3 Canonical Forms

3.1 Jordan Forms & Generalized Eigenvectors

Throughout this course we've concerned ourselves with variations of a general question: for a given map $\mathrm{T} \in \mathcal{L}(V)$, find a basis β such that the matrix $[\mathrm{T}]_\beta$ is as close to diagonal as possible. In this chapter we see what is possible when T is non-diagonalizable.

Example 3.1. The matrix $A = \left(\begin{smallmatrix} -8 & 4 \\ -25 & 12 \end{smallmatrix}\right) \in M_2(\mathbb{R})$ has characteristic equation

$$p(t) = (-8-t)(12-t) + 4 \cdot 25 = t^2 - 4t + 4 = (t-2)^2$$

and thus a single eigenvalue $\lambda = 2$. It is non-diagonalizable since the eigenspace is one-dimensional

$$E_2 = \mathcal{N}\begin{pmatrix} -10 & 4 \\ -25 & 10 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

However, if we consider a basis $\beta = \{\mathbf{v}_1, \mathbf{v}_2\}$ where $\mathbf{v}_1 = \left(\begin{smallmatrix} 2 \\ 5 \end{smallmatrix}\right)$ is an eigenvector, then $[\mathrm{L}_A]_\beta$ is upper-triangular, which is better than nothing! How simple can we make this matrix? Let $\mathbf{v}_2 = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$, then

$$A\mathbf{v}_2 = \begin{pmatrix} -8x+4y \\ -25x+12y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -10x+4y \\ -25x+10y \end{pmatrix} = 2\mathbf{v}_2 + (-5x+2y)\mathbf{v}_1$$
$$\implies [\mathrm{L}_A]_\beta = \begin{pmatrix} 2 & -5x+2y \\ 0 & 2 \end{pmatrix}$$

Since $\mathbf{v}_2$ cannot be parallel to $\mathbf{v}_1$, the only thing we cannot have is a diagonal matrix. The next best thing is for the upper right corner be 1; for instance we could choose

$$\beta = \{\mathbf{v}_1, \mathbf{v}_2\} = \left\{\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\} \implies [\mathrm{L}_A]_\beta = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Definition 3.2. A *Jordan block* is a square matrix of the form

$$J = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$$

where all non-indicated entries are zero. Any 1×1 matrix is also a Jordan block.

A *Jordan canonical form* is a block-diagonal matrix $\operatorname{diag}(J_1, \ldots, J_m)$ where each J_k is a Jordan block.

A *Jordan canonical basis* for $\mathrm{T} \in \mathcal{L}(V)$ is a basis β of V such that $[\mathrm{T}]_\beta$ is a Jordan canonical form.

If a map is diagonalizable, then any eigenbasis is Jordan canonical and the corresponding Jordan canonical form is diagonal. What about more generally? Does every non-diagonalizable map have a Jordan canonical basis? If so, how can we find such?

Example 3.3. It can easily be checked that $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ is a Jordan canonical basis for

$$A = \begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{pmatrix}$$

(really $\mathrm{L}_A \in \mathcal{L}(\mathbb{R}^3)$). Indeed

$$A\mathbf{v}_1 = 2\mathbf{v}_1, \quad A\mathbf{v}_2 = 3\mathbf{v}_2, \quad A\mathbf{v}_3 = \begin{pmatrix} 4 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 1+3 \\ 2+3 \\ 0+3 \end{pmatrix} = \mathbf{v}_2 + 3\mathbf{v}_3 \implies [\mathrm{L}_A]_\beta = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

Generalized Eigenvectors

Example 3.3 was easy to check, but how would we go about finding a suitable β if we were merely given A? We brute-forced this in Example 3.1, but such is not a reasonable approach in general. Eigenvectors get us some of the way:

- $\mathbf{v}_1$ is an eigenvector in Example 3.1, but $\mathbf{v}_2$ is not.
- $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors in Example 3.3, but $\mathbf{v}_3$ is not.

The practical question is how to fill out a Jordan canonical basis once we have a maximal independent set of eigenvectors. We now define the necessary objects.

Definition 3.4. Suppose $\mathrm{T} \in \mathcal{L}(V)$ has an eigenvalue λ. Its *generalized eigenspace* is

$$K_\lambda := \{\mathbf{x} \in V : (\mathrm{T} - \lambda\mathrm{I})^k(\mathbf{x}) = \mathbf{0} \text{ for some } k \in \mathbb{N}\} = \bigcup_{k \in \mathbb{N}} \mathcal{N}(\mathrm{T} - \lambda\mathrm{I})^k$$

A *generalized eigenvector* is any non-zero $\mathbf{v} \in K_\lambda$.

As with eigenspaces, the generalized eigenspaces of $A \in M_n(\mathbb{F})$ are those of the map $\mathrm{L}_A \in \mathcal{L}(\mathbb{F}^n)$.

It is easy to check that our earlier Jordan canonical bases consist of generalized eigenvectors.

Example 3.1: We have one eigenvalue $\lambda = 2$. Since $(A - 2I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero matrix, every non-zero vector is a generalized eigenvector; plainly $K_2 = \mathbb{R}^2$.

Example 3.3: We see that

$$(A - 2I)\mathbf{v}_1 = \mathbf{0}, \qquad (A - 3I)\mathbf{v}_2 = \mathbf{0}, \qquad (A - 3I)^2\mathbf{v}_3 = (A - 3I)\mathbf{v}_2 = \mathbf{0}$$

whence β is a basis of generalized eigenvectors. Indeed

$$K_3 = \mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2\}, \quad K_2 = E_2 = \mathrm{Span}\{\mathbf{v}_3\}$$

though verifying this with current technology is a little awkward…

In order to easily compute generalized eigenspaces, it is useful to invoke the main result of this section. We postpone the proof for a while due to its meatiness.

Theorem 3.5. *Suppose that the characteristic polynomial of* $\mathrm{T} \in \mathcal{L}(V)$ *splits over* $\mathbb{F}$*:*

$$p(t) = (\lambda_1 - t)^{m_1} \cdots (\lambda_k - t)^{m_k}$$

where the λ_j *are the* distinct *eigenvalues of* T *with algebraic multiplicities* m_j*. Then:*

1. *For each eigenvalue; (a)* $K_\lambda = \mathcal{N}(\mathrm{T} - \lambda \mathrm{I})^m$ *and (b)* $\dim K_\lambda = m$*.*
2. $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$*: there exists a basis of generalized eigenvectors.*

Compare this with the statement on diagonalizability from the start of the course.

With regard to part 2; we shall eventually be able to *choose* this to be a Jordan canonical basis. In conclusion: a map has a Jordan canonical basis if and only if its characteristic polynomial splits.

