2 Sequences

Sequences are the fundamental tool in our approach to analysis.

2.7 Limits of Sequences

Definition 2.1. A *sequence* of real numbers is a list indexed by the natural numbers

$$(s_n) = (s_1, s_2, s_3, \ldots)$$

We call s_1 the *initial term/element.*

More formally, we could view a sequence as a function $s_n : \mathbb{N} \to \mathbb{R}$. Other letters may be used (a_n, b_n, etc.), though s_n is typical in the abstract. It is also common to have sequences which start with a different initial term (e.g., $n = 0$). If you need to be explicit, write, e.g., $(s_n)_{n=0}^{\infty}$.

Examples 2.2. 1. Explicit sequences are often defined by providing a formula for the n^{th} term. For instance, $s_n = \left(1 + \frac{1}{n}\right)^n$ defines a sequence whose first three terms are

$$s_1 = 2, \quad s_2 = \tfrac{9}{4}, \quad s_3 = \tfrac{64}{27}, \quad \ldots$$

Since each term is a rational number, (s_n) could be described as a *rational sequence.*

2. Sequences can be defined inductively. For instance, if $t_1 = 1$ and $t_{n+1} = 3t_n - 1$, then

$$(t_n) = (1, 2, 5, 14, 41, \ldots)$$

3. $u_n = \frac{1}{n^2-4}$ defines a sequence with initial term $u_3 = \frac{1}{5}$:

$$(u_n)_{n=3}^{\infty} = \left(\tfrac{1}{5}, \tfrac{1}{12}, \tfrac{1}{21}, \ldots\right)$$

Limits We are typically most interested in what happens to the terms of a sequence when n gets *large* (one reason it is common to be non-explicit as to the initial term). In elementary calculus you should have become used to examples such as[9]

$$\lim \frac{2n^2 + 3n - 1}{3n^2 - 2} = \frac{2}{3}$$

which encapsulates the idea that the expression $s_n = \frac{2n^2+3n-1}{3n^2-2}$ gets 'close to' $\frac{2}{3}$ when n is large. We can easily convince ourselves of this with a calculator/computer: to 4 decimal places,

$$(s_n) = (4,\ 1.3,\ 1.04,\ 0.9348,\ 0.8767,\ 0.8396,\ 0.8138,\ 0.7947,\ \ldots), \qquad s_{1000} = 0.6677$$

Our primary business is to make this idea logically watertight, the major issue being what is meant by 'close to.' In Section 2.8 we will do so by developing the formal definition of limit. Before seeing this, we quickly refresh a few simple examples. All these examples can be made formal later, but for the present just rely on your intuition and experience: it is essential to have a good idea of the correct answer *before* you try to prove it!

[9] If there are multiple letters around, writing $\lim\limits_{n\to\infty}$ with a subscript can aid the reader.

Examples 2.3. 1. $\lim \frac{1}{n} = 0$. Our instinct is that $s_n = \frac{1}{n}$ becomes arbitrarily small as n becomes large.

2. $\lim \frac{7n+9}{2n-4} = \frac{7}{2}$. To convince yourself of this, you might write $\frac{7n+9}{2n-4} = \frac{7+\frac{9}{n}}{2-\frac{4}{n}}$ and observe that the $\frac{1}{n}$ terms become tiny as n increases.

3. The sequence with n^{th} term $s_n = (-1)^n$ does not converge to anything: it *diverges*.

$$(s_n)_{n=0}^{\infty} = (1, -1, 1, -1, 1, -1, \ldots)$$

4. If $c_n = \frac{1}{n}\cos\left(\frac{\pi n}{6}\right)$, then $\lim c_n = 0$. To see this, observe that the cosine term lies between ± 1, while $\frac{1}{n}$ has limit 0.

5. The sequence defined inductively by $s_0 = 2,\ s_{n+1} := \frac{1}{2}s_n + 3$ begins

$$(s_n) = \left(2, 4, 5, \tfrac{11}{2}, \tfrac{23}{4}, \tfrac{47}{8}, \ldots\right)$$

This appears to have limit $\lim s_n = 6$. It is not hard to spot the pattern $s_n = 6 - \frac{4}{2^n}$, which may easily be verified by induction: for the induction step, observe that

$$\frac{1}{2}s_n + 3 = \frac{1}{2}\left(6 - \frac{4}{2^n}\right) + 3 = 6 - \frac{4}{2^{n+1}}$$

Exercises 2.7. *Key concepts: Sequences, Use your intuition!*

1. Decide whether each sequence converges; if it does, state the limit. No proofs are required; if you're unsure what's going on, try writing out the first few terms.

   (a) $a_n = \dfrac{1}{3n+1}$ (b) $b_n = \dfrac{3n+1}{4n-1}$ (c) $c_n = \dfrac{n}{3^n}$ (d) $d_n = \sin\left(\dfrac{n\pi}{4}\right)$

2. Repeat the previous question for sequences whose n^{th} term is as follows:

   (a) $\dfrac{n^2+3}{n^2-3}$ (b) $1 + \dfrac{2}{n}$ (c) $2^{1/n}$ (d) $(-1)^n n$ (e) $\dfrac{7n^3+8n}{2n^3-31}$

   (f) $\sin\left(\dfrac{n\pi}{2}\right)$ (g) $\sin\left(\dfrac{2n\pi}{3}\right)$ (h) $\dfrac{2^{n+1}+5}{2^n-7}$ (i) $\left(1+\dfrac{1}{n}\right)^2$ (j) $\dfrac{6n+4}{9n^2+7}$

3. Give an example of:

   (a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.
   (b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

4. Prove by induction that the sequence defined in Example 2.2.2 has n^{th} term $t_n = \frac{1}{2}\left(3^{n-1}+1\right)$.

5. In future courses, you'll meet sequences of *functions*. For instance, we could define a sequence (f_n) of functions $f_n : \mathbb{R} \to \mathbb{R}$ inductively via

$$f_0(x) \equiv 1, \qquad f_{n+1}(x) := 1 + \int_0^x f_n(t)\,\mathrm{d}t$$

   Compute the functions f_1, f_2 and f_3. The sequence (f_n) should seem familiar if you think back to elementary calculus; why?

2.8 The Formal Definition of Limit

While intuition regarding limits is essential, mathematics is about *proving* things rigorously; for this one needs a formal *definition*. The essential difficulty is that 'close to' is poorly defined without context; we get round this by considering all possible measures of closeness simultaneously.

Definition 2.4. To say that a sequence (s_n) *converges to a limit* $s \in \mathbb{R}$ means,[10]

$$\forall \epsilon > 0, \ \exists N \text{ such that } n > N \implies |s_n - s| < \epsilon$$

We write $\lim s_n = s$ or simply $s_n \to s$; both are read "s_n *approaches* (or *tends to*) s."
A sequence *converges* if it has a limit, and *diverges* otherwise.

