Math 13: Homework 2

Submit these and questions 2 and 10 from section 2.2, and questions 6 and 8 from section 2.3 in the notes at the discussion on Thursday 24th January

1. Consider the following proposition, where $x$ is assumed to be a real number.

$$x^3 - 3x^2 - 2x + 6 = 0 \implies x = 3$$  \hspace{1cm} (*)

(a) Is the proposition (*) true or false? Justify your answer. Is its converse true?

(b) Repeat part (a) for the proposition

$$x^3 - 3x^2 - 2x + 6 = 0 \implies x \neq 3$$

(c) Does anything change about the truth status of (*) if we assume that it is a statement about rational numbers $x$? Explain.

2. Consider the following proposition.

$$\forall m, n \in \mathbb{R}, \ m > n \implies m^2 > n^2.$$ \hspace{1cm} (*)

(a) What is the negation of (*)?

(b) Prove that (*) is false.

(c) Suppose you rewrite the proposition as follows

$$\forall m, n \in A, \ m > n \implies m^2 > n^2.$$  

What is the largest set of real numbers $A$ for which the proposition is true? Justify your answer.

3. First try question 2.3.7 from the notes. Here is an extension.

**Conjecture.** Between any two real numbers there exists a rational number.

Let $\lceil x \rceil$, the ceiling of $x$, denote the smallest integer greater than or equal to $x$. E.g. $\lceil 3.2 \rceil = 4$, $\lceil 7 \rceil = 7$ and $\lceil -8.4 \rceil = -8$.

(a) Suppose that $x$ and $y$ are real numbers with $x < y$. Use the ceiling function to show that there exists a positive integer $n$ for which $n(y - x) > 1$.

(b) Prove or disprove: $\forall x, y \in \mathbb{R}$ with $x < y$, $\exists m, n \in \mathbb{Z}$ for which $nx < m < ny$.

(c) Is the conjecture true or false? Prove your assertion.

(d) Challenge: is the true that between any two real numbers there exists an irrational number? If so, prove it.