Examples 3.6. 1. Observe how Example 3.3 works in this language:

$$A = \begin{pmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{pmatrix} \implies p(t) = (2-t)^1(3-t)^2$$

$$K_2 = \mathcal{N}(A - 2I)^1 = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \implies \dim K_2 = 1$$

$$K_3 = \mathcal{N}(A - 3I)^2 = \mathcal{N}\begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ 2 & -1 & -1 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\} \implies \dim K_3 = 2$$

$$\mathbb{R}^3 = K_2 \oplus K_3$$

2. We find the generalized eigenspaces of the matrix $A = \left(\begin{smallmatrix} 5 & 2 & -1 \\ 0 & 0 & 0 \\ 9 & 6 & -1 \end{smallmatrix}\right)$

   The characteristic polynomial is

$$p(t) = \det(A - \lambda I) = -t\begin{vmatrix} 5-t & -1 \\ 9 & -1-t \end{vmatrix} = -t(t^2 - 5t + t - 5 + 9) = -(0-t)^1(2-t)^2$$

   - $\lambda = 0$ has multiplicity 1; indeed $K_0 = \mathcal{N}(A - 0I)^1 = \mathcal{N}(A) = \operatorname{Span}\left(\begin{smallmatrix} 1 \\ -1 \\ 3 \end{smallmatrix}\right)$ is just the eigenspace E_0.
   - $\lambda = 2$ has multiplicity 2,

$$K_2 = \mathcal{N}(A - 2I)^2 = \mathcal{N}\begin{pmatrix} 3 & 2 & -1 \\ 0 & -2 & 0 \\ 9 & 6 & -3 \end{pmatrix}^2 = \mathcal{N}\begin{pmatrix} 0 & -4 & 0 \\ 0 & 4 & 0 \\ 0 & -12 & 0 \end{pmatrix} = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

   In this case the corresponding eigenspace is *one*-dimensional, $E_2 = \operatorname{Span}\left(\begin{smallmatrix} 1 \\ 0 \\ 3 \end{smallmatrix}\right) \lneq K_2$, and the matrix is non-diagonalizable.

   Observe also that $\mathbb{R}^3 = K_0 \oplus K_2$ in accordance with the Theorem.

Properties of Generalized Eigenspaces and the Proof of Theorem 3.5

A lot of work is required to justify our main result. Feel free to skip the proofs at first reading.

Lemma 3.7. *Let λ be an eigenvalue of $\mathrm{T} \in \mathcal{L}(V)$. Then:*

1. *E_λ is a subspace of K_λ, which is itself a subspace of V.*
2. *K_λ is T-invariant.*
3. *Suppose K_λ is finite-dimensional and $\mu \neq \lambda$. Then:*
   (a) *K_λ is $(\mathrm{T} - \mu\mathrm{I})$-invariant and the restriction of $\mathrm{T} - \mu\mathrm{I}$ to K_λ is an isomorphism.*
   (b) *If μ is another eigenvalue, then $K_\lambda \cap K_\mu = \{\mathbf{0}\}$. In particular K_λ contains no eigenvectors other than those in E_λ.*

Proof. 1. These are an easy exercise.

2. Let $\mathbf{x} \in K_\lambda$, then $\exists k$ such that $(\mathrm{T} - \lambda\mathrm{I})^k(\mathbf{x}) = \mathbf{0}$. But then

$$\begin{aligned}(\mathrm{T} - \lambda\mathrm{I})^k\big(\mathrm{T}(\mathbf{x})\big) &= (\mathrm{T} - \lambda\mathrm{I})^k\big(\mathrm{T}(\mathbf{x}) - \lambda\mathbf{x} + \lambda\mathbf{x}\big)\\ &= (\mathrm{T} - \lambda\mathrm{I})^{k+1}(\mathbf{x}) + \lambda(\mathrm{T} - \lambda\mathrm{I})^k(\mathbf{x}) = \mathbf{0}\end{aligned}$$

   Otherwise said, $\mathrm{T}(\mathbf{x}) \in K_\lambda$.

3. (a) Let $\mathbf{x} \in K_\lambda$. Part 2 tells us that

$$(\mathrm{T} - \mu\mathrm{I})(\mathbf{x}) = \mathrm{T}(\mathbf{x}) - \mu\mathbf{x} \in K_\lambda$$

   whence K_λ is $(\mathrm{T} - \mu\mathrm{I})$-invariant.

   Suppose, for a contradiction, that $\mathrm{T} - \mu\mathrm{I}$ is *not injective* on K_λ. Then

$$\exists \mathbf{y} \in K_\lambda \setminus \{\mathbf{0}\} \text{ such that } (\mathrm{T} - \mu\mathrm{I})(\mathbf{y}) = \mathbf{0}$$

   Let $k \in \mathbb{N}$ be minimal such that $(\mathrm{T} - \lambda\mathrm{I})^k(\mathbf{y}) = \mathbf{0}$ and let $\mathbf{z} = (\mathrm{T} - \lambda\mathrm{I})^{k-1}(\mathbf{y})$. Plainly $\mathbf{z} \neq \mathbf{0}$, for otherwise k is not minimal. Moreover,

$$(\mathrm{T} - \lambda\mathrm{I})(\mathbf{z}) = (\mathrm{T} - \lambda\mathrm{I})^k(\mathbf{y}) = \mathbf{0} \implies \mathbf{z} \in E_\lambda$$

   Since $\mathrm{T} - \mu\mathrm{I}$ and $\mathrm{T} - \lambda\mathrm{I}$ commute, we can also compute the effect of $\mathrm{T} - \mu\mathrm{I}$;

$$(\mathrm{T} - \mu\mathrm{I})(\mathbf{z}) = (\mathrm{T} - \mu\mathrm{I})(\mathrm{T} - \lambda\mathrm{I})^{k-1}(\mathbf{y}) = (\mathrm{T} - \lambda\mathrm{I})^{k-1}(\mathrm{T} - \mu\mathrm{I})(\mathbf{x}) = \mathbf{0}$$

   which says that $\mathbf{z}$ is an eigenvector in E_μ; if μ isn't an eigenvalue, then we already have our contradiction! Even if μ is an eigenvalue, $E_\mu \cap E_\lambda = \{\mathbf{0}\}$ provides the desired contradiction.

   We conclude that $(\mathrm{T} - \mu\mathrm{I})_{K_\lambda} \in \mathcal{L}(K_\lambda)$ is injective. Since $\dim K_\lambda < \infty$, the restriction is automatically an isomorphism.

   (b) This is another exercise.

■

Now to prove Theorem 3.5: remember that the characteristic polynomial of T is assumed to split.

Proof. (Part 1(a)) Fix an eigenvalue λ. By definition, we have $\mathcal{N}(\mathrm{T}-\lambda\mathrm{I})^m \le K_\lambda$.

For the converse, parts 2 and 3 of the Lemma tell us (why?) that

$$p_\lambda(t) = (\lambda - t)^{\dim K_\lambda} \quad \text{from which} \quad \dim K_\lambda \le m \tag{$*$}$$

By the Cayley–Hamilton Theorem, T_{K_λ} satisfies its characteristic polynomial, whence

$$\forall \mathbf{x} \in K_\lambda,\ (\lambda\mathrm{I}-\mathrm{T})^{\dim K_\lambda}(\mathbf{x}) = \mathbf{0} \implies K_\lambda \le \mathcal{N}(\mathrm{T}-\lambda\mathrm{I})^m$$

(Parts 1(b) and 2) We prove simultaneously by induction on the number of distinct eigenvalues of T.

(*Base case*) If T has only one eigenvalue, then $p(t) = (\lambda - t)^m$. Another appeal to Cayley–Hamilton says $(\mathrm{T}-\lambda\mathrm{I})^m(\mathbf{x}) = \mathbf{0}$ for all $\mathbf{x} \in V$. Thus $V = K_\lambda$ and $\dim K_\lambda = m$.

(*Induction step*) Fix k and suppose the results hold for maps with k distinct eigenvalues. Let T have distinct eigenvalues $\lambda_1, \ldots, \lambda_k, \mu$, with multiplicities $m_1, \ldots, m_k, m$ respectively. Define[1]

$$W = \mathcal{R}(\mathrm{T}-\mu\mathrm{I})^m$$

The subspace W has the following properties, the first two of which we leave as exercises:

- W is T-invariant.
- $W \cap K_\mu = \{\mathbf{0}\}$ so that μ is not an eigenvalue of the restriction T_W.
- Each $K_{\lambda_j} \le W$: since $(\mathrm{T}-\mu\mathrm{I})_{K_{\lambda_j}}$ is an isomorphism (Lemma part 3), we can invert,

$$\mathbf{x} \in K_{\lambda_j} \implies \mathbf{x} = (\mathrm{T}-\mu\mathrm{I})^m \left((\mathrm{T}-\mu\mathrm{I})^{-1}_{K_{\lambda_j}} \right)^m (\mathbf{x}) \in \mathcal{R}(\mathrm{T}-\mu\mathrm{I})^m = W$$

We conclude that λ_j is an eigenvalue of the restriction T_W with generalized eigenspace K_{λ_j}.