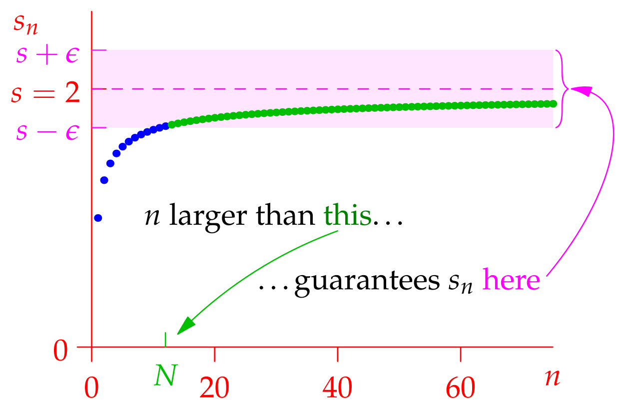
Try reading it this way: given any measure of closeness, we can make n large enough so that (by our given measure) s_n is close to s. This isn't as hard as it looks, though a *lot* of examples will likely be necessary for it to sink in...

Example 2.5. We prove that the sequence with

$$s_n = 2 - \frac{1}{\sqrt{n}}$$

converges to $s = 2$.

By plotting the sequence, we see how ϵ controls the distance (closeness) from s_n to the limit $s = 2$; no matter the size of ϵ, we must show that (s_n) has some tail whose terms are closer to s.



Proving such 'for all, there exists' statements requires a specific argument structure:

- Suppose $\epsilon > 0$ has been given and describe N as a function of ϵ (ϵ smaller means N larger).
- Verify algebraically that $n > N \implies |s_n - s| < \epsilon$.

Scratch work. To find a suitable N, start with what you want to be true and let it inspire you.

Whenever $n > N$, we want $\epsilon > |s_n - s| = \left|\left(2 - \frac{1}{\sqrt{n}}\right) - 2\right| = \left|\frac{1}{\sqrt{n}}\right|$.

Choosing $N = \frac{1}{\epsilon^2}$ should be enough to complete the proof!

Warning! "$N = \frac{1}{\epsilon^2}$" is not the correct conclusion! To make it clear that we've satisfied the definition, we rearrange our scratch work: these last three lines are all you *need* to write!

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \frac{1}{\epsilon^2}$. Then

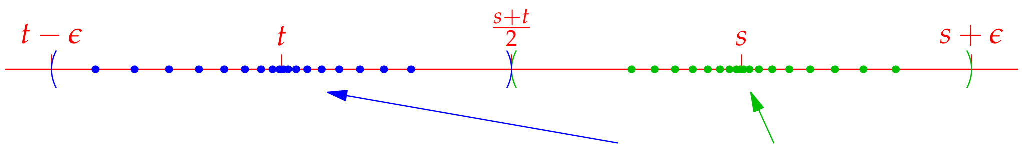
$$n > N \implies |s_n - s| = \left|2 - \frac{1}{\sqrt{n}} - 2\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \epsilon$$

Thus $\lim s_n = 2$, as required.

[10] N may be a real number or a natural number (equivalent by the Archimedean property, Theorem 1.22). It tends to be easier to use $N \in \mathbb{R}$ for convergence and $N \in \mathbb{N}$ when directly proving *divergence* (see Definition 2.9 and Examples 2.10).

Lemma 2.6 (Uniqueness of Limit). *If (s_n) converges, then its limit is unique.*

The proof structure should be familiar from other uniqueness arguments: assume there are two limits $s \neq t$ and obtain a contradiction. The picture explains the strategy: by choosing $\epsilon = \frac{|s-t|}{2}$ in the definition we obtain a *tail* of the sequence which must be simultaneously close to *both limits*.



For all $n > N$, s_n must lie both here and here!

Proof. Suppose $s \neq t$ are two limits. Take $\epsilon = \frac{|s-t|}{2}$ and apply Definition 2.4 twice: $\exists N_1, N_2$ such that

$$n > N_1 \implies |s_n - s| < \frac{|s-t|}{2} \qquad \text{and} \qquad n > N_2 \implies |s_n - t| < \frac{|s-t|}{2}$$

Define $N := \max(N_1, N_2)$. Taking any $n > N$ quickly yields a contradiction:

$$\begin{aligned} n > N \implies |s-t| &= |s - s_n + s_n - t| \leq |s_n - s| + |s_n - t| && (\triangle\text{-inequality}) \\ &< \frac{|s-t|}{2} + \frac{|s-t|}{2} = |s-t| \end{aligned}$$

■

Theorem 2.7. *If $k > 0$ is constant, then* $\lim \frac{1}{n^k} = 0$.

Proof. In this, and the examples that follow, only the formal arguments are required; scratch work is included to show the thought process.

Scratch work. We want $n > N \implies \frac{1}{n^k} < \epsilon$. That is, $n > \frac{1}{\epsilon^{1/k}}$. It should be enough to choose $N = \frac{1}{\epsilon^{1/k}}$.

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \frac{1}{\epsilon^{1/k}}$. Then

$$n > N \implies \left| \frac{1}{n^k} - 0 \right| = \frac{1}{n^k} < \frac{1}{N^k} = \epsilon$$

■

We conclude that $\lim \frac{1}{n^k} = 0$.

Examples 2.8. 1. We prove that $\lim\left(\sqrt{n+4} - \sqrt{n}\right) = 0$.

Scratch work. We use a (hopefully) familiar trick for manipulating surd expressions:

$$\sqrt{n+4} - \sqrt{n} = \frac{4}{\sqrt{n+4} + \sqrt{n}} < \frac{4}{2\sqrt{n}} = \frac{2}{\sqrt{n}}$$

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \frac{4}{\epsilon^2}$. Then

$$n > N \implies \left| \sqrt{n+4} - \sqrt{n} \right| = \frac{4}{\sqrt{n+4} + \sqrt{n}} < \frac{4}{2\sqrt{n}} = \frac{2}{\sqrt{n}} < \frac{2}{\sqrt{N}} = \epsilon$$

Thus $\lim\left(\sqrt{n+4} - \sqrt{n}\right) = 0$.

2. We prove that $\lim \frac{3n+1}{n-7} = 3$.

   Scratch work. Given $\epsilon > 0$, we want to choose N such that

$$n > N \implies \left|\frac{3n+1}{n-7} - 3\right| = \left|\frac{(3n+1)-3(n-7)}{n-7}\right| = \left|\frac{22}{n-7}\right| < \epsilon \qquad (*)$$

   For large n ($n > 7$) everything is positive, so it is sufficient to have $n - 7 > \frac{22}{\epsilon}\ldots$

   Formal argument 1. Suppose $\epsilon > 0$ is given and let $N = 7 + \frac{22}{\epsilon}$. Then

$$n > N \implies \left|\frac{3n+1}{n-7} - 3\right| = \frac{22}{n-7} < \frac{22}{N-7} = \epsilon$$

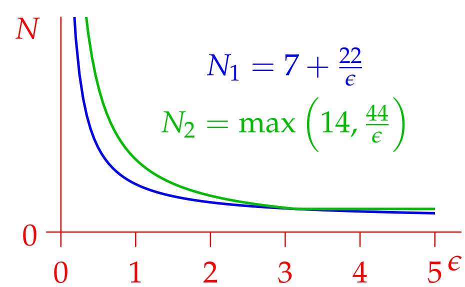
   The absolute values are dropped since $n > 7$. We conclude that $\lim \frac{3n+1}{n-7} = 3$.

