Math 130B Homework 4

Hand in the following questions at discussion on Thursday 18th February

1. A company sells radios. In a batch of 1000 radios, 20 are known to be broken but are well-mixed with the working radios.

   (a) The company sells 500 of the radios. Let the random variables Y_i be defined by

   $$Y_i = \begin{cases} 1 & \text{if broken radio } i \text{ is sold in the batch of 500} \\ 0 & \text{otherwise} \end{cases}$$

   Let Y be the number of broken radios in the batch. Use the Y_i to compute $\mathbb{E}(Y)$.

   (b) Check your answer using the fact that Y has a hypergeometric distribution. What is the standard deviation of the number of broken radios sold?

   (c) Suppose instead that the company starts selling radios and keeps going until it sells its 10th broken radio. How many radios does the company expect to sell? What is the standard deviation of the number of radios sold?

2. A total of n balls, numbered 1 through n, are put into n urns, also numbered 1 through n in such a way that ball i is equally likely to go into any of the urns $1, 2, \ldots, i$. Find

   (a) The expected number of urns that are empty.

   (b) The probability that none of the urns are empty.

3. A function $\langle \, , \, \rangle$ on a real vector space is an *inner product* if it satisfies the following conditions for all vectors X, Y, Z and scalars a, b:

   - $\langle X, X \rangle \geq 0$ with equality if and only if $X = 0$
   - $\langle X, Y \rangle = \langle Y, X \rangle$
   - $\langle aX + bY, Z \rangle = a \langle X, Z \rangle + b \langle Y, Z \rangle$

   (a) Let X and Y be non-zero vectors and $\lambda \in \mathbb{R}$. It is immediate from the second and third properties of an inner product that

   $$\langle X - \lambda Y, X - \lambda Y \rangle = \langle X, X \rangle - 2\lambda \langle X, Y \rangle + \lambda^2 \langle Y, Y \rangle \tag{$*$}$$

   Use this to prove the Cauchy–Schwarz inequality:

   $$|\langle X, Y \rangle|^2 \leq \langle X, X \rangle \langle Y, Y \rangle \tag{$\dagger$}$$

   (*think about* $(*)$ *as a quadratic in* $\lambda \ldots$).

   (b) Also prove the following: ($\dagger$) is an *equality* if and only if X and Y are *parallel*.

   (c) For any random variables X, Y, define the inner product $\langle X, Y \rangle = \mathbb{E}(XY)$. Use the Cauchy–Schwarz inequality to prove that

   $$|\mathrm{Cov}(X, Y)|^2 \leq \mathrm{Var}\, X \, \mathrm{Var}\, Y$$

(d) The *correlation* of random variables X, Y is the number

$$\rho(X, Y) = \frac{\operatorname{Cov}(X, Y)}{\sqrt{\operatorname{Var} X \operatorname{Var} Y}} \in [-1, 1]$$

Prove that $\rho(X, Y) = \pm 1$ if and only if X and Y have a *linear relationship*: that is $X = aY + b$ or $Y = aX + b$ for some constants a, b.