Since T_W has k distinct eigenvalues, the induction hypotheses apply:

$$W = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k} \quad \text{and} \quad p_W(t) = (\lambda_1 - t)^{\dim K_{\lambda_1}} \cdots (\lambda_k - t)^{\dim K_{\lambda_k}}$$

Since $W \cap K_\mu = \{\mathbf{0}\}$ it is enough finally to use the rank–nullity theorem and count dimensions:

$$\begin{aligned} \dim V &= \operatorname{rank}(\mathrm{T}-\mu\mathrm{I})^m + \operatorname{null}(\mathrm{T}-\mu\mathrm{I})^m = \dim W + \dim K_\mu = \sum_{j=1}^k \dim K_{\lambda_j} + \dim K_\mu \\ &\overset{(*)}{\le} m_1 + \cdots + m_k + m = \deg(p(t)) = \dim V \end{aligned}$$

The inequality is thus an *equality*; each $\dim K_{\lambda_j} = m_j$ and $\dim K_\mu = m$. We conclude that

$$V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k} \oplus K_\mu$$

which completes the induction step and thus the proof. Whew! ∎

[1] This is yet another argument where we consider a suitable subspace to which we can apply an induction hypothesis; recall the spectral theorem, Schur's lemma, bilinear form diagonalization, etc. Theorem 3.12 will provide one more!

Cycles of Generalized Eigenvectors

By Theorem 3.5, for every linear map whose characteristic polynomial splits there exists generalized eigenbasis. This isn't the same as a Jordan canonical basis, but we're very close!

Example 3.8. The matrix $A = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix} \in M_3(\mathbb{R})$ is a single Jordan block, whence there is a single generalized eigenspace $K_5 = \mathbb{R}^3$ and the standard basis $\epsilon = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is Jordan canonical.
The crucial observation for what follows is that one of these vectors $\mathbf{e}_3$ *generates* the others via repeated applications of $A - 5I$:

$$\mathbf{e}_2 = (A-5I)\mathbf{e}_3, \qquad \mathbf{e}_1 = (A-5I)\mathbf{e}_2 = (A-5I)^2\mathbf{e}_3$$

Definition 3.9. A *cycle of generalized eigenvectors* for a linear operator T is a set

$$\beta_{\mathbf{x}} := \left\{ (\mathrm{T}-\lambda\mathrm{I})^{k-1}(\mathbf{x}), \ldots, (\mathrm{T}-\lambda\mathrm{I})(\mathbf{x}), \mathbf{x} \right\}$$

where the *generator* $\mathbf{x} \in K_\lambda$ is non-zero and k is minimal such that $(\mathrm{T}-\lambda\mathrm{I})^k(\mathbf{x}) = \mathbf{0}$.

Note that the first element $(\mathrm{T}-\lambda\mathrm{I})^{k-1}(\mathbf{x})$ is an *eigenvector*.
Our goal is to show that K_λ has a basis consisting of cycles of generalized eigenvectors; putting these together results in a Jordan canonical basis.

Lemma 3.10. *Let $\beta_{\mathbf{x}}$ be a cycle of generalized eigenvectors of* T *with length k. Then:*

1. *$\beta_{\mathbf{x}}$ is linearly independent and thus a basis of* Span $\beta_{\mathbf{x}}$.
2. Span $\beta_{\mathbf{x}}$ *is* T*-invariant. With respect to $\beta_{\mathbf{x}}$, the matrix of the restriction of* T *is the $k \times k$ Jordan block* $[\mathrm{T}_{\operatorname{Span}\beta_{\mathbf{x}}}]_{\beta_{\mathbf{x}}} = \begin{pmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix}$.

In what follows, it will be useful to consider the linear map $\mathrm{U} = \mathrm{T} - \lambda\mathrm{I}$. Note the following:

- The nullspace of U is the *eigenspace*: $\mathcal{N}(\mathrm{U}) = E_\lambda \le K_\lambda$.
- T commutes with U: that is $\mathrm{TU} = \mathrm{UT}$.
- $\beta_{\mathbf{x}} = \{\mathrm{U}^{k-1}(\mathbf{x}), \ldots, \mathrm{U}(\mathbf{x}), \mathbf{x}\}$; that is, Span $\beta_{\mathbf{x}} = \langle \mathbf{x} \rangle$ is the U-cyclic subspace generated by $\mathbf{x}$.

Proof. 1. Feed the linear combination $\sum_{j=0}^{k-1} a_j \mathrm{U}^j(\mathbf{x}) = \mathbf{0}$ to U^{k-1} to obtain

$$a_0 \mathrm{U}^{k-1}(\mathbf{x}) = \mathbf{0} \implies a_0 = 0$$

Now feed the same combination to U^{k-2}, etc., to see that all coefficients $a_j = 0$.

2. Since T and U commute, we see that

$$\mathrm{T}\big(\mathrm{U}^j(\mathbf{x})\big) = \mathrm{U}^j\big(\mathrm{T}(\mathbf{x})\big) = \mathrm{U}^j\big((\mathrm{U}+\lambda\mathrm{I})(\mathbf{x})\big) = \mathrm{U}^{j+1}(\mathbf{x}) + \lambda \mathrm{U}^j(\mathbf{x}) \in \operatorname{Span}\beta_{\mathbf{x}}$$

This justifies both T-invariance and the Jordan block claim! ∎

The basic approach to finding a Jordan canonical basis is to find the generalized eigenspaces and play with cycles until you find a basis for each K_λ. *Many* choices of canonical basis exist for a given map! We'll consider a more systematic method in the next section.

Examples 3.11. 1. The characteristic polynomial of $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 6 \\ 6 & -2 & 1 \end{pmatrix} \in M_3(\mathbb{R})$ splits:

$$p(t) = (1-t)\begin{vmatrix} 1-t & 6 \\ -2 & 1-t \end{vmatrix} + 2\begin{vmatrix} 0 & 1-t \\ 6 & -2 \end{vmatrix} = (1-t)\left((1-t)^2 + 12 - 12\right) = (1-t)^3$$

With only one eigenvalue we see that $K_1 = \mathbb{R}^3$. Simply choose any vector in $\mathbb{R}^3$ and see what $U = A - I$ does to it! For instance, with $\mathbf{x} = \mathbf{e}_1$,

$$\beta_{\mathbf{x}} = \left\{ \mathrm{U}^2 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathrm{U} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 12 \\ 36 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

provides a Jordan canonical basis of $\mathbb{R}^3$. We conclude

$$A = QJQ^{-1} = \begin{pmatrix} 12 & 0 & 1 \\ 36 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 12 & 0 & 1 \\ 36 & 0 & 0 \\ 0 & 6 & 0 \end{pmatrix}^{-1}$$

In practice, almost any choice of $\mathbf{x} \in \mathbb{R}^3$ will generate a cycle of length three!

2. The matrix $B = \begin{pmatrix} 7 & 1 & -4 \\ 0 & 3 & 0 \\ 8 & 1 & -5 \end{pmatrix} \in M_3(\mathbb{R})$ has characteristic equation

$$p(t) = (3-t)(t^2 - 2t - 3) = -(t+1)^1 (t-3)^2$$

$\dim K_{-1} = 1 \implies K_{-1} = E_{-1} = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, spanned by a cycle of length one.