   Scratch work 2. An alternative approach is available if we play with $(*)$ a little. By insisting that $n \geq 14$, we can simplify the denominator $n - 7 \geq \frac{1}{2}n \implies \frac{22}{n-7} \leq \frac{44}{n}$.

   Formal argument 2. Suppose $\epsilon > 0$ is given and let $N = \max\left(14, \frac{44}{\epsilon}\right)$. Then

$$\begin{aligned} n > N \implies \left|\frac{3n+1}{n-7} - 3\right| &= \left|\frac{22}{n-7}\right| \leq \frac{22}{\frac{1}{2}n} = \frac{44}{n} && (\text{since } n \geq 14) \\ &< \frac{44}{N} \leq \epsilon && (\text{since } N \geq \tfrac{44}{\epsilon}) \end{aligned}$$

   We again conclude that $\lim \frac{3n+1}{n-7} = 3$.



   The plot illustrates the two choices of N as functions of ϵ. The second is always larger than the first: if $N = N_1(\epsilon)$ works in a proof, then any larger choice $N_2(\epsilon)$ will work also; use this to your advantage to produce simpler arguments!

3. Given $s_n = \frac{2n^4-3n+1}{3n^4+n^2+4}$, we prove that $\lim s_n = \frac{2}{3}$.

   Scratch work. We want to conclude that $\left|\frac{2n^4-3n+1}{3n^4+n^2+4} - \frac{2}{3}\right| = \left|\frac{-2n^2-9n-5}{3(3n^4+n^2+4)}\right| < \epsilon$. Attempting to solve for n is crazy! Instead we simplify the fraction using $n \geq 1$ and the $\triangle$-inequality:

$$\left|\frac{-2n^2-9n-5}{3(3n^4+n^2+4)}\right| \overset{\triangle}{\leq} \frac{16n^2}{3(3n^4+n^2+4)} < \frac{16n^2}{9n^4} < \frac{2}{n^2} \qquad (1 \leq n \leq n^2 \text{ and } n^2+4 > 0)$$

   The final simplification is merely for additional cleanliness.

   Formal argument. Suppose $\epsilon > 0$ is given and let $N = \sqrt{\frac{2}{\epsilon}}$. Then,

$$\begin{aligned} n > N \implies \left|\frac{2n^4-3n+1}{3n^4+n^2+4} - \frac{2}{3}\right| = \left|\frac{-2n^2-9n-5}{3(3n^4+n^2+4)}\right| &\overset{\triangle}{\leq} \frac{16n^2}{3(3n^4+n^2+4)} && (n \geq 1) \\ &< \frac{16n^2}{9n^4} < \frac{2}{n^2} < \frac{2}{N^2} = \epsilon \end{aligned}$$

   Other choices of N are feasible (e.g., Exercise 3); everything depends on how you want to simplify things in your scratch work.

Divergent sequences

By negating Definition 2.4, we obtain explicit criteria for divergence.

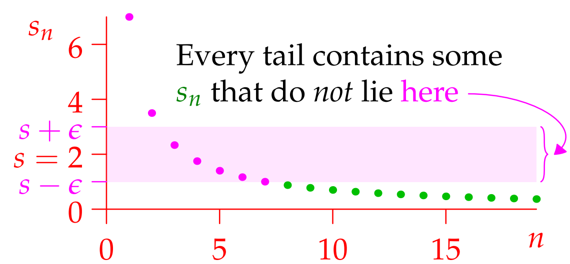
Definition 2.9. A sequence (s_n) *does not converge to* $s \in \mathbb{R}$ if,

$$\exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

A sequence is *divergent* if it does not converge to *any* limit $s \in \mathbb{R}$. Otherwise said,

$$\forall s \in \mathbb{R},\ \exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

Examples 2.10. 1. We prove that the sequence with $s_n = \frac{7}{n}$ does not converge to $s = 2$.



Scratch work. Of course the limit is really 0.

If ϵ is anything smaller than 2, then s_n will eventually be further than ϵ from $s = 2$.

Choosing $\epsilon = 1$ should be enough. Indeed, since we only care about large n,

$$|s_n - 2| = \left|\frac{7}{n} - 2\right| = 2 - \frac{7}{n} \geq 1 \iff n \geq 7$$

Direct proof. Let $\epsilon = 1$. Given $N \in \mathbb{N}$, let $n = \max(7, N+1)$. Then $n > N$ and

$$|s_n - 2| = \left|\frac{7}{n} - 2\right| = 2 - \frac{7}{n} \geq 1 = \epsilon$$

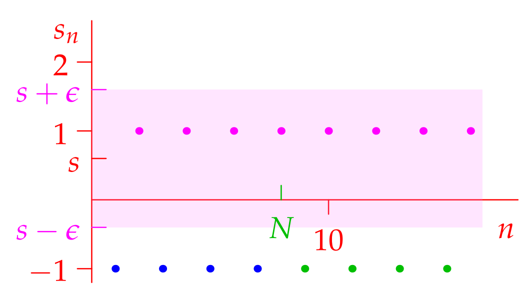
Contradiction proof. For an alternative approach, suppose $\lim s_n = 2$ and let $\epsilon = 1$ in Definition 2.4. Then $\exists N$ such that

$$n > N \implies \left|\frac{7}{n} - 2\right| < 1 \implies 1 < \frac{7}{n} < 3 \implies \frac{7}{3} < n < 7$$

This last is false for large n, in particular for $n := \max(7, N+1)$. Contradiction.

While the two arguments are similar, the contradiction approach has the advantage in that you need only remember *one definition*!

2. We prove (just by contradiction this time) that $s_n = (-1)^n$ defines a divergent sequence.



Suppose that $\lim s_n = s$ and let $\epsilon = 1$ in the definition of limit. Then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} n > N &\implies |(-1)^n - s| < 1 \\ &\implies (-1)^n - 1 < s < 1 + (-1)^n \end{aligned}$$

Choosing any even $n > N$ forces $0 < s$; any odd $n > N$ forces $s < 0$: contradiction.

The choice of $\epsilon = 1$ was suggested by the picture: if $\epsilon = 1$, then, regardless of N, there exist $n > N$ with $|s_n - s| > 1$.

3. We prove directly that the sequence defined by $s_n = \ln n$ is divergent.

   Scratch work. Since logarithms increase unboundedly,[11] for large n we should have $\ln n \geq s + 1$, for any purported limit $s \in \mathbb{R}$.

