1, 5 \in \mathbb{R}$ and is diagonalizable.
2. $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) \in M_2(\mathbb{C})$ has distinct eigenvalues $\lambda = \pm i \in \mathbb{C}$ and is diagonalizable.
3. $\mathrm{T} \in \mathcal{L}(P_2(\mathbb{R}))$ defined by $\mathrm{T}(f)(x) = \int_0^2 f(x)\,\mathrm{d}x + (x-3)f'(x)$ has only two distinct eigenvalues $\lambda = 1, 2$ and is non-diagonalizable.

[1]From algebra, every degree n polynomial has *at most* n distinct roots.

After reviewing the examples, it might feel as if the Corollary should be biconditional. However, a trivial example says not: the identity map $\mathrm{I}_V \in \mathcal{T}(V)$ has only one eigenvalue $\lambda = 1$ but is plainly diagonalizable (by *any* basis!). We now develop a *necessary* condition for diagonalizability.

Definition 5.11. A degree n polynomial $p(t)$ *splits over a field* $\mathbb{F}$ if it factorizes completely over $\mathbb{F}$. Otherwise said, $\exists c, \alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that

$$p(t) = c(t - \alpha_1) \cdots (t - \alpha_n)$$

The values $\alpha_1, \ldots, \alpha_n$ are the *roots* or *zeros* of the polynomial.

Example 5.12. $p(t) = t^2 + 4 = (t - 2i)(t + 2i)$ does not split over $\mathbb{R}$, but does split over $\mathbb{C}$.

Theorem 5.13. *If* T *is diagonalizable, then its characteristic polynomial splits.*

Proof. Let $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an eigenbasis, then $[\mathrm{T}]_\beta$ is diagonal with the eigenvalues down the diagonal. But then the characteristic polynomial of T splits:

$$p(t) = \det([\mathrm{T}]_\beta - tI) = (\lambda_1 - t) \cdots (\lambda_n - t)$$

∎

Putting Corollary 5.10 and Theorem 5.13 together, we have

$$p(t) \text{ has distinct roots} \implies \mathrm{T} \text{ diagonalizable} \implies p(t) \text{ splits}$$

Our ‘identity’ observation above shows that these conditions are not equivalent. Here is another example of repeated eigenvalues.

Examples 5.14. The polynomial $p(t) = (6 - t)(4 - t)^2$ splits but does not have three distinct roots. This is not an idle example, for p is the characteristic polynomial of *many* linear maps, some diagonalizable, some not. For instance:

1. $A = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is diagonalizable (it's already diagonal!) with eigenbasis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$.
2. $B = \begin{pmatrix} 6 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}$ is non-diagonalizable. To verify this, observe that

$$\mathcal{N}(B - 6I) = \mathcal{N}\begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$\mathcal{N}(B - 4I) = \mathcal{N}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

   We can therefore find only two independent eigenvectors $\mathbf{v}_1 = \mathbf{i}$ and $\mathbf{v}_2 = \mathbf{j}$.

To obtain a fuller description of diagonalizability, we need to come to terms with the discrepancy above: $p(t)$ has root ($\lambda = 4$) with multiplicity *two*, but we can only find *one* independent eigenvector ($\mathbf{v}_2 = \mathbf{j}$) with this eigenvalue $B\mathbf{v}_2 = 4\mathbf{v}_2$.

Definition 5.15. Suppose V is finite-dimensional and that $\mathrm{T} \in \mathcal{L}(V)$ has an eigenvalue λ.

1. The *geometric multiplicity* of λ is the dimension $\dim E_\lambda$ of its *eigenspace*[a]

$$E_\lambda := \mathcal{N}(\mathrm{T} - \lambda \mathrm{I})$$

2. The *algebraic multiplicity* $\mathrm{mult}(\lambda)$ of λ is the highest power m for which $(t-\lambda)^m$ is a factor of the characteristic polynomial $p(t)$. Otherwise said, there exists a polynomial $q(t)$ such that

$$p(t) = (t-\lambda)^m q(t) \quad \text{and} \quad q(\lambda) \neq 0$$

[a] $\mathbf{v}$ is an eigenvector with eigenvalue λ if and only if $\mathbf{v} \in E_\lambda$ is *non-zero*.