Since $\dim K_3 = 2$, we have

$$K_3 = \mathcal{N}(B - 3I)^2 = \mathcal{N}\begin{pmatrix} 4 & 1 & -4 \\ 0 & 0 & 0 \\ 8 & 1 & -8 \end{pmatrix}^2 = \mathcal{N}\begin{pmatrix} -16 & 0 & 16 \\ 0 & 0 & 0 \\ -32 & 0 & 32 \end{pmatrix} = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

This is spanned by a cycle of length two: $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector and

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (B - 3I)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

We conclude that $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a Jordan canonical basis for B, and that

$$B = QJQ^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}^{-1}$$

3. Let $\mathrm{T} = \frac{\mathrm{d}}{\mathrm{d}x}$ on $P_3(\mathbb{R})$. With respect to the standard basis $\epsilon = \{1, x, x^2, x^3\}$,

$$A = [\mathrm{T}]_\epsilon = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

With only one eigenvalue $\lambda = 0$, we have a single generalized eigenspace $K_0 = P_3(\mathbb{R})$. It is easy to check that $f(x) = x^3$ generates a cycle of length three and thus a Jordan canonical basis:

$$\beta = \{6, 6x, 3x^2, x^3\} \implies [\mathrm{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Our final results state that this process works generally.

Theorem 3.12. *Let* $\mathrm{T} \in \mathcal{L}(V)$ *have an eigenvalue* λ. *If* $\dim K_\lambda < \infty$, *then there exists a basis* $\beta_\lambda = \beta_{\mathbf{x}_1} \cup \cdots \cup \beta_{\mathbf{x}_n}$ *of* K_λ *consisting of finitely many linearly independent cycles.*

Intuition suggests that we create cycles $\beta_{\mathbf{x}_j}$ by starting with a basis of the eigenspace E_λ and extending backwards: for each $\mathbf{x}$, if $\mathbf{x} = (\mathrm{T} - \lambda\mathrm{I})(\mathbf{y})$, then $\mathbf{x} \in \beta_{\mathbf{y}}$; now repeat until you have a maximum length cycle. This is essentially what we do, though a sneaky induction is required to make sure we keep track of everything and guarantee that the result really is a basis of K_λ.

Proof. We prove by induction on $m = \dim K_\lambda$.

(*Base case*) If $m = 1$, then $K_\lambda = E_\lambda = \operatorname{Span} \mathbf{x}$ for some eigenvector $\mathbf{x}$. Plainly $\{\mathbf{x}\} = \beta_{\mathbf{x}}$.

(*Induction step*) Fix $m \geq 2$. Write $n = \dim E_\lambda \leq m$ and $\mathrm{U} = (\mathrm{T} - \lambda\mathrm{I})_{K_\lambda}$.

(i) For the induction hypothesis, suppose *every* generalized eigenspace with dimension $< m$ (for *any* linear map!) has a basis consisting of independent cycles of generalized eigenvectors.

(ii) Define $W = \mathcal{R}(\mathrm{U}) \cap E_\lambda$: that is

$$\mathbf{w} \in W \iff \begin{cases} \mathrm{U}(\mathbf{w}) = \mathbf{0} & \text{and} \\ \mathbf{w} = \mathrm{U}(\mathbf{v}) & \text{for some } \mathbf{v} \in K_\lambda \end{cases}$$

Let $k = \dim W$, choose a complementary subspace X such that $E_\lambda = W \oplus X$ and select a basis $\{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$ of X. If $k = 0$, the induction step is finished (why?). Otherwise we continue…

(iii) The calculation in the proof of Lemma 3.10 (take $j = 1$) shows that $\mathcal{R}(\mathrm{U})$ is T-invariant; it is therefore the single generalized eigenspace $\tilde{K}_\lambda$ of $\mathrm{T}_{\mathcal{R}(\mathrm{U})}$.

(iv) By the rank–nullity theorem,

$$\dim \mathcal{R}(\mathrm{U}) = \operatorname{rank} \mathrm{U} = \dim K_\lambda - \operatorname{null} \mathrm{U} = m - \dim E_\lambda < m$$

By the induction hypothesis, $\mathcal{R}(\mathrm{U})$ has a basis of independent cycles. Since the last non-zero element in each cycle is an eigenvector, this basis consists of k distinct cycles $\beta_{\hat{\mathbf{x}}_1} \cup \cdots \cup \beta_{\hat{\mathbf{x}}_k}$ whose terminal vectors form a basis of W.

(v) Since each $\hat{\mathbf{x}}_j \in \mathcal{R}(\mathrm{U})$, there exist vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ such that $\hat{\mathbf{x}}_j = \mathrm{U}(\mathbf{x}_j)$. Including the length-one cycles generated by the basis of X, the cycles $\beta_{\mathbf{x}_1}, \ldots, \beta_{\mathbf{x}_n}$ now contain

$$\dim \mathcal{R}(\mathrm{U}) + k + (n - k) = \operatorname{rank} \mathrm{U} + \operatorname{null} \mathrm{U} = m$$

vectors. We leave as an exercise the verification that these vectors are linearly independent. ∎

Corollary 3.13. *Suppose that the characteristic polynomial of* $\mathrm{T} \in \mathcal{L}(V)$ *splits (necessarily* $\dim V < \infty$*). Then there exists a Jordan canonical basis, namely the union of bases* β_λ *from Theorem 3.12.*

Proof. By Theorem 3.5, V is the direct sum of generalized eigenspaces. By the previous result, each K_λ has a basis β_λ consisting of finitely many cycles. By Lemma 3.10, the matrix of T_{K_λ} has Jordan canonical form with respect to β_λ. It follows that $\beta = \bigcup \beta_\lambda$ is a Jordan canonical basis for T. ∎

Exercises 3.1 1. For each matrix, find the generalized eigenspaces K_λ, find bases consisting of unions of disjoint cycles of generalized eigenvectors, and thus find a Jordan canonical form J and invertible Q so that the matrix may be expressed as QJQ^{-1}.

(a) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ (b) $B = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ (c) $C = \begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}$

(d) $D = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & -1 & 3 \end{pmatrix}$

2. If $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a Jordan canonical basis, what can you say about $\mathbf{v}_1$? Briefly explain why the linear map $\mathrm{L}_A \in \mathcal{L}(\mathbb{R}^2)$ where $A = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ has no Jordan canonical form.

3. Find a Jordan canonical basis for each linear map T:

   (a) $\mathrm{T} \in \mathcal{L}(P_2(\mathbb{R}))$ defined by $\mathrm{T}(f(x)) = 2f(x) - f'(x)$

   (b) $\mathrm{T}(f) = f'$ defined on $\mathrm{Span}\{1, t, t^2, e^t, te^t\}$

   (c) $\mathrm{T}(A) = 2A + A^T$ defined on $M_2(\mathbb{R})$

4. In Example 3.11.1, suppose $\mathbf{x} = \left(\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}\right)$. Show that almost any choice of a, b, c produces a Jordan canonical basis $\beta_{\mathbf{x}}$.

5. We complete the proof of Lemma 3.7.

   (a) Prove part 1: that $E_\lambda \leq K_\lambda \leq V$.

   (b) Verify that $\mathrm{T} - \mu\mathrm{I}$ and $\mathrm{T} - \lambda\mathrm{I}$ commute.

   (c) Prove part 3(b): generalized eigenspaces for distinct eigenvalues have trivial intersection.