   Proof. Suppose $s \in \mathbb{R}$ is given and let $\epsilon = 1$. Given $N \in \mathbb{N}$, define $n = \max(N+1, e^{s+1})$. Then

$$\begin{aligned} n > N \quad \text{and} \quad & \ln n \geq \ln(e^{s+1}) = s+1 \qquad (\ln \text{ is } \textit{increasing}) \\ \implies & |s_n - s| = \ln n - s \geq 1 = \epsilon \end{aligned}$$

   We conclude that (s_n) is divergent.

Bounded Sequences

As a first taste using the limit definition abstractly, we consider several related results regarding the boundedness of sequences.

Theorem 2.11. *Suppose (s_n) is convergent: $\lim s_n = s$.*

1. *If $s_n \geq m$ for all large[12] n, then $\lim s_n \geq m$.*
2. *If $s_n \leq M$ for all large n, then $\lim s_n \leq M$.*
3. *(s_n) is bounded ($\exists M$ such that $\forall n, |s_n| \leq M$).*

Proof. 1. We prove the contrapositive. Suppose $s < m$ and let $\epsilon = \frac{m-s}{2} > 0$. Then $\exists N$ such that

$$\begin{aligned} n > N \implies |s_n - s| < \frac{m-s}{2} \implies s_n - s < \frac{m-s}{2} \\ \implies s_n - m < \frac{s-m}{2} < 0 \implies s_n < m \end{aligned}$$

2. This is almost identical to part 1.

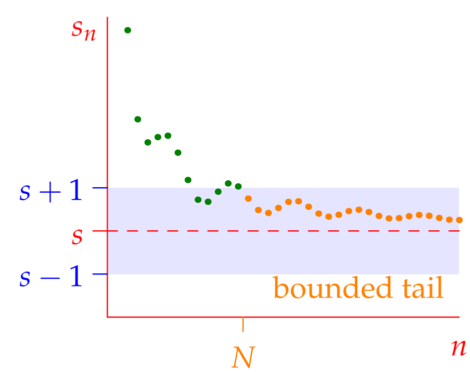
3. The picture shows the strategy: taking $\epsilon = 1$ in the limit definition bounds an infinite tail of the sequence; the finitely many terms that come before are a non-issue.

   Let $\epsilon = 1$ in the definition of limit. Then $\exists N$ such that

$$\begin{aligned} n > N \implies |s_n - s| < 1 \implies s - 1 < s_n < s + 1 \\ \implies |s_n| < \max\{|s-1|, |s+1|\} \end{aligned}$$

   It follows that every term of the sequence is bounded by

$$M := \max\big(|s-1|, |s+1|, |s_n| : n \leq N\big)$$



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[11] Definition 2.19 will state what it means for a sequence to *diverge to* ∞: this isn't (yet) what we're trying to demonstrate.

[12] Equivalently $s_n \geq m$ for all but finitely many n. In the language of the proof, for all n expect perhaps when $n \leq N$. Since we typically care only about large n, this caveat is sometimes left unstated: e.g., $s_n \geq m \implies \lim s_n \geq m$.

Theorem 2.12 (Squeeze Theorem). *Suppose sequences satisfy $a_n \le s_n \le b_n$ (for all large n), where the two outer sequences converge to s. Then $\lim s_n = s$.*

Proof. Subtract s from the assumed inequality to obtain

$$a_n - s \le s_n - s \le b_n - s \implies |s_n - s| \le \max\big(|a_n - s|, |b_n - s|\big)$$

Suppose $\epsilon > 0$ is given. Since $\lim a_n = s = \lim b_n$, there exist N_a, N_b such that

$$n > N_a \implies |a_n - s| < \epsilon \qquad \text{and} \qquad n > N_b \implies |b_n - s| < \epsilon$$

Now let $N = \max(N_a, N_b)$ to see that

$$n > N \implies |s_n - s| \le \max\big(|a_n - s|, |b_n - s|\big) < \epsilon$$

■

Example 2.13. Since $0 \le \frac{1+\sin n}{n} \le \frac{2}{n}$ for all n, the squeeze theorem forces $\lim \frac{1+\sin n}{n} = 0$.

Exercises 2.8. *Key concepts: ϵ–N definition, divergence, boundedness, squeeze theorem*

1. For each sequence, determine the limit and prove your claim.

   (a) $a_n = \frac{n}{n^2+1}$ (b) $b_n = \frac{7n-19}{3n+7}$ (c) $c_n = \frac{4n+3}{7n-5}$

   (d) $d_n = \frac{2n+4}{5n+2}$ (e) $e_n = \frac{1}{n}\sin n$ (f) $f_n = \frac{n^2+n-1}{3n^2-10}$

2. Prove:

   (a) $\lim[\sqrt{n^2+1} - n] = 0$ (b) $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$ (c) $\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$

3. (a) Show that $n \ge 2 \implies 2n^2 + 9n + 5 \le 9n^2$.

   (b) (Example 2.8.3) Give another proof that $\lim \frac{2n^4-3n+1}{3n^4+n^2+4} = \frac{2}{3}$ by choosing $N = \max(2, \frac{1}{\sqrt{\epsilon}})$.

4. (a) Prove that the sequence with n^{th} term $s_n = \frac{2}{n^2}$ does not converge to -1.

   (b) Prove that (s_n) does not converge to 1.

5. Prove that the sequence defined by $t_n = n^2$ diverges.

6. (Example 2.10.3) Prove by contradiction that $(\ln n)$ diverges.

7. True or false: if (s_n) is bounded, then it is convergent. Explain.

8. Let (t_n) be bounded, and let (s_n) satisfy $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

9. We extend Theorem 2.11.

   (a) Suppose $\lim s_n = s$ where every $s_n > m$. Can we conclude that $s > m$? Explain.

   (b) Let (s_n) be convergent and suppose $\lim s_n > a$. Prove that $\exists N$ such that $n > N \implies s_n > a$.

10. Suppose $s \in \mathbb{R}$. Prove:

    (a) $\lim s_n = s \iff \lim(s_n - s) = 0$

    (b) $\lim s_n = s \implies \lim |s_n| = |s|$

2.9 Limit Theorems for Sequences

We'd like to develop rules for limits so that we don't have to resort to ϵ–N proofs every time. The rough idea of these results is that limits respect the basic rules of algebra. Rather than dive straight into abstract proofs, we start with two special cases that illustrate commonly encountered tricks.

Examples 2.14. Suppose that (s_n) converges to s. We prove that $\lim 5s_n = 5s$ and that $\lim s_n^2 = s^2$.

1. The sequence $(5s_n)$ is obtained by multiplying the original terms by 5. To prove $\lim 5s_n = 5s$ we must show:

   $$\forall \epsilon > 0, \exists N \text{ such that } n > N \implies |5s_n - 5s| < \epsilon \tag{$*$}$$

   This last amounts to observing that $|s_n - s| < \frac{\epsilon}{5}$. Since $\frac{\epsilon}{5}$ is simply another small number, $(*)$ is just the statement $\lim s_n = s$ in disguise! Here is a formal argument.

