Math 130B Homework 6

Hand in the following questions at discussion on Thursday 3rd March

1. The joint density of X and Y is given by

$$f(x,y) = \frac{e^{-y}}{y}, \qquad 0 < x < y, \quad 0 < y < \infty$$

   Compute $\mathbb{E}(X^3 | Y = y)$.

2. A coin having probability p of coming up heads is continually flipped until both heads and tails have appeared. Find:
   (a) The expected number of flips.
   (b) The probability that the last flip lands on heads.

3. Consider the general miner's dilemma problem. The miner has an equal choice of three tunnels A, B, C.
   - Tunnel A takes a hours before leading the miner back to their cell.
   - Tunnel B takes b hours before leading the miner back to their cell.
   - Tunnel C takes c hours and leads the miner to the exit.

   Compute the following:
   (a) $\mathbb{E}(X)$ where X is the time taken for the miner to escape.
   (b) Var X.

   Consider conditioning on the initial choice of tunnel. Both of your answers will depend on a, b, c. However, your answer to part (b) shouldn't depend on c: why is this obvious?

4. Consider the following dice game. Players 1 and 2 each roll a single regular fair die. The bank then rolls a fair die to determine the outcome according to the following rule: Player i wins if his roll is strictly greater than the bank's. For $i = 1, 2$, let

$$I_i = \begin{cases} 1 & \text{if } i \text{ wins} \\ 0 & \text{otherwise} \end{cases}$$

   Compute the correlation of I_1 and I_2. Why should it have been obvious that the correlation would turn out to be positive.

5. Consider a sequence $U_1, U_2, \ldots$ of independent $(0, 1)$ uniform random variables. In class we considered the random variable

$$N(x) = \min \left\{ n : \sum_{i=1}^{n} U_i > x \right\}$$

   whenever $x \in [0, 1]$ and computed its expectation to be $m(x) = \mathbb{E}(N(x)) = e^x$.
   Now we think about what happens if $x \geq 1$.

(a) Show that $m(x) = 1 + \int_0^1 m(x-y)\,\mathrm{d}y$, and that the substitution $u = x - y$ followed by the Fundamental Theorem of Calculus leads us to the differential equation

$$m'(x) = m(x) - m(x-1)$$

(b) Suppose that $x \in [1,2]$, then $x - 1 \in [0,1]$ and so $m(x-1) = e^{x-1}$, as we saw in class. Solve the resulting linear equation, and therefore calculate the expected number of uniform $(0,1)$ variables we need to sum in order to exceed $x = 2$.