Example (5.14, mark II). Here are the eigenspaces and multiplicities for B: note how the algebraic and geometric multiplicities differ.

eigenvalue λ	6	4
algebraic multiplicity $\mathrm{mult}(\lambda)$	1	2
eigenspace E_λ	Span $\mathbf{i}$	Span $\mathbf{j}$
geometric multiplicity $\dim E_\lambda$	1	1

We can now state the main result.

Theorem 5.16. *Suppose* $\dim V = n$ *and that* $\mathrm{T} \in \mathcal{L}(V)$ *has* distinct *eigenvalues* $\lambda_1, \ldots, \lambda_k$.

1. *For each eigenvalue* λ_i*, we have* $\dim E_{\lambda_i} \leq \mathrm{mult}(\lambda_i)$.
2. *The following are equivalent:*
   (a) T *is diagonalizable.*
   (b) *The characteristic polynomial of* T *splits and* $\dim E_{\lambda_i} = \mathrm{mult}(\lambda_i)$ *for each i;*

$$p(t) = (\lambda_1 - t)^{\dim E_{\lambda_1}} \cdots (\lambda_k - t)^{\dim E_{\lambda_k}}$$

   (c) $\sum_{i=1}^k \dim E_{\lambda_i} = n$.
   (d) $V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$.

We'll prove this shortly, but first, here are two examples where the calculations have been omitted.

Examples 5.17. 1. $A = \left(\begin{smallmatrix} 2&0&0&0\\0&3&1&0\\0&0&3&1\\0&0&0&3 \end{smallmatrix}\right)$ is non-diagonalizable: $p(t) = (2-t)(3-t)^3$ splits, but,

λ	2	3
$\mathrm{mult}(\lambda)$	1	3
E_λ	Span $\mathbf{e}_1$	Span $\mathbf{e}_2$
$\dim E_\lambda$	1	1

$$\dim E_3 \neq \mathrm{mult}(3), \quad \text{and} \quad \sum_{i=1}^{2} \dim E_{\lambda_i} = 2 < 4$$

2. Let $B = \left(\begin{smallmatrix} -1 & 6 & 0 & 0 \\ -2 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{smallmatrix}\right)$ is diagonalizable. Indeed $p(t) = (2-t)(3-t)^3$ splits, and we have

λ	2	3
$\text{mult}(\lambda)$	1	3
E_λ	$\text{Span}(2\mathbf{e}_1 + \mathbf{e}_2)$	$\text{Span}\{3\mathbf{e}_1 + 2\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$
$\dim E_\lambda$	1	3

and $\mathbb{R}^4 = E_2 \oplus E_3$

From the table, we can read off an eigenbasis with respect to which the map is diagonal

$$\beta = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \implies [\mathrm{L}_B]_\beta = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Proof. 1. Let $r = \dim E_\lambda$ and extend a basis β_λ of E_λ to a basis $\beta = \beta_\lambda \cup \gamma$ of V.
Since $\mathrm{T}(\mathbf{v}) = \lambda \mathbf{v}$ for all $\mathbf{v} \in E_\lambda$, we see that the matrix of T has block form $[\mathrm{T}]_\beta = \left(\begin{array}{c|c} \lambda I_r & A \\ \hline O & B \end{array}\right)$ for some matrices A, B, from which the characteristic polynomial of T is

$$p(t) = \det \left(\begin{array}{c|c} (\lambda - t) I_r & A \\ \hline O & B - tI_{n-r} \end{array}\right) = (\lambda - t)^r \det(B - tI_{n-r})$$

It follows that $(\lambda - t)^{\dim E_\lambda}$ divides $p(t)$, and so $\dim E_\lambda \leq \text{mult}(\lambda)$.