6. Consider the induction step in the proof of Theorem 3.5.

   (a) Prove that W is T-invariant.

   (b) Explain why $W \cap K_\mu = \{\mathbf{0}\}$.

   (c) The assumption $p_W(t) = (\lambda_1 - t)^{\dim K_{\lambda_1}} \cdots (\lambda_k - t)^{\dim K_{\lambda_k}}$ near the end of the proof is the induction hypothesis for part 1(b). Why can't we also assume that $\dim K_{\lambda_j} = m_j$ and thus tidy the inequality argument near the end of the proof?

7. We finish some of the details of Theorem 3.12.

   (a) In step (ii), suppose $\dim W = k = 0$. Explain why $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is in fact a basis of K_λ, so that the rest of the proof is unnecessary.

   (b) In step (v), prove that the m vectors in the cycles $\beta_{\mathbf{x}_1}, \ldots, \beta_{\mathbf{x}_n}$ are linearly independent.

   (*Hint: model your argument on part 1 of Lemma 3.10*)

3.2 Cycle Patterns and the Dot Diagram

In this section we obtain a useful result that helps us compute Jordan forms more efficiently and systematically. To give us some clues how to proceed, here is a lengthy example.

Example 3.14. Precisely three Jordan canonical forms $A, B, C \in M_3(\mathbb{R})$ correspond to the characteristic polynomial $p(t) = (5-t)^3$:

$$A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix} \qquad C = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 5 \end{pmatrix}$$

In all three cases the standard basis $\beta = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is Jordan canonical, so how do we distinguish things? By considering the *number* and *lengths* of the cycles of generalized eigenvectors.

- A has eigenspace $E_5 = K_5 = \mathbb{R}^3$. Since $(A-5I)\mathbf{v} = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^3$, we have maximum cycle-length one. We therefore need three distinct cycles to construct a Jordan basis, e.g.

$$\beta_{\mathbf{e}_1} = \{\mathbf{e}_1\}, \quad \beta_{\mathbf{e}_2} = \{\mathbf{e}_2\}, \quad \beta_{\mathbf{e}_3} = \{\mathbf{e}_3\} \implies \beta = \beta_{\mathbf{e}_1} \cup \beta_{\mathbf{e}_2} \cup \beta_{\mathbf{e}_3} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

- B has eigenspace $E_5 = \operatorname{Span}\{\mathbf{e}_1, \mathbf{e}_3\}$. By computing

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies (B-5I)\mathbf{v} = \begin{pmatrix} b \\ 0 \\ 0 \end{pmatrix} \implies (B-5I)^2\mathbf{v} = \mathbf{0}$$

  we see that $\beta_{\mathbf{v}}$ is a cycle with maximum length two, provided $b \neq 0$ ($\mathbf{v} \notin E_5$). We therefore need two distinct cycles, of lengths two and one, to construct a Jordan basis, e.g.

$$\beta_{\mathbf{e}_2} = \{(B-5I)\mathbf{e}_2,\, \mathbf{e}_2\} = \{\mathbf{e}_1, \mathbf{e}_2\}, \quad \beta_{\mathbf{e}_3} = \{\mathbf{e}_3\} \implies \beta = \beta_{\mathbf{e}_2} \cup \beta_{\mathbf{e}_3} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

- C has eigenspace $E_5 = \operatorname{Span} \mathbf{e}_1$. This time

$$\mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \implies (C-5I)\mathbf{v} = \begin{pmatrix} b \\ c \\ 0 \end{pmatrix}, \quad (C-5I)^2\mathbf{v} = \begin{pmatrix} c \\ 0 \\ 0 \end{pmatrix}, \quad (C-5I)^3\mathbf{v} = \mathbf{0}$$

  generates a cycle with maximum length two provided $c \neq 0$. Indeed this cycle is a Jordan basis, so one cycle is all we need:

$$\beta = \beta_{\mathbf{e}_3} = \{(C-5I)^2\mathbf{e}_3,\, (C-5I)\mathbf{e}_3,\, \mathbf{e}_3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$$

Why is the example relevant? Suppose that $\dim_{\mathbb{R}} V = 3$ and that $\mathrm{T} \in \mathcal{L}(V)$ has characteristic polynomial $p(t) = (5-t)^3$. Theorem 3.12 tells us that T has a Jordan canonical form, and that is is moreover one of the above matrices A, B, C. Our goal is to develop a method whereby the pattern of cycle-lengths can be determined, thus allowing us to be able to discern which Jordan form is correct. As a side-effect, this will also demonstrate that the pattern of cycle lengths for a given T is independent of the Jordan basis so that, up to some reasonable restriction, the Jordan *form* of T is unique. To aid us in this endeavor, we require some terminology. . .

Definition 3.15. Let V be finite dimensional and K_λ a generalized eigenspace of $\mathrm{T} \in \mathcal{L}(V)$. Following the Theorem 3.12, assume that $\beta_\lambda = \beta_{\mathbf{x}_1} \cup \cdots \cup \beta_{\mathbf{x}_n}$ is a Jordan canonical basis of T_{K_λ}, where the cycles are arranged in *non-increasing* length. That is:

1. $\beta_{\mathbf{x}_j} = \{(\mathrm{T} - \lambda \mathrm{I})^{k_j - 1}(\mathbf{x}_j), \ldots, \mathbf{x}_j\}$ has length k_j, and
2. $k_1 \geq k_2 \geq \cdots \geq k_n$

The *dot diagram* of T_{K_λ} is a representation of the elements of β_λ, one dot for each vector: the j^{th} column represents the elements of $\beta_{\mathbf{x}_j}$ arranged vertically with $\mathbf{x}_j$ at the bottom.

Given a linear map, our eventual goal is to identify the dot diagram as an intermediate step in the computation of a Jordan basis. First, however, we observe how the conversion of dot diagrams to a Jordan form is essentially trivial.

Example 3.16. Suppose $\dim V = 14$ and that $\mathrm{T} \in \mathcal{L}(V)$ has the following eigenvalues and dot diagrams:

$$\begin{array}{cccc} \multicolumn{4}{c}{\lambda_1 = -4} \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \\ & & & \end{array} \qquad \begin{array}{cc} \multicolumn{2}{c}{\lambda_2 = 7} \\ \bullet & \bullet \\ \bullet & \bullet \\ \bullet & \end{array} \qquad \begin{array}{ccc} \multicolumn{3}{c}{\lambda_3 = 12} \\ \bullet & \bullet & \bullet \\ & & \\ & & \end{array}$$

Then generalized eigenspaces of T satisfy:

- $K_{-4} = \mathcal{N}(\mathrm{T} + 4\mathrm{I})^2$ and $\dim K_{-4} = 6$;
- $K_7 = \mathcal{N}(\mathrm{T} - 7\mathrm{I})^3$ and $\dim K_7 = 5$;
- $K_{12} = \mathcal{N}(\mathrm{T} - 12\mathrm{I}) = E_{12}$ and $\dim K_{12} = 3$;

T has a Jordan canonical basis β with respect to which its Jordan canonical form is

$$[\mathrm{T}]_\beta = \begin{pmatrix} -4 & 1 & & & & & & & & & & & & \\ 0 & -4 & & & & & & & & & & & & \\ & & -4 & 1 & & & & & & & & & & \\ & & 0 & -4 & & & & & & & & & & \\ & & & & -4 & & & & & & & & & \\ & & & & & -4 & & & & & & & & \\ & & & & & & 7 & 1 & 0 & & & & & \\ & & & & & & 0 & 7 & 1 & & & & & \\ & & & & & & 0 & 0 & 7 & & & & & \\ & & & & & & & & & 7 & 1 & & & \\ & & & & & & & & & 0 & 7 & & & \\ & & & & & & & & & & & 12 & & \\ & & & & & & & & & & & & 12 & \\ & & & & & & & & & & & & & 12 \end{pmatrix}$$

Note how the sizes of the Jordan blocks are *non-increasing* within each eigenvalue. For instance, for $\lambda_1 = -4$, the sequence of cycle lengths (k_j) is $2 \geq 2 \geq 1 \geq 1$.

