# Math 130A: Homework 6

Submit your answers to questions 1, 3, 5, 6, 8 & 9 on Gradescope by Wednesday 25th May

1. Suppose $X$ is a random variable with density

$$f(x) = \begin{cases} c(1-x^2) & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find $c$ and the cumulative distribution function $F(x) = \mathbb{P}\{X \le x\}$.

2. The density function of $X$ is given by

$$f(x) = \begin{cases} a + bx^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) If $\mathbb{E}[X] = \frac{5}{8}$, find $a$ and $b$.
   (b) Find the density and distribution functions of the random variable $Y = \sqrt{X}$.
   (c) Compute $\mathbb{E}[\sqrt{X}]$ in two ways:
      i. Using $\mathbb{E}[g(X)] = \int_{\mathbb{R}} g(x)f(x)\,\mathrm{d}x$.
      ii. Using the density function for $Y$; $\mathbb{E}[Y] = \int_{\mathbb{R}} y f_Y(y)\,\mathrm{d}y$

3. The lifetime in hours of an electronic tube is a random variable with density

$$f(x) = xe^{-x}, \quad x \ge 0$$

   Find the expected lifetime of such a tube and its standard deviation.

   (*Hint: You might find it helpful to compute* $I_n = \int_0^\infty x^n e^{-x}\,\mathrm{d}x$ *recursively by parts*)

4. A bus travels between cities $A$ and $B$ which are 100 miles apart. If the bus breaks down, the distance from the breakdown to $A$ is a uniform variable $X \sim U(0, 100)$. There is a bus service station in both cities and at the midpoint of the route. It is suggested that it would be more efficient to have the stations located at 25, 50 and 75 miles from city $A$. Do you agree? Why?

   (*If you want a challenge, find the optimal locations for the three service stations?*)

5. A point is chosen at random on a line segment of length $L$. Interpret this statement and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{3}$.

6. If $X \sim U(a, b)$ is a uniform variable, compute $\mathbb{E}[X^n]$.

7. For a non-negative random variable $X$, prove that

$$\mathbb{E}[X^n] = \int_0^\infty nx^{n-1}\mathbb{P}\{X > x\}\,\mathrm{d}x$$

   (*Hint: use the fact that* $\mathbb{E}[Y] = \int_0^\infty \mathbb{P}\{Y > y\}\,\mathrm{d}y$ *if* $Y$ *is non-negative*)

8. If $X$ has standard deviation $\sigma = \sqrt{\operatorname{Var} X}$, find the standard deviation of the random variable $Y = aX + b$.

9. Let $X$ be a random variable that takes values between 0 and $c$. Show that $\mathbb{E}[X^2] \le c\mathbb{E}[X]$ and thus conclude that $\operatorname{Var} X \le \frac{c^2}{4}$