2. We give a brief summary:

(a) $\implies$ (b) If T is diagonalizable, then $p(t)$ splits by Theorem 5.13, whence $\sum \text{mult}(\lambda_i) = n$. The cardinality n of an eigenbasis is *at most* $\sum \dim E_{\lambda_i}$. Combined with part 1, we have equality of multiplicities:

$$n \leq \sum \dim E_{\lambda_i} \leq \sum \text{mult}(\lambda_i) = n \implies \dim E_{\lambda_i} = \text{mult}(\lambda_i)$$

(b) $\implies$ (c) $p(t)$ splits $\implies n = \sum \text{mult}(\lambda_i) = \sum \dim E_{\lambda_i}$

(c) $\implies$ (d) This requires an induction on the number of distinct eigenvalues.
For the induction step, fix $j < k$ and let $\mathbf{v}_{j+1}$ be an eigenvector with eigenvalue λ_{j+1}. If $\mathbf{v}_{j+1} \in E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_j}$ then there exist eigenvectors $\mathbf{v}_i \in E_{\lambda_i}$ and $a_i \in \mathbb{F}$ for which

$$\mathbf{v}_{j+1} = a_1 \mathbf{v}_1 + \cdots + a_j \mathbf{v}_j$$

But this contradicts the linear independence of the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{j+1}\}$ (Theorem 5.9).

By induction, $E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_k}$ exists; by assumption it has dimension $n = \dim V$ and thus equals V.

(d) $\implies$ (a) is trivial since (d) says there exists a basis of eigenvectors.

■

Exercises 5.2 1. For each matrix, find its characteristic polynomial, its eigenvalues/spaces, its algebraic and geometric multiplicities and decide if it is diagonalizable.

(a) $A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & -6 & 0 & 6 \end{pmatrix}$

2. Let $\mathrm{T} = \frac{\mathrm{d}}{\mathrm{d}x}$ be the derivative operator.

   (a) If we consider $\mathrm{T} = \mathcal{L}(P_2(\mathbb{R}))$, show that T is *not* diagonalizable.

   (b) More generally, what is the characteristic polynomial of $\mathrm{T} \in \mathcal{L}(P_n(\mathbb{R}))$? Why is it clear that T is non-diagonalizable?

3. Diagonalize $A = \left(\begin{smallmatrix} 1 & 4 \\ 2 & 3 \end{smallmatrix}\right) \in M_2(\mathbb{R})$, and thus find an expression for A^n for any $n \in \mathbb{N}$.

4. Show that the characteristic polynomial of $A = \left(\begin{smallmatrix} 3 & -4 \\ 4 & 3 \end{smallmatrix}\right)$ does not split over $\mathbb{R}$. Diagonalize A over $\mathbb{C}$.

5. Suppose T is a linear operator on a finite dimensional vector space V and that β is a basis of V with respect to which $[\mathrm{T}]_\beta$ is diagonal. Prove that the characteristic polynomial of T splits.

6. Suppose $\mathrm{T} \in \mathcal{L}(V)$ is invertible with eigenvalue λ. Prove that λ^{-1} is an eigenvalue of T^{-1} with the same eigenspace. If T is diagonalizable, prove that T^{-1} is diagonalizable.

7. If $p(t)$ splits, prove that $\det \mathrm{T} = \lambda_1^{\mathrm{mult}(\lambda_1)} \cdots \lambda_k^{\mathrm{mult}(\lambda_k)}$ is the product of its distinct eigenvalues up to (algebraic) multiplicity.

5.3 Invariant Subspaces and the Cayley–Hamilton Theorem

Eigenspaces of a linear map provide a simple example of a special type of subspace.

Definition 5.18. Suppose $\mathrm{T} \in \mathcal{L}(V)$. A subspace W of V is T-*invariant* if $\mathrm{T}(W) = \{\mathrm{T}(\mathbf{w}) : \mathbf{w} \in W\}$ is a subspace of W. In such a case we define the *restriction* $\mathrm{T}_W \in \mathcal{L}(W)$ by

$$\mathrm{T}_W : W \to W : \mathbf{w} \mapsto \mathrm{T}(\mathbf{w})$$

Much can often be understood about a linear map by considering its invariant subspaces.
We start by extending the proof of Theorem 5.16 (part 1) to *any* invariant subspace.