Theorem 3.17. *Suppose β_λ is a Jordan canonical basis of T_{K_λ} as described in Definition 3.15, and suppose the i^{th} row of the dot diagram has r_i entries. Then:*

1. *For each $r \in \mathbb{N}$, the vectors associated to the dots in the first r rows form a basis of $\mathcal{N}(\mathrm{T}-\lambda\mathrm{I})^r$.*
2. $r_1 = \mathrm{null}(\mathrm{T}-\lambda\mathrm{I}) = \dim V - \mathrm{rank}(\mathrm{T}-\lambda\mathrm{I})$
3. *When $i > 1$, $r_i = \mathrm{null}(\mathrm{T}-\lambda\mathrm{I})^i - \mathrm{null}(\mathrm{T}-\lambda\mathrm{I})^{i-1} = \mathrm{rank}(\mathrm{T}-\lambda\mathrm{I})^{i-1} - \mathrm{rank}(\mathrm{T}-\lambda\mathrm{I})^i$*

Example (3.14 cont). We describe the dot diagrams of the three matrices A, B, C, along with the corresponding vectors in the Jordan canonical basis β and the values r_i.

A : • • • $\mathbf{x}_1$ $\mathbf{x}_2$ $\mathbf{x}_3$ $\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$

Since $A-5I$ is the zero matrix, $r_1 = 3 - \mathrm{rank}(A-5I) = 3$. The dot diagram has one row, corresponding to three independent cycles of length one: $\beta = \beta_{\mathbf{e}_1} \cup \beta_{\mathbf{e}_2} \cup \beta_{\mathbf{e}_3}$.

B : • • $(B-5I)\mathbf{x}_1$ $\mathbf{x}_2$ $\mathbf{e}_1$ $\mathbf{e}_3$
• $\mathbf{x}_1$ $\mathbf{e}_2$

Row 1: $B - 5I = \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix} \implies \mathrm{rank}(B-5I) = 1$ and $r_1 = 3-1 = 2$. The first row $\{\mathbf{e}_1, \mathbf{e}_3\}$ is a basis of $E_5 = \mathcal{N}(B-5I)$.

Row 2: $(B-5I)^2$ is the zero matrix, whence $r_2 = \mathrm{rank}(B-5I) - \mathrm{rank}(B-5I)^2 = 1-0 = 1$.

The dot diagram corresponds to $\beta = \beta_{\mathbf{e}_2} \cup \beta_{\mathbf{e}_3} = \{\mathbf{e}_1, \mathbf{e}_2\} \cup \{\mathbf{e}_3\}$.

C : • $(C-5I)^2\mathbf{x}_1$ $\mathbf{e}_1$
• $(C-5I)\mathbf{x}_1$ $\mathbf{e}_2$
• $\mathbf{x}_1$ $\mathbf{e}_3$

Row 1: $C - 5I = \begin{pmatrix} 0&1&0\\0&0&1\\0&0&0 \end{pmatrix} \implies r_1 = 3 - \mathrm{rank}(C-5I) = 1$. The first row $\{\mathbf{e}_1\}$ is a basis of $E_5 = \mathcal{N}(C-5I)$.

Row 2: $(C-5I)^2 = \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix} \implies r_2 = \mathrm{rank}(C-5I) - \mathrm{rank}(C-5I)^2 = 2-1 = 1$. The first two rows $\{\mathbf{e}_1, \mathbf{e}_2\}$ form a basis of $\mathcal{N}(C-5I)^2$.

Row 3: $(C-5I)^3$ is the zero matrix, whence $r_3 = \mathrm{rank}(C-5I)^2 - \mathrm{rank}(C-5I)^3 = 1-0 = 1$.

Proof. As previously, let $\mathrm{U} = \mathrm{T} - \lambda\mathrm{I}$.

1. Since each dot represents a basis vector $\mathrm{U}^p(\mathbf{v}_j)$, any $\mathbf{v} \in K_\lambda$ may be written uniquely as a linear combination of the dots. Applying U simply moves all the dots up a row and all dots in the top row to $\mathbf{0}$. It follows that $\mathbf{v} \in \mathcal{N}(\mathrm{U}^r) \iff$ it lies in the span of the first r rows. Since the dots are linearly independent, they form a basis.
2. By part 1, $r_1 = \dim \mathcal{N}(\mathrm{U}) = \mathrm{null}(\mathrm{T}-\lambda\mathrm{I}) = \dim V - \mathrm{rank}(\mathrm{T}-\lambda\mathrm{I})$.
3. More generally,

$$\begin{aligned} r_i &= (r_1 + \cdots + r_i) - (r_1 + \cdots + r_{i-1}) = \dim \mathcal{N}(\mathrm{U}^i) - \dim \mathcal{N}(\mathrm{U}^{i-1}) \\ &= \mathrm{null}(\mathrm{U}^i) - \mathrm{null}(\mathrm{U}^{i-1}) = \mathrm{rank}(\mathrm{T}-\lambda\mathrm{I})^{i-1} - \mathrm{rank}(\mathrm{T}-\lambda\mathrm{I})^i \end{aligned}$$

■

Since the ranks of maps $(\mathrm{T}-\lambda\mathrm{I})^i$ are independent of basis, so also is the dot diagram...

Corollary 3.18. *For any eigenvalue λ, the dot diagram is uniquely determined by* T *and λ. If we list Jordan blocks for each eigenspace in non-increasing order, then the Jordan form of a linear map is unique up to the order of the eigenvalues.*

We now have a slightly more systematic method for finding Jordan canonical bases.

Example 3.19. The matrix $A = \begin{pmatrix} 6 & 2 & -4 & -6 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 2 & 1 & -2 & -1 \end{pmatrix}$ has characteristic equation

$$p(t) = (3-t)^2 \begin{vmatrix} 6-t & -6 \\ 2 & -1-t \end{vmatrix} = (2-t)(3-t)^3$$

We have two generalized eigenspaces:

- $K_2 = E_2 = \mathcal{N}(A-2I) = \mathcal{N}\begin{pmatrix} 4 & 2 & -4 & -6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 1 & -2 & -3 \end{pmatrix} = \mathrm{Span}\begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \end{pmatrix}$. The trivial dot diagram • corresponds to this single eigenvector.
- $K_3 = \mathcal{N}(A-3I)^3$. To find the dot diagram, compute powers of $A-3I$:

  Row 1: $A - 3I = \begin{pmatrix} 3 & 2 & -4 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & -4 \end{pmatrix}$ has rank 2 and the first row has $r_1 = 4-2 = 2$ entries.

  Row 2: $(A-3I)^2 = \begin{pmatrix} -3 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 4 \end{pmatrix}$ has rank 1 and the second row has $r_2 = 2-1 = 1$ entry.

  Since we now have three dots (equalling $\dim K_3$), the algorithm terminates and the dot diagram for K_3 is
  • •
  •

  For the single dot in the second row, we choose something in $\mathcal{N}(A-3I)^2$ which isn't an eigenvector; perhaps the simplest choice is $\mathbf{x}_1 = \mathbf{e}_2$, which yields the two-cycle

  $$\beta_{\mathbf{x}_1} = \{(A-3I)\mathbf{x}_1, \mathbf{x}_1\} = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

  To complete the first row, choose any eigenvector to complete the span: for instance $\mathbf{x}_2 = \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix}$.

