Math 130B Homework 5

Hand in the following questions at discussion on Thursday 25th February

1. If $\mathbb{E}(X) = 1$ and $\operatorname{Var} X = 5$, find
   (a) $\mathbb{E}[(2+X)^2]$
   (b) $\operatorname{Var}(4+3X)$
2. Let X be the number of 1's and Y the number of 2's that occur in n rolls of a fair die. Compute $\operatorname{Cov}(X,Y)$.
3. A fair die is rolled twice. Let X equal the sum of the outcomes and let Y equal the first outcome minus the second. Compute $\operatorname{Cov}(X,Y)$.
4. If X_1, X_2, X_3, X_4 are pairwise uncorrelated random variables, each having mean 0 and variance 1, compute the correlations of
   (a) $X_1 + X_2$ and $X_2 + X_3$
   (b) $X_1 + X_2$ and $X_3 + X_4$
5. A pond contains 100 fish, of which 30 are carp. If 20 fish are caught, what are the mean and variance of the number of carp among the 20? What assumptions are you making?
6. A deck of n cards labelled 1 through n is thoroughly shuffled so that all possible $n!$ orderings are equally likely to be dealt. Suppose that you make n guesses sequentially, where the ith guess is of the number of the card dealt in position i. Let N denote the number of correct guesses.
   (a) If you are not given any information about your earlier guesses, show that for any strategy, $\mathbb{E}(N) = 1$.
   (b) Suppose that after each guess you are shown the card that was in the position in question. What do you think is the best strategy? Show that under this strategy

   $$\mathbb{E}(N) = \frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 \approx \int_1^n \frac{1}{x}\,\mathrm{d}x = \ln n$$

   (c) (Don't submit!) Suppose that you are told after each guess whether you are right or wrong. In this case it can be shown that the strategy that maximizes $\mathbb{E}(N)$ is the one that keeps on guessing the same card until you are told that you are correct and then changes to a new card. For this strategy, show that

   $$\mathbb{E}(N) = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} \approx e - 1$$