   Let $\epsilon > 0$ be given. Since $\lim s_n = s$, we know that[13]

   $$\exists N \text{ such that } n > N \implies |s_n - s| < \frac{\epsilon}{5} \implies |5s_n - 5s| < \epsilon$$

2. The challenge is to make $\left|s_n^2 - s^2\right| = |s_n - s|\,|s_n + s|$ small. The first term can be made arbitrarily small by the hypothesis $\lim s_n = s$. To control the second, we use the triangle-inequality:

   $$|s_n + s| = |s_n - s + 2s| \leq |s_n - s| + 2\,|s|$$

   Assuming $|s_n - s| \leq 1$ gives a bound $|s + s_n| \leq 1 + 2\,|s|$. We now have enough for a proof.

   Suppose $\lim s_n = s$, that $\epsilon > 0$ is given and let $\delta = \min(1, \frac{\epsilon}{1+2|s|})$. Then $\exists N$ such that,

   $$\begin{aligned} n > N &\implies |s_n - s| < \delta \leq 1 \\ &\implies \left|s_n^2 - s^2\right| = |s_n - s|\,|s_n + s| \overset{\triangle}{\leq} |s_n - s|\,(|s_n - s| + 2\,|s|) \\ &\qquad\qquad < \delta(1 + 2\,|s|) \leq \epsilon \end{aligned}$$

Theorem 2.15 (Limit laws). *Limits respect algebraic operations: $\pm, \times, \div$ and rational roots. More specifically, if (s_n) converges to s and (t_n) to t, then,*

1. $\lim(s_n \pm t_n) = s \pm t$
2. $\lim(s_n t_n) = st$; *in particular, if k is constant, then* $\lim ks_n = ks$
3. *If $t \neq 0$, then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$
4. *If $k \in \mathbb{N}$, then* $\lim \sqrt[k]{s_n} = \sqrt[k]{s}$*, provided the roots exist ($s_n, s \geq 0$ if k even)*

By part 4 and induction on part 2, $\lim s_n^q = s^q$ *for any $q \in \mathbb{Q}$.*

Examples 2.14 are special cases of part 2 ($k = 5$ and then $t_n = s_n$).

[13] It is non-standard, but if this approach makes you squeamish, you can introduce a second $\tilde{\epsilon}$:

Given $\epsilon > 0$, let $\tilde{\epsilon} = \frac{\epsilon}{5}$, then $\exists N$ such that $n > N \implies |s_n - s| < \tilde{\epsilon} \implies |5s_n - 5s| < \epsilon$.

Rigorously proving the limit laws takes some work. Before engaging in some of this, we advertise their benefit by performing some calculations as you might have seen in elementary calculus.

Examples 2.16. 1. We evaluate $\lim \frac{3n^2+2\sqrt{n}-1}{5n^2-2}$ using the limit laws.

$$\begin{aligned}
\lim \frac{3n^2+2\sqrt{n}-1}{5n^2-2} &= \lim \frac{3+\frac{2}{n^{3/2}}-\frac{1}{n^2}}{5-\frac{2}{n^2}} && (n>0)\\
&= \frac{\lim\left(3+\frac{2}{n^{3/2}}-\frac{1}{n^2}\right)}{\lim\left(5-\frac{2}{n^2}\right)} && \text{(part 3)}\\
&= \frac{\lim 3+\lim\frac{2}{n^{3/2}}-\lim\frac{1}{n^2}}{\lim 5-\lim\frac{2}{n^2}} && \text{(part 1)}\\
&= \frac{3+0-0}{5-0}=\frac{3}{5} && \text{(parts 2, 4 and Theorem 2.7))}
\end{aligned}$$

The calculation involves some generally accepted sleight of hand: none of the limits should really be written until one knows they exist. To be completely logical would require us to rewrite the argument upside down, though to sacrifice readability in this manner would be ill-advised.

2. Suppose (s_n) is defined inductively via $s_1=2$ and $s_{n+1}=\frac{1}{2}(s_n+\frac{2}{s_n})$:

$$(s_n)=\left(2,\frac{3}{2},\frac{17}{12},\frac{577}{408},\ldots\right)$$

This sequence in fact converges, though a proof requires ideas from Section 2.10. Given this fact, the limit laws allow us to compute the limit s:

$$s=\lim s_{n+1}=\frac{1}{2}\left(\lim s_n+\frac{2}{\lim s_n}\right)=\frac{1}{2}\left(s+\frac{2}{s}\right)\implies \frac{1}{2}s=\frac{1}{s}\implies s^2=2$$

Since s_n is plainly always positive, we conclude that $\lim s_n=\sqrt{2}$.

We now commence our assault on the limit laws.

Proof. 1. We use a trick similar to that in Example 2.14.1: control both sequences so that both $|s_n-s|,|t_n-t|<\frac{\epsilon}{2}$, then sum.

Suppose $\epsilon>0$ is given. Since $\lim s_n=s$ and $\lim t_n=t$, we see that $\exists N_1,N_2$ such that

$$n>N_1\implies |s_n-s|<\frac{\epsilon}{2}\qquad\text{and}\qquad n>N_2\implies |t_n-t|<\frac{\epsilon}{2}$$

Let $N=\max(N_1,N_2)$, then

$$n>N\implies |s_n+t_n-(s+t)|\overset{\triangle}{\leq}|s_n-s|+|t_n-t|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

The argument for s_n-t_n is almost identical.

2. Exercise 2.8.8 deals with (and extends) the case when $s = 0$. We therefore suppose $s \neq 0$. The approach is similar to part 1, we just need to be a bit cleverer to break up $|s_n t_n - st|$.

   Suppose $\epsilon > 0$ is given. By Theorem 2.11, (t_n) is bounded:

   $$\exists M \text{ such that } \forall n,\ |t_n| \leq M$$

   We make assume, WLOG, that $M > 0$. Since $\lim s_n = s$ and $\lim t_n = t$, $\exists N_1, N_2$ such that

   $$n > N_1 \implies |s_n - s| < \frac{\epsilon}{2M} \qquad \text{and} \qquad n > N_2 \implies |t_n - t| < \frac{\epsilon}{2|s|}$$

   Let $N = \max(N_1, N_2)$, then

   $$|s_n t_n - st| = |s_n t_n - st_n + st_n - st| \overset{\triangle}{\leq} |s_n - s|\,|t_n| + |s|\,|t_n - t| < \frac{\epsilon}{2M}M + |s|\frac{\epsilon}{2|s|} = \epsilon$$

3 & 4.: See Exercise 7.

■

With a few general examples, the limit laws allow us to rapidly compute a great variety of limits.

Theorem 2.17. 1. *If* $|a| < 1$ *then* $\lim a^n = 0$

2. *If* $a > 0$ *then* $\lim a^{1/n} = 1$

3. $\lim n^{1/n} = 1$

Examples 2.18. 1. $\lim (3n)^{2/n} = (\lim 3^{1/n})^2 (\lim n^{1/n})^2 = 1$.