We now have suitable cycles and a Jordan canonical basis/form:

$$\beta = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad A = QJQ^{-1} = \begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 \end{pmatrix}^{-1}$$

Other choices are available! For instance, if we'd chosen the two-cycle generated by $\mathbf{x}_1 = \mathbf{e}_3$, we'd obtain a different Jordan basis but the same canonical form J:

$$\tilde{\beta} = \left\{ \begin{pmatrix} 3 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \\ 0 \end{pmatrix} \right\}, \qquad A = \begin{pmatrix} 3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & -2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 2 & -2 & 0 & 0 \end{pmatrix}^{-1}$$

We do one final example for a non-matrix map.

Example 3.20. Let $\epsilon = \{1, x, y, x^2, y^2, xy\}$ and define $\mathrm{T}\big(f(x,y)\big) = 2\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}$ as a linear operator on $V = \operatorname{Span}_{\mathbb{R}} \epsilon$. The matrix and characteristic polynomial of T is easy to compute:

$$[\mathrm{T}]_\epsilon = \begin{pmatrix} 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \implies p(t) = t^6, \qquad [\mathrm{T}^2]_\epsilon = \begin{pmatrix} 0 & 0 & 0 & 8 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad [\mathrm{T}^3]_\epsilon = O$$

There is only one eigenvalue $\lambda = 0$ and therefore one generalized eigenspace $K_0 = V$. We could keep working with matrices, but it is easy to translate the nullspaces of the matrices back to subspaces of V, from which the necessary data can be read off:

$$\begin{aligned} \mathcal{N}(\mathrm{T}) &= \operatorname{Span}\{1, x+2y, x^2+4y^2+4xy\} & \operatorname{null}\mathrm{T} &= 3,\ \operatorname{rank}\mathrm{T} = 3,\ r_1 = 3 \\ \mathcal{N}(\mathrm{T}^2) &= \operatorname{Span}\{1, x, y, x^2+2xy, 2y^2+xy\} & \operatorname{null}\mathrm{T}^2 &= 5,\ \operatorname{rank}\mathrm{T}^2 = 1,\ r_2 = 3-1 = 2 \end{aligned}$$

We now have five dots; since $\dim K_0 = 6$, the last row has one, and the dot diagram is $\begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array}$

Since the first two rows span $\mathcal{N}(\mathrm{T}^2)$, we may choose any $f_1 \notin \mathcal{N}(\mathrm{T}^2)$ for the final dot: $f_1 = xy$ is suitable, from which the first column of the dot diagram becomes

$$\begin{array}{ccc} \mathrm{T}^2(xy) & \bullet & \bullet \\ \mathrm{T}(xy) & \bullet & \\ xy & & \end{array} \qquad \begin{array}{ccc} -4 & \bullet & \bullet \\ 2y-x & \bullet & \\ xy & & \end{array}$$

Now choose the second dot on the second row to be anything in $\mathcal{N}(\mathrm{T}^2)$ such that the first two rows span $\mathcal{N}(\mathrm{T}^2)$: this time $f_2 = x^2 - 4y^2$ is suitable, and the diagram becomes:

$$\begin{array}{ccc} \mathrm{T}^2(xy) & \mathrm{T}(x^2-4y^2) & \bullet \\ \mathrm{T}(xy) & x^2-4y^2 & \\ xy & & \end{array} \qquad \begin{array}{ccc} -4 & 4x+8y & \bullet \\ 2y-x & x^2-4y^2 & \\ xy & & \end{array}$$

The final dot is now chosen so that the first row spans $\mathcal{N}(\mathrm{T})$: this time $f_3 = x^2 + 4y^2 + 4xy$ works. The result is a Jordan canonical basis and form for T

$$\beta = \big\{-4,\ 2y-x,\ xy,\ 4x+8y,\ x^2-4y^2,\ x^2+4y^2+4xy\big\}, \qquad J = [\mathrm{T}]_\beta = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & & \\ & & & 0 & 1 & \\ & & & 0 & 0 & \\ & & & & & 0 \end{pmatrix}$$

As previously, many other choices of cycle-generators f_1, f_2, f_3 are available; while these result in different Jordan canonical bases, Corollary 3.18 assures us that we'll always obtain the same canonical form J.

Exercises 3.2 1. Let T be a linear operator whose characteristic polynomial splits. Suppose the eigenvalues and the dot diagrams for the generalized eigenspaces K_{λ_i} are as follows:

$$\begin{array}{ccc} \lambda_1 = 2 & \lambda_2 = 4 & \lambda_3 = -3 \\ \begin{array}{ccc} \bullet & \bullet & \bullet \\ \bullet & \bullet & \\ \bullet & & \end{array} & \begin{array}{cc} \bullet & \bullet \\ \bullet & \\ \bullet & \end{array} & \begin{array}{cc} \bullet & \bullet \\ & \\ & \end{array} \end{array}$$

Find the Jordan form J of T.

2. Suppose T has Jordan canonical form

$$J = \begin{pmatrix} 2 & 1 & 0 & & & & \\ 0 & 2 & 1 & & & & \\ 0 & 0 & 2 & & & & \\ & & & 2 & 1 & & \\ & & & 0 & 2 & & \\ & & & & & 3 & \\ & & & & & & 3 \end{pmatrix}$$

   (a) Find the characteristic polynomial of T.
   (b) Find the dot diagram for each eigenvalue.
   (c) For each eigenvalue find the smallest k_j such that $K_{\lambda_j} = \mathcal{N}(\mathrm{T} - \lambda_j \mathrm{I})^{k_j}$.

3. For each matrix A find a Jordan canonical form and an invertible Q such that $A = QJQ^{-1}$.

$$\text{(a)}\ A = \begin{pmatrix} -3 & 3 & -2 \\ -7 & 6 & -3 \\ 1 & -1 & 2 \end{pmatrix} \qquad \text{(b)}\ A = \begin{pmatrix} 0 & 1 & -1 \\ -4 & 4 & -2 \\ -2 & 1 & 1 \end{pmatrix} \qquad \text{(c)}\ A = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}$$

4. For each linear operator T, find a Jordan canonical form J and basis β:
   (a) $\mathrm{T}(f) = f'$ on $\mathrm{Span}_{\mathbb{R}}\{e^t, te^t, t^2e^t, e^{2t}\}$
   (b) $\mathrm{T}\big(f(x)\big) = xf''(x)$ on $P_3(\mathbb{R})$
   (c) $\mathrm{T}(f) = af_x + bf_y$ on $\mathrm{Span}_{\mathbb{R}}\{1, x, y, x^2, y^2, xy\}$. How does your answer depend on a, b?

5. (Generalized Eigenvector Method for ODEs) Let $A \in M_n(\mathbb{R})$ have an eigenvalue λ and suppose $\beta_{\mathbf{v}_0} = \{\mathbf{v}_{k-1}, \ldots, \mathbf{v}_1, \mathbf{v}_0\}$ is a cycle of generalized eigenvectors for this eigenvalue. Show that

$$\mathbf{x}(t) := e^{\lambda t} \sum_{j=0}^{k-1} b_j(t) \mathbf{v}_j \quad \text{satisfies} \quad \mathbf{x}'(t) = A\mathbf{x} \iff \begin{cases} b_0'(t) = 0, \text{ and} \\ b_j'(t) = b_{j-1}(t) \text{ when } j \geq 1 \end{cases}$$

Use this method to solve the system of differential equations

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \mathbf{x}$$

3.3 The Rational Canonical Form (non-examinable)

We finish the course with a very quick discussion of what can be done when the characteristic polynomial of a linear map does not split. In such a situation, we may assume that

$$p(t) = (-1)^n \big(\phi_1(t)\big)^{m_1} \cdots \big(\phi_k(t)\big)^{m_k} \tag{$*$}$$

where each $\phi_j(t)$ is an *irreducible monic polynomial* over the field.