Theorem 5.19. *Suppose $\mathrm{T} \in \mathcal{L}(V)$, that $\dim V$ is finite and that $W \leq V$ is T-invariant. Then the characteristic polynomial $p_W(t)$ of T_W divides that of T.*

Proof. Extend a basis β_W of W to a basis $\beta = \beta_W \cup \gamma$ of V. Then $\exists A, B$ such that

$$[\mathrm{T}]_\beta = \left(\begin{array}{c|c} [\mathrm{T}_W]_{\beta_W} & A \\ \hline O & B \end{array}\right) \implies p(t) = \det([\mathrm{T}_W]_{\beta_W} - tI)\det(B - tI) = p_W(t)\det(B - tI)$$

■

Examples 5.20. 1. Every eigenspace E_λ is T-invariant: $\forall \mathbf{w} \in E_\lambda$, we have, $\mathrm{T}(\mathbf{w}) = \lambda\mathbf{w} \in E_\lambda$.
   Restricted to the eigenspace, the linear map is simply $\mathrm{T}_{E_\lambda} = \lambda \mathrm{I}$, with characteristic polynomial $p_\lambda(t) = (\lambda - t)^{\dim E_\lambda}$. This divides $p(t)$, as seen in Theorem 5.16.

2. If $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{pmatrix}$, then $\mathrm{L}_A \in \mathcal{L}(\mathbb{R}^3)$ has an invariant subspace $W = \operatorname{Span}\{\mathbf{i}, \mathbf{j}\}$. It is easy to check that, with respect to the standard basis, the restriction of L_A to W has matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ Since both this and A are upper triangular, we quickly verify that

$$p(t) = (1 - t)(2 - t)(3 - t) = (2 - t)p_W(t)$$

We now consider a more general type of invariant subspace.

Definition 5.21. Let $\mathrm{T} \in \mathcal{L}(V)$ and $\mathbf{v} \in V$. The T-*cyclic subspace generated by* $\mathbf{v}$ is the span

$$\langle \mathbf{v} \rangle = \operatorname{Span}\{\mathbf{v}, \mathrm{T}(\mathbf{v}), \mathrm{T}^2(\mathbf{v}), \ldots\}$$

The T-cyclic subspace $\langle \mathbf{v} \rangle$ is the *smallest* T-invariant subspace containing $\mathbf{v}$ (see Exercise 5.3.4).

Examples 5.22. 1. If $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & -4 \end{pmatrix}$, then the eigenspaces are L_A-cyclic subspaces:

$$E_5 = \operatorname{Span}\mathbf{i} = \langle \mathbf{i} \rangle, \qquad E_{-4} = \operatorname{Span}\mathbf{j} = \langle \mathbf{j} \rangle$$

   There are other examples, for instance $\langle \mathbf{k} \rangle = \operatorname{Span}\{\mathbf{j}, \mathbf{k}\}$ is L_A-cyclic, but is not an eigenspace.

2. $\dim \langle \mathbf{v} \rangle = 1 \iff \mathbf{v}$ is an eigenvector of T.

3. Not every T-invariant subspace is T-cyclic: for instance, if $\mathrm{T} = \mathrm{I}$ is the identity, then every subspace is T-invariant, however only the *one-dimensional* subspaces are T-cyclic!

For T-cyclic subspaces, we can extend Theorem 5.19 further.

Theorem 5.23. *Let V be finite dimensional, $\mathrm{T} \in \mathcal{L}(V)$, and suppose $W = \langle \mathbf{w} \rangle$ is T-invariant with $\dim W = k$. Then:*

1. *$\beta_W = \{\mathbf{w}, \mathrm{T}(\mathbf{w}), \ldots, \mathrm{T}^{k-1}(\mathbf{w})\}$ is a basis of W.*
2. *If $\mathrm{T}^k(\mathbf{w}) + a_{k-1}\mathrm{T}^{k-1}(\mathbf{w}) + \cdots + a_0\mathbf{w} = \mathbf{0}$, then the characteristic polynomial of T_W is*
$$p_W(t) = (-1)^k \left(t^k + a_{k-1}t^{k-1} + \cdots + a_1 t + a_0\right)$$
3. *$p_W(\mathrm{T}_W) = 0$.*

Proof. 1. Let i be maximal such that $\{\mathbf{w}, \mathrm{T}(\mathbf{w}), \ldots, \mathrm{T}^{i-1}(\mathbf{w})\}$ is linearly independent. Observe:

- Plainly i exists since a maximal linearly independent set is finite ($\dim W < \infty$).
- By the maximality of i, $\mathrm{T}^i(\mathbf{w}) \in \mathrm{Span}\{\mathbf{w}, \mathrm{T}(\mathbf{w}), \ldots, \mathrm{T}^{i-1}(\mathbf{w})\}$; by induction this extends to
$$j \geq i \implies \mathrm{T}^j(\mathbf{w}) \in \mathrm{Span}\{\mathbf{w}, \mathrm{T}(\mathbf{w}), \ldots, \mathrm{T}^{i-1}(\mathbf{w})\}$$
It follows that $W = \mathrm{Span}\{\mathbf{w}, \mathrm{T}(\mathbf{w}), \ldots, \mathrm{T}^{i-1}(\mathbf{w})\}$.

We conclude that $\{\mathbf{w}, \mathrm{T}(\mathbf{w}), \ldots, \mathrm{T}^{i-1}(\mathbf{w})\}$ is a basis of W, whence $i = k$.

2. Expand the characteristic polynomial along the first row:
$$p_W(t) = \det([\mathrm{T}_W]_{\beta_W} - tI_k) = \begin{vmatrix} -t & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & -t & 0 & & 0 & -a_1 \\ 0 & 1 & -t & & 0 & -a_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & -t & -a_{k-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{vmatrix}$$
$$= -t \begin{vmatrix} -t & 0 & & 0 & -a_1 \\ 1 & -t & & 0 & -a_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & -t & -a_{k-2} \\ 0 & 0 & \cdots & 1 & -a_{k-1} - t \end{vmatrix} + (-1)^k a_0 \begin{vmatrix} 1 & -t & 0 & \cdots & 0 \\ 0 & 1 & -t & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & & \ddots & -t \\ 0 & 0 & \cdots & & 1 \end{vmatrix}$$
The second matrix has determinant 1, yielding the $(-1)^k a_0$ term. The first is $-t$ multiplied by a determinant of the same type but one dimension lower. An induction finishes things off.

3. Write $\mathrm{S} \in \mathcal{L}(V)$ for the linear map
$$\mathrm{S} := p_W(\mathrm{T}) = \mathrm{T}^k + a_{k-1}\mathrm{T}^{k-1} + \cdots + a_0\mathrm{I}$$
Part 2 says $\mathrm{S}(\mathbf{w}) = \mathbf{0}$. Since S is a polynomial in T, it commutes with all powers of T:
$$\forall i,\ \mathrm{S}(\mathrm{T}^i(\mathbf{w})) = \mathrm{T}^i(\mathrm{S}(\mathbf{w})) = \mathbf{0}$$
Since S is zero on the basis β_W of W, we see that S_W is the zero function. ∎

While the previous result is a little intense, the punchline of the discussion is thankfully much cleaner.

Corollary 5.24 (Cayley–Hamilton). *A linear map satisfies its characteristic polynomial.*

Proof. Let $\mathbf{v} \in V$ and consider the T-cyclic subspace $W = \langle \mathbf{v} \rangle$ generated by $\mathbf{v}$. By Theorem 5.19, the characteristic polynomial $p_W(t)$ of the restriction T_W satisfies

$$p(t) = q_W(t) p_W(t)$$

for some polynomial $q_W(t)$. However, Theorem 5.23 part 3 says that $p_W(\mathrm{T})(\mathbf{v}) = \mathbf{0}$, whence

$$p(\mathrm{T})(\mathbf{v}) = \mathbf{0}$$

Since we may apply this reasoning to *any* $\mathbf{v} \in V$, we conclude that $p(\mathrm{T}) \equiv 0$ is the *zero function.* ■

The Cayley–Hamilton Theorem is used extensively to develop the idea of diagonalizability in inner product spaces and in the discussion of Jordan canonical forms. We will simply apply it to the calculation of inverses and large powers of a linear map.

