Extra practice questions for the final

1. Type i lightbulbs function for a random amount of time, having mean μ_i and standard deviation σ_i ($i = 1, 2$). A light bulb is randomly chosen from a bin of bulbs: the bulb is type 1 with probability p and type 2 with probability $1 - p$. Let X denote the lifetime of this bulb. By conditioning on the bulb chosen, find:

   (a) $\mathbb{E}(X)$

   (b) $\operatorname{Var} X$

2. An urn contains 15 balls, of which 5 are red and 4 blue. From this urn, 6 balls are randomly withdrawn. Let X denote the number of red and Y the number of blue balls that are withdrawn. Find $\operatorname{Cov}(X, Y)$ by defining appropriate indicator variables X_i, Y_j such that

   $$X = \sum_{i=1}^{5} X_i, \qquad Y = \sum_{j=1}^{4} Y_j$$

   Hint: X (and Y) each have a hypergeometric distribution: for X we want the number of red balls in a sample of $n = 6$ from $N = 15$, where $m = 5$ balls in total are red. Thus $\mathbb{E}(X) = \frac{mn}{N}\ldots$

3. If X has variance σ^2, then σ, the positive square root of the variance, is called the *standard deviation*. If X has mean μ and standard deviation σ, prove that

   $$\mathbb{P}(|X - \mu| \geq k\sigma) \leq \frac{1}{k^2}$$

4. Let $X_1, \ldots, X_{20}$ be independent Poisson random variables with mean 1.

   (a) Use the Markov inequality to obtain a bound on

   $$\mathbb{P}\left\{\sum_{i=1}^{20} X_i > 15\right\}$$

   (b) Use the central limit theorem to approximate

   $$\mathbb{P}\left\{\sum_{i=1}^{20} X_i > 15\right\}$$

5. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for 2000 hours is at least 0.95?

6. If Z is a standard normal random variable, compute $\operatorname{Cov}(Z, Z^2)$.

7. Let X have moment generating function $M(t)$ and define $\Psi(t) = \ln M(t)$. Prove that

   $$\Psi''(0) = \operatorname{Var} X$$

8. The best quadratic predictor of Y with respect to X is $a + bX + cX^2$ where a, b, c are chosen to minimize

$$\mathbb{E}[(Y - (a + bX + cX^2))^2]$$

Determine a, b and c.

9. If X and Y are independent, and identically distributed with mean μ and variance σ^2, find

$$\mathbb{E}[(X - Y)^2]$$

10. How many times would you expect to roll a fair die until all 6 faces appeared at least once?

11. Cards from an ordinary (well-shuffled) deck are turned face up one at a time. Compute the expected number of cards that need to be turned up in order to obtain

    (a) 2 aces;

    (b) 5 spades;

    (c) all 13 hearts.

12. The random variables X and Y have a joint density function given by

$$f(x,y) = \begin{cases} 2x^{-1}e^{-2x} & \text{if } 0 < x < \infty \text{ and } 0 < y < x \\ 0 & \text{otherwise} \end{cases}$$

Compute $\text{Cov}(X, Y)$.

13. For a group of 100 people, compute:

    (a) The expected number of days of the year that are birthdays of exactly 3 people.

    (b) The expected number of distinct birthdays.