Example 3.21. The following matrix has characteristic equation $p(t) = (t^2+1)^2(3-t)$

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix} \in M_5(\mathbb{R})$$

This doesn't split over $\mathbb{R}$ since $t^2+1=0$ has no real roots. It is, however, diagonalizable over $\mathbb{C}$.

A couple of basic facts from algebra:

- Every polynomial splits over $\mathbb{C}$: every $A \in M_n(\mathbb{C})$ therefore has a Jordan form.
- Every polynomial over $\mathbb{R}$ factorizes into linear or irreducible quadratic factors.

The question is how to deal with non-linear irreducible factors in the characteristic polynomial.

Definition 3.22. The monic polynomial $t^k + a_{k-1}t^{k-1} + \cdots + a_0$ has *companion matrix*

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & & 0 & -a_1 \\ 0 & 1 & 0 & & 0 & -a_2 \\ \vdots & & & \ddots & & \vdots \\ 0 & 0 & 0 & & 0 & -a_{k-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{k-1} \end{pmatrix} \qquad \text{(when } k=1\text{, this is the } 1\times 1 \text{ matrix } (-a_0)\text{)}$$

If $\mathrm{T} \in \mathcal{L}(V)$ has characteristic polynomial $(*)$, then a *rational canonical basis* is a basis for which

$$[\mathrm{T}]_\beta = \begin{pmatrix} C_1 & O & \cdots & O \\ O & C_2 & & O \\ \vdots & & \ddots & \vdots \\ O & O & \cdots & C_r \end{pmatrix}$$

where each C_j is a companion matrix of some $(\phi_j(t))^{s_j}$ where $s_j \le m_j$. We call $[\mathrm{T}]_\beta$ a *rational canonical form* of T.

We state the main result without proof:

Theorem 3.23. *A rational canonical basis exists for* any *linear operator* T *on a finite-dimensional vector space V. The canonical form is unique up to ordering of companion matrices.*

Example (3.21 cont). The matrix A is *already in rational canonical form*: the standard basis is rational canonical with three companion blocks,

$$C_1 = C_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C_3 = (3)$$

Example 3.24. Let $A = \begin{pmatrix} 4 & -3 \\ 2 & 2 \end{pmatrix} \in M_2(\mathbb{R})$. Its characteristic polynomial

$$p(t) = t^2 - 6t + 14 = (t-3)^2 + 5$$

doesn't split over $\mathbb{R}$ and so it has no eigenvalues. Instead simply pick a vector, $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (say), define $\mathbf{y} = A\mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, let $\beta = \{\mathbf{x}, \mathbf{y}\}$ and observe that

$$[\mathrm{L}_A]_\beta = \begin{pmatrix} 0 & -14 \\ 1 & 6 \end{pmatrix}$$

is a rational canonical form. Indeed this works for *any* $\mathbf{x} \neq \mathbf{0}$: if $\beta := \{\mathbf{x}, A\mathbf{x}\}$, then Cayley–Hamilton forces

$$A^2\mathbf{x} = (6A - 14I)\mathbf{x} = -14\mathbf{x} + 6A\mathbf{x} \implies [\mathrm{L}_A]_\beta = \begin{pmatrix} 0 & -14 \\ 1 & 6 \end{pmatrix}$$

whence β is a rational canonical basis and the form $[\mathrm{L}_A]_\beta$ is independent of $\mathbf{x}$!

A systematic approach to finding rational canonical forms is similar to that for Jordan forms: for each irreducible divisor of $p(t)$, the subspace $K_\phi = \mathcal{N}\big(\phi(\mathrm{T})\big)^m$ plays a role analogous to a generalized eigenspace; indeed $K_\lambda = K_\phi$ for the *linear* irreducible factor $\phi(t) = \lambda - t$!

We finish with two examples; hopefully the approach is intuitive, even without theoretical justification.

Examples 3.25. If the characteristic polynomial of $\mathrm{T} \in \mathcal{L}(\mathbb{R}^4)$ is

$$p(t) = (\phi(t))^2 = (t^2 - 2t + 3)^2 = t^4 - 4t^3 + 10t^2 - 12t + 9$$

then there are two possible rational canonical forms; here is an example of each.

1. If $A = \begin{pmatrix} 0 & -15 & 0 & -9 \\ 2 & 2 & -3 & 0 \\ 0 & -9 & 0 & -6 \\ -3 & 0 & 5 & 2 \end{pmatrix}$, then $\phi(A) = O$ is the zero matrix, whence $\mathcal{N}(\phi(A)) = \mathbb{R}^4$. Since $\phi(t)$ isn't the full characteristic polynomial, we expect there to be *two* independent cycles of length two in the canonical basis. Start with something simple as a guess:

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \implies \mathbf{x}_2 = A\mathbf{x}_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \\ -3 \end{pmatrix} \implies A\mathbf{x}_2 = \begin{pmatrix} -3 \\ 4 \\ 0 \\ -6 \end{pmatrix} = -3\mathbf{x}_1 + 2\mathbf{x}_2$$

   Now make another choice that isn't in the span of $\{\mathbf{x}_1, \mathbf{x}_2\}$:

$$\mathbf{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \implies \mathbf{x}_4 = A\mathbf{x}_3 = \begin{pmatrix} 0 \\ -3 \\ 0 \\ 5 \end{pmatrix} \implies A\mathbf{x}_4 = \begin{pmatrix} 0 \\ -6 \\ -3 \\ 10 \end{pmatrix} = -3\mathbf{x}_3 + 2\mathbf{x}_4$$

   We therefore have a rational canonical basis $\beta = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ and

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 5 \end{pmatrix} \begin{pmatrix} 0 & -3 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & 0 & 5 \end{pmatrix}^{-1}$$

   Over $\mathbb{C}$, this example is diagonalizable. Indeed each of the 2×2 companion matrices is diagonalizable over $\mathbb{C}$.

2. Let $B = \begin{pmatrix} 0 & 0 & 2 & 1 \\ 1 & 1 & -1 & -1 \\ 0 & 1 & -2 & -16 \\ 0 & 0 & 1 & 5 \end{pmatrix}$. This time

$$\phi(B) = B^2 - 2B + 3I = \begin{pmatrix} 3 & 2 & -7 & -29 \\ -1 & 1 & 4 & 13 \\ 1 & -3 & -6 & -17 \\ 0 & 1 & 1 & 2 \end{pmatrix} \implies \mathcal{N}(\phi(B)) = \operatorname{Span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 11 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Anything not in this span will suffice as a generator for a single cycle of length four: e.g.,

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_2 = B\mathbf{x}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = B\mathbf{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{x}_4 = B\mathbf{x}_3 = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$B\mathbf{x}_4 = \begin{pmatrix} -1 \\ 2 \\ -14 \\ 4 \end{pmatrix} = -9 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + 12 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - 10 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + 4 \begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

We therefore have a rational canonical basis $\beta = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$ and

$$B = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & -9 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1}$$

In contrast to the first example, B isn't diagonalizable over $\mathbb{C}$. It has Jordan form $J = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \overline{\lambda} & 1 \\ 0 & 0 & 0 & \overline{\lambda} \end{pmatrix}$ where $\lambda = 1 + i\sqrt{2}$.