Examples 5.25. 1. (Example 5.2) $A = \left(\begin{smallmatrix} 2 & 1 \\ 3 & 4 \end{smallmatrix}\right)$ has $p(t) = t^2 - 6t + 5$ and we confirm:

$$A^2 - 6A = \begin{pmatrix} 7 & 6 \\ 18 & 19 \end{pmatrix} - 6 \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = -5I$$

It may seem like a strange thing to do for this matrix, but the characteristic equation can be used to calculate the inverse of A:

$$A^2 - 6A + 5I = 0 \implies A(A - 6I) = -5I \implies A^{-1} = \frac{1}{5}(6I - A) = \frac{1}{5}\begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$$

2. (Example 5.7.3) We use the Cayley–Hamilton Theorem to compute T^4 when

$$A = [\mathrm{T}]_\epsilon = \left(\begin{smallmatrix} 2 & -1 & \frac{8}{3} \\ 0 & 1 & -6 \\ 0 & 0 & 2 \end{smallmatrix}\right) \quad \text{with} \quad p(t) = (2-t)^2(1-t) = 4 - 8t + 5t^2 - t^3$$

By Cayley–Hamilton,

$$\begin{aligned} \mathrm{T}^4 &= \mathrm{T} \circ \mathrm{T}^3 = \mathrm{T} \circ (5\mathrm{T}^2 - 8\mathrm{T} + 4I) = 5\mathrm{T}^3 - 8\mathrm{T}^2 + 4\mathrm{T} = 5(5\mathrm{T}^2 - 8\mathrm{T} + 4\mathrm{I}) - 8\mathrm{T}^2 + 4\mathrm{T} \\ &= 17\mathrm{T}^2 - 36\mathrm{T} + 20\mathrm{I} \end{aligned}$$

We can easily(!) compute the matrix:

$$[\mathrm{T}^4]_\epsilon = 17A^2 - 36A + 20I = 17\left(\begin{smallmatrix} 4 & -3 & \frac{50}{3} \\ 0 & 1 & -18 \\ 0 & 0 & 4 \end{smallmatrix}\right) - 36\left(\begin{smallmatrix} 2 & -1 & \frac{8}{3} \\ 0 & 1 & -6 \\ 0 & 0 & 2 \end{smallmatrix}\right) + 20\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 16 & -15 & \frac{562}{3} \\ 0 & 1 & -90 \\ 0 & 0 & 16 \end{smallmatrix}\right)$$

It follows, for example, that

$$\mathrm{T}^4(35 - 3x^2) = 35 \cdot 16 - 3\left(\frac{562}{3} - 90x + 16x^2\right) = -2 + 270x - 48x^2$$

3. The linear map $\mathrm{T} \in \mathcal{L}(P_2(\mathbb{R}))$ defined by $\mathrm{T}(f(x)) = f(x) + (x-1)f'(x)$ has characteristic polynomial

$$p(t) = (1-t)(2-t)(3-t) = -t^3 + 6t^2 - 11t + 6$$

as is easily seen by computing the matrix of T with respect to the standard basis $\{1, x, x^2\}$. By Cayley–Hamilton, we conclude that $\mathrm{T}^3 = 6\mathrm{T}^2 - 11\mathrm{T} + 6\mathrm{I}$. You can also check this explicitly, after first computing

$$\begin{aligned}\mathrm{T}^2(f)(x) &= f(x) + 3(x-1)f'(x) + (x-1)^2 f''(x)\\ \mathrm{T}^3(f)(x) &= f(x) + 7(x-1)f'(x) + 6(x-1)^2 f''(x)\end{aligned}$$

We can also apply Cayley–Hamilton to find the inverse of T:

$$\begin{aligned}\mathrm{I} = \frac{1}{6}(\mathrm{T}^3 - 6\mathrm{T}^2 + 11\mathrm{T}) \implies \mathrm{T}^{-1} &= \frac{1}{6}(\mathrm{T}^2 - 6\mathrm{T} + 11\mathrm{I})\\ \implies \mathrm{T}^{-1}(f)(x) &= f(x) - \frac{1}{2}(x-1)f'(x) + \frac{1}{6}(x-1)^2 f''(x)\end{aligned}$$

Warning! This is only the inverse of T viewed as a linear transformation of $P_2(\mathbb{R})$! If we change the vector space, the formula for the inverse will also change…

