Math 130A: Homework 3

Submit your answers to questions 1, 2, 3, 4, 5 & 6 on Gradescope by Wednesday 20th April

1. Two fair dice are rolled. For each $n = 2, 3, 4 \ldots, 12$, find;
   (a) The conditional probability that the *first* die is a 2 given that the sum of the dice is n.
   (b) The conditional probability that *at least one* die is a 2 given that the sum of the dice is n.
2. Suppose that $\mathbb{P}(A) > 0$. Prove that $\mathbb{P}(A \cap B | A) \geq \mathbb{P}(A \cap B | A \cup B)$
3. Prove that if E, F are mutually exclusive, then
$$\mathbb{P}(E|E \cup F) = \frac{\mathbb{P}(E)}{\mathbb{P}(E) + \mathbb{P}(F)}$$
4. Let $A \subseteq B$. Supposing that $\mathbb{P}(A) \neq 0$ and $\mathbb{P}(B) \neq 1$, express the following as simply as possible (as a number or in terms only of $\mathbb{P}(A)$ and $\mathbb{P}(B)$)
$$\mathbb{P}(A|B), \qquad \mathbb{P}(A|B^{\mathsf{C}}), \qquad \mathbb{P}(B|A), \qquad \mathbb{P}(B|A^{\mathsf{C}})$$
5. An urn contains 10 balls, of which 6 are white and 4 red.
   (a) A sample of size 4 is drawn with replacement. What is the conditional probability that the first and third balls drawn will be white given that the sample contains exactly three white balls?
   (b) Repeat part (a), but where the sample is drawn *without* replacement.
6. Suppose that an ordinary deck of 52 cards is randomly divided into four hands of 13 cards each. Let E_i be the event that the i^{th} hand contains exactly one ace. Use the multiplication rule to determine the probability $\mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4)$ that each hand has an ace.
7. An urn contains n red and m blue balls. They are withdrawn one at a time until a total of $r \leq n$ red balls have been withdrawn.
   (a) Suppose $m = 3$, $n = 4$, $r = 3$ and $k = 5$. List all the arrangements of all *seven* the balls such that the $r = 3^{\text{rd}}$ red ball is the $k = 5^{\text{th}}$ to be drawn. Hence find the probability that a total of five balls are withdrawn.
   (*Hint: think about how to order the first four balls and the last two*)
   (b) In general, find the probability that exactly $k \geq r$ balls need to be withdrawn.
8. Consider the hat-matching problem from chapter 3.
   (a) It is correct to claim that $\mathbb{P}(E^{\mathsf{C}}_{k+1} | E_1 \cap \cdots \cap E_k) = \frac{n-k-1}{n-k}$, since there are $n - k$ hats remaining, exactly $n - k - 1$ of which do not belong to person $k + 1$. (As in the notes,) When $n = 4$, and for each of $k = 1, 2, 3$, verify explicitly which arrangements of hats a, b, c, d correspond to this situation.
   (b) Explain why we *cannot* continue this argument; that is, why, in general, do we have the following?
$$\mathbb{P}(E^{\mathsf{C}}_{k+2} | E_1 \cap \cdots \cap E_k \cap E^{\mathsf{C}}_{k+1}) \neq \frac{n-k-2}{n-k-1}$$