2. Observe from the squeeze theorem (2.12) that $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \to 0$. We conclude:

   $$\lim \frac{n^{2/n} + \left(3 - n^{-1}\sin n\right)^{1/5}}{4n^{-3/2} + 7} = \frac{(\lim n^{1/n})^2 - \left(3 - \lim \frac{\sin n}{n}\right)^{1/5}}{4\lim \frac{1}{n^{3/2}} + 7} = \frac{1 - \sqrt[5]{3}}{7}$$

Proof. 1. The $a = 0$ case is trivial. Otherwise, given $\epsilon > 0$, let $N = \log_{|a|} \epsilon$, then

   $$n > N \implies |a^n| < \left|a^N\right| = |a|^N = \epsilon$$

2. Suppose $a \geq 1$ and let $s_n = a^{1/n} - 1$. Since $s_n > 0$, the binomial theorem[14] shows that

   $$a = (1 + s_n)^n \geq 1 + ns_n \implies 0 < s_n \leq \frac{a-1}{n}$$

   The squeeze theorem shows that $\lim s_n = 0$, whence $\lim a^{1/n} = 1$.

   The $a < 1$ case and part 3 are in Exercise 8.

■

[14] $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{2\cdot 3}x^3 + \cdots + x^n$.

Divergence to $\pm\infty$ and the 'divergence laws'

We consider unbounded sequences and provide a positive definition of a type of divergence.

Definition 2.19. We say that (s_n) *diverges to* ∞ if,

$$\forall M > 0,\ \exists N \text{ such that } n > N \implies s_n > M$$

We write $\lim s_n = \infty$, or $s_n \to \infty$, and say that s_n 'tends to' ∞. The definition for $s_n \to -\infty$ is similar. If (s_n) neither converges nor diverges to $\pm\infty$, we say that it *diverges by oscillation*: you likely wrote $\lim s_n = \text{DNE}$ ("does not exist") in elementary calculus.

Consider how M describes "closeness" to infinity analogously to how ϵ measures closeness to s in the original definition of limit (2.4).

Examples 2.20. As with convergence arguments, some scratch work might be helpful.

1. We show that $\lim(n^2 + 4n) = \infty$.

   Let $M > 0$ be given, and let $N = \sqrt{M}$. Then

   $$n > N \implies n^2 + 4n > n^2 > N^2 = M$$

2. We prove that $s_n = n^5 - n^4 - 2n + 1$ diverges to ∞.

   The negative terms cause some trouble, though our solution should be familiar from previous calculations:

   $$s_n > \frac{1}{2}n^5 \iff n^5 > 2(n^4 + 2n - 1) \iff n > 2 + \frac{4}{n^3} - \frac{1}{n^4}$$

   Certainly this holds if $n > 6$. We can now complete the proof.

   Let $M > 0$ be given, and let $N = \max(6, \sqrt[5]{2M})$. Then

   $$n > N \implies s_n > \frac{1}{2}n^5 > \frac{1}{2}(2M) = M$$

3. We prove that $s_n = n^2 - n^3$ diverges to $-\infty$.

   First observe that

   $$s_n = n^2(1-n) < -\frac{1}{2}n^3 \iff 1 - n < -\frac{1}{2}n \iff n \geq 2$$

   Now let $M > 0$ be given,[15] and define $N = \max(2, \sqrt[3]{2M})$. Then

   $$n > N \implies n > 2 \implies s_n < -\frac{1}{2}n^3 < -\frac{1}{2}N^3 \leq -M$$

[15] You may prefer to phrase $\lim s_n = -\infty$ instead as

$$\forall m < 0,\ \exists N \text{ such that } n > N \implies s_n < m \qquad \text{(in our argument } M = -m\text{)}$$

Several of the limit laws can be adapted to sequences which diverge to $\pm\infty$.

Theorem 2.21. *Suppose* $\lim s_n = \infty$.

1. *If* $t_n \geq s_n$ *for all (large)* n*, then* $\lim t_n = \infty$
2. *If* $\lim t_n$ *exists and is* finite*, then* $\lim s_n + t_n = \infty$.
3. *If* $\lim t_n > 0$ *then* $\lim s_n t_n = \infty$.
4. $\lim \frac{1}{s_n} = 0$
5. *If* $\lim t_n = 0$ *and* $t_n > 0$ *for all (large)* n*, then* $\lim \frac{1}{t_n} = \infty$

Similar statements when $\lim s_n = -\infty$ *should be clear.*

Proof. We prove two parts; try the rest yourself.

2. Since (t_n) converges, it is bounded (below): $\exists m$ such that $\forall n,\ t_n \geq m$. Let M be given. Since $\lim s_n = \infty$, $\exists N$ such that
$$n > N \implies s_n > M - m \implies s_n + t_n > M - m + m = M$$
4. Let $\epsilon > 0$ be given, and let $M = \frac{1}{\epsilon}$. Then $\exists N$ such that
$$n > N \implies s_n > M = \frac{1}{\epsilon} \implies \frac{1}{s_n} < \epsilon$$

∎

Rational Sequences

We can now describe the limit of any sequence $\frac{p_n}{q_n}$ where (p_n), (q_n) are polynomials in n.

Example 2.22. By applying Theorem 2.21 (part 3) to

$$s_n := 3n + 4n^{-2} \to \infty \quad \text{and} \quad t_n = \frac{1}{2 - n^{-2}} \to \frac{1}{2}$$

we see that

$$\lim \frac{3n^3 + 4}{2n^2 - 1} = \lim \frac{3n + 4n^{-2}}{2 - n^{-2}} = \lim(3n + 4n^{-2}) \cdot \lim \frac{1}{2 - n^{-2}} = \infty$$

More generally, you should be able to confirm a familiar result from elementary calculus:

Corollary 2.23. *If* p_n, q_n *are polynomials in* n *with leading coefficients* p, q *respectively then*

$$\lim \frac{p_n}{q_n} = \begin{cases} 0 & \text{if } \deg(p_n) < \deg(q_n) \\ \frac{p}{q} & \text{if } \deg(p_n) = \deg(q_n) \\ \operatorname{sgn}(\frac{p}{q})\infty & \text{if } \deg(p_n) > \deg(q_n) \end{cases}$$

Exercises 2.9. *Key concepts: Divergence to $\pm\infty$, Limit/divergence laws,* $\lim a^{1/n} = 1 = \lim n^{1/n}$

1. Suppose $\lim x_n = 3$, $\lim y_n = 7$ and that all y_n are non-zero. Determine the following:

   (a) $\lim(x_n + y_n)$ (b) $\lim \frac{3y_n - x_n}{y_n^2}$ (c) $\lim \sqrt{x_n y_n + 4}$

2. Suppose $s \in \mathbb{R}$. Prove that $\lim s_n = s \implies \lim s_n^3 = s^3$ by mimicking Example 2.14.2.

3. Let $s_n = (100n)^{\frac{100}{n}}$. Describe s_1 and s_{10} (1 followed by how many zeros?). Now compute $\lim s_n$.

4. Define (s_n) inductively via $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for $n \geq 1$.

   (a) List the first four terms of (s_n).