Exercises 5.3 1. Find a basis for the T-cyclic subspace $\langle \mathbf{v} \rangle$ of the given linear map:

(a) $\mathrm{T}\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} = \begin{pmatrix} a+b\\ b-c\\ a+c\\ a+d\end{pmatrix}$ on $\mathbb{R}^4$, where $\mathbf{v} = \begin{pmatrix} 1\\ 0\\ 0\\ 0\end{pmatrix}$

(b) $\mathrm{T}(f)(x) = f''(x)$ on $P_3(\mathbb{R})$, where $\mathbf{v} = x^3$

(c) $\mathrm{T}(f)(x) = f''(x) + f(x)$ on $\operatorname{Span}\{1, \sin x, \cos x, x\sin x, x\cos x\}$ where $\mathbf{v} = 1 + x\sin x$.

2. If $A = \begin{pmatrix} 4&1&0&0\\ 0&4&0&0\\ 0&0&4&0\\ 0&0&0&3\end{pmatrix}$, find *three distinct* invariant subspaces $W \leq \mathbb{R}^4$ such that $\dim W = 3$.
(*Hint: What is* $A\mathbf{e}_2$*?*)

3. Let $\mathrm{T} \in \mathcal{L}(V)$ and $\mathbf{v} \in V$. Prove that $\dim \langle \mathbf{v} \rangle = 1 \iff \mathbf{v}$ is an eigenvector of T.

4. We earlier remarked that the T-cyclic subspace $\langle \mathbf{v} \rangle$ is the smallest T-invariant subspace of V containing $\mathbf{v}$. To flesh this out, prove the following explicitly:

   (a) $\langle \mathbf{v} \rangle$ is T-invariant.

   (b) If $W \leq V$ is T-invariant and $\mathbf{v} \in W$, then $\langle \mathbf{v} \rangle \leq W$.

5. Consider the linear map $\mathrm{L}_A : \mathbb{R}^3 \to \mathbb{R}^3$ where $A = \begin{pmatrix} 3&0&0\\ 0&2&4\\ 0&0&2\end{pmatrix}$

   (a) Find the L_A-cyclic subspace generated by each $\mathbf{v} \in \mathbb{R}^3$. In particular, prove that $\langle \mathbf{v} \rangle = \mathbb{R}^3 \iff ac \neq 0$.
   (*Hint: first compute* $\det(\mathbf{v}\ A\mathbf{v}\ A^2\mathbf{v})$ *for any* $\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$)

   (b) Check that the Cayley–Hamilton Theorem is satisfied for L_A.

6. Let $\mathrm{T}(f)(x) = f'(x) + \frac{1}{x}\int_0^x f(t)\,\mathrm{d}t$ be a linear map $\mathrm{T} \in \mathcal{L}(P_2(\mathbb{R}))$.

   (a) Find the characteristic polynomial of T and identify its eigenspaces. Is it diagonalizable?

   (b) Find $a, b, c \in \mathbb{R}$ such that $\mathrm{T}^3 = a\mathrm{T}^2 + b\mathrm{T} + c\mathrm{I}$.

   (c) What are $\dim \mathcal{L}(P_2(\mathbb{R}))$ and $\dim \operatorname{Span}\{\mathrm{T}^k : k \in \mathbb{N}_0\}$? Explain.

7. Recall Exercise 5.25.3. Find an explicit expression for $\mathrm{T}^{-1}(f)(x)$ (i.e. using derivatives!) when T is viewed as a linear transformation of $P_1(\mathbb{R})$.

8. For any matrix $A \in M_n(\mathbb{F})$, prove that

$$\dim \operatorname{Span}\{I, A, A^2, \ldots\} \leq n$$

9. Suppose $A \in M_n(\mathbb{F})$ has characteristic polynomial

$$p(t) = (-1)^n t^n + c_{n-1}t^{n-1} + \cdots + c_0$$

   (a) Prove that if A is invertible, then

$$A^{-1} = -\frac{1}{c_0}\left((-1)^n A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_1 I\right)$$

   (b) Use this to find the inverse of T in Exercise 6.

   (c) If A is upper-triangular and invertible, prove that A^{-1} is also upper-triangular.