   (b) It turns out that (s_n) converges. Assume this and prove that $\lim s_n = \frac{1}{2}(1 + \sqrt{5})$.

5. Prove:

   (a) $\lim(n^3 - 98n) = \infty$ (b) $\lim\left(\sqrt{n} - n + \frac{4}{n}\right) = -\infty$

6. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

   (a) Show that if (x_n) converges with limit a, then $a = \frac{1}{3}$ or $a = 0$.

   (b) What is $\lim x_n$? Prove your assertion and explain what is going on.

7. We prove parts 3 and 4 of the limit laws (Theorem 2.15). Assume $\lim s_n = s$ and $\lim t_n = t$.

   (a) Suppose $t \neq 0$. Explain why $\exists N_1$ such that $n > N_1 \implies |t_n| > \frac{1}{2}|t|$.

   (b) Let $\epsilon > 0$ be given. Since $\lim t_n = t$, $\exists N_2$ such that $n > N_2 \implies |t_n - t| < \frac{1}{2}|t|^2 \epsilon$. Combine N_1 and N_2 to prove that $\lim \frac{1}{t_n} = \frac{1}{t}$.

   (c) Explain how to conclude part 3: $\lim \frac{s_n}{t_n} = \frac{s}{t}$.

   (d) Use the following inequality (valid when $s_n, s > 0$) to construct a proof for part 4

   $$\left|s_n^{1/k} - s^{1/k}\right| = \frac{|s_n - s|}{s_n^{\frac{k-1}{k}} + s_n^{\frac{k-2}{k}} s^{\frac{1}{k}} + \cdots + s^{\frac{k-1}{k}}} \leq \frac{|s_n - s|}{s^{\frac{k-1}{k}}}$$

8. We finish the proof of Theorem 2.17.

   (a) Suppose $0 < a < 1$. By considering $b = \frac{1}{a}$, prove that $\lim a^{1/n} = 1$.

   (b) Let $s_n = n^{1/n} - 1$. Apply the binomial theorem to $n = (1 + s_n)^n$ to prove that $s_n < \sqrt{\frac{2}{n-1}}$. Hence conclude that $\lim n^{1/n} = 1$.

9. Prove the remaining parts of Theorem 2.21.

10. Assume $s_n \neq 0$ for all n and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

    (a) Show that if $L < 1$, then $\lim s_n = 0$.
    (*Hint: if $L < a < 1$, obtain N so that $n > N \implies |s_n| < a^{n-N}|s_N|$*)

    (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.
    (*Hint: apply (a) to the sequence $t_n = \frac{1}{|s_n|}$*)

11. Let $p > 0$ and $a \in \mathbb{R}$ be given. How does $\lim\limits_{n\to\infty} \frac{a^n}{n^p}$ depend on the value of a?

2.10 Monotone and Cauchy Sequences

The definition of limit (Definition 2.4) has a major weakness: to demonstrate the convergence of a sequence we must already know its limit! What we'd like is a method for determining whether a sequence converges *without* first guessing a suitable limit.[16] In this section we consider two important classes of sequence for which this can be done.

Definition 2.24. A sequence (s_n) is said to be:

- *Monotone-up*[17] if $s_{n+1} \geq s_n$ for all n.
- *Monotone-down* if $s_{n+1} \leq s_n$ for all n.
- *Monotone* if either of the above is true.

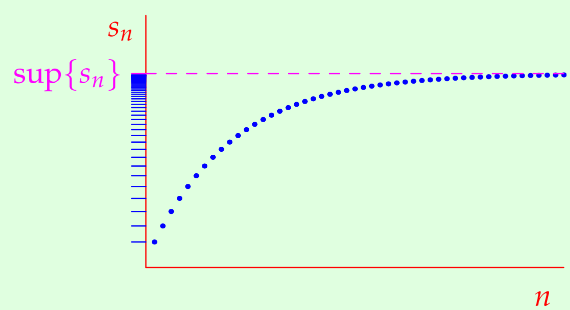
Examples 2.25. 1. The sequence with n^{th} term $s_n = \frac{7}{n} + 4$ is (strictly) monotone-down:

$$s_{n+1} = \frac{7}{n+1} < \frac{7}{n} = s_n$$

2. A constant sequence $(s_n) = (s, s, s, s, \dots)$ is *both* monotone-up and monotone-down.

Theorem 2.26 (Monotone Convergence).
Every bounded monotone sequence is convergent. Specifically:

- *If (s_n) is bounded above and monotone-up, then $\lim s_n$ exists and equals $\sup\{s_n\}$.*
- *If (s_n) is bounded below and monotone-down, then $\lim s_n$ exists and equals $\inf\{s_n\}$.*



In fact the conclusion $\lim s_n = \sup\{s_n\}$ holds for all monotone-up sequences: if unbounded above, then the result is ∞ (Exercise 5).

Proof. If (s_n) is bounded above, then $s := \sup\{s_n\}$ exists by the completeness axiom (s is finite!). Let $\epsilon > 0$ be given. By Lemma 1.20, there exists some $s_N > s - \epsilon$. Since (s_n) is monotone-up,

$$n > N \implies s_n \geq s_N > s - \epsilon \implies 0 \leq s - s_n < \epsilon \implies |s - s_n| < \epsilon$$

The monotone-down case is similar. ∎

[16] This gets at a typical application of sequences: given a sequence whose elements have a useful property, one demonstrates the existence of a new object (the limit) to which (hopefully!) the useful property transfers. For instance, if (f_n) is a sequence of differentiable functions, we'd like to know if $\lim f_n(x)$ exists and is itself differentiable with derivative $\lim f_n'(x)$. Discussions of this ilk dominate Math 140B.

[17] Some authors describe a monotone-up sequence as either *non-decreasing* or *increasing*. We prefer *monotone-up* since it directly describes the direction of any possible movement in the sequence and prevents confusion over whether the inequality is strict. If necessary, a sequence with $s_{n+1} > s_n$ may be described as *strictly increasing* or *strictly monotone-up*.

Examples 2.27. 1. Define (s_n) via $s_n = 1$ and $s_{n+1} = \frac{1}{5}(s_n + 8)$:

$$(s_n) = \big(1, 1.8, 1.96, 1.992, 1.9984, 1.99968, \ldots\big)$$

This sequence certainly appears to be monotone-up and converging to 2. Here is a proof:

Bounded above: $s_n < 2 \implies s_{n+1} < \frac{1}{5}[2+8] = 2$. By induction, (s_n) is bounded above by 2.

Monotone-up: $s_{n+1} - s_n = \frac{4}{5}[2 - s_n] > 0$ since $s_n < 2$.

Convergence: By monotone convergence, $s = \lim s_n$ exists. Now use the limit laws to find s:

$$s = \lim s_{n+1} = \frac{1}{5}(\lim s_n + 8) = \frac{1}{5}(s+8) \implies s = 2$$

2. (Example 2.16.2, cont.) Let $s_1 = 2$ and $s_{n+1} = \frac{1}{2}\big(s_n + \frac{2}{s_n}\big)$.

Bounded below: The sequence is plainly always positive and thus bounded below by zero.

Monotone-down: We first obtain an improved lower bound:

$$s_{n+1}^2 = \frac{1}{4}\left(s_n + \frac{2}{s_n}\right)^2 = 2 + \frac{1}{4}\left(s_n - \frac{2}{s_n}\right)^2 \ge 2$$

shows[18] that $s_n^2 \ge 2$ for all n. It follows that

$$\frac{s_{n+1}}{s_n} = \frac{1}{2}\left(1 + \frac{2}{s_n^2}\right) \le 1 \implies s_{n+1} \le s_n$$

Convergence: By monotone convergence, $s = \lim s_n$ exists. Example 2.16.2 provides the limit:

$$s = \frac{1}{2}\left(s + \frac{2}{s}\right) \implies s = \sqrt{2}$$

This shows the necessity of completeness: (s_n) is a monotone, bounded sequence of *rational* numbers, but its limit is *irrational*.

3. A *decimal number* $d_0.d_1d_2d_3\ldots$ is the limit of a monotone-up sequence of *rational numbers*:

$$d_0.d_1d_2d_3\ldots = d_0 + \lim_{n\to\infty}\sum_{k=1}^{n}\frac{d_k}{10^k}$$

This is bounded above (by $d_0 + 1 \in \mathbb{Z}$) and so converges.

4. The sequence with $s_n = \big(1 + \frac{1}{n}\big)^n$ is particularly famous. In Exercise 10 we show that (s_n) is monotone-up and bounded above. The limit provides, arguably, the oldest definition of e:

$$e := \lim\left(1 + \frac{1}{n}\right)^n$$

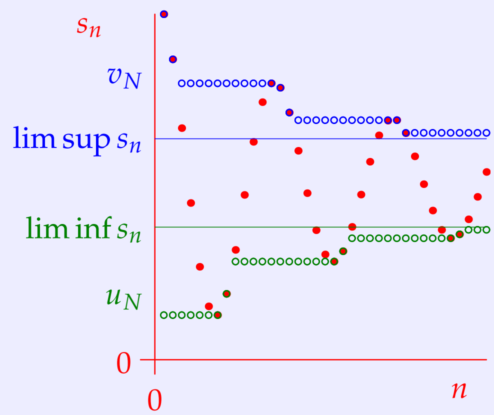
[18] This is the famous AM–GM inequality $\frac{x+y}{2} \ge \sqrt{xy}$ with $x = s_n$ and $y = \frac{2}{s_n}$.

Limits Superior and Inferior

One interpretation of $\lim s_n$ is that it approximately describes s_n for large n. Even when a sequence does not have a limit, it is useful to be able to describe its long-term behavior.

Definition 2.28. Let (s_n) be a sequence and define two related sequences (v_N) and (u_N):

$$v_N := \sup\{s_n : n \geq N\}, \qquad u_N := \inf\{s_n : n \geq N\}$$



1. The *limit superior* of (s_n) is

$$\limsup s_n = \begin{cases} \lim\limits_{N\to\infty} v_N & \text{if } (s_n) \text{ bounded above} \\ \infty & \text{if } (s_n) \text{ unbounded above} \end{cases}$$

2. The *limit inferior* of (s_n) is

$$\liminf s_n = \begin{cases} \lim\limits_{N\to\infty} u_N & \text{if } (s_n) \text{ bounded below} \\ -\infty & \text{if } (s_n) \text{ unbounded below} \end{cases}$$

The original sequence (s_n) is wedged between (v_n) and (u_n) in a manner reminiscent of the squeeze theorem (though lim sup and lim inf need not be equal). The next result summarizes the situation more formally; we omit the proof since these claims should be clear from the definition and previous results, particularly the monotone convergence theorem.

Lemma 2.29. 1. *(v_N) is monotone-down and (u_N) monotone-up.*

2. *$\limsup s_n$ and $\liminf s_n$ exist for any sequence (they might be infinite).*

3. *If $n \geq N$, then $u_N \leq s_n \leq v_N$.*

4. *$\liminf s_n \leq \limsup s_n$.*

Examples 2.30. 1. If $s_n = \frac{1}{n}$, then $v_N = s_N$ and $u_N = 0$, whence $\limsup s_n = \liminf s_n = 0$.

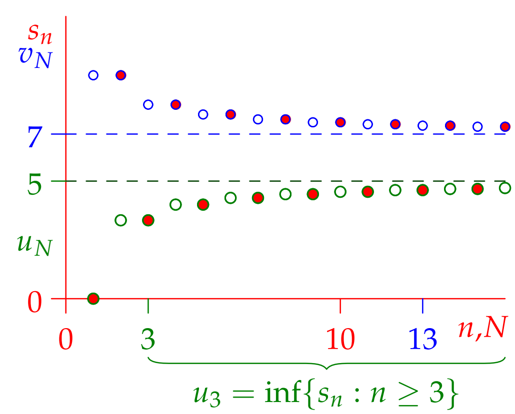
2. The picture shows the sequences (s_n), (u_N) and (v_N) when $s_n = 6 + (-1)^n \left(1 + \frac{5}{n}\right)$

We won't compute everything precisely, but the picture suggests (s_n) has two "sub"-sequences: the odd terms increase while the even terms decrease towards, respectively

$$\liminf s_n = 5, \qquad \limsup s_n = 7$$

Here is one value from each derived sequence:

$$u_3 = \inf\{s_n : n \geq 3\} = s_3 \approx 3.333$$
$$v_{13} = \sup\{s_n : n \geq 13\} = s_{14} \approx 7.357$$



3. Let $s_n = (-1)^n$. This time the calculation is easy: for any N,

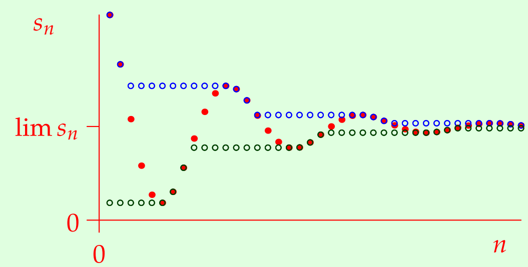
$$u_N = \inf\{s_n : n \geq N\} = -1 \quad \text{and} \quad v_N = \sup\{s_n : n \geq N\} = 1$$

Therefore $\limsup s_n = 1$ and $\liminf s_n = -1$.

Theorem 2.31. *For any sequence (s_n),*

$$\limsup s_n = \liminf s_n \iff \lim s_n \text{ exists}$$

In such a case all three values are equal.
Note that the limits can be $\pm\infty$.



Proof. ($\Rightarrow$) Suppose $s := \limsup s_n = \liminf s_n$.

- If s is finite, apply the squeeze theorem to $u_n \leq s_n \leq v_n$ (both extremes converge to s).
- If $s = \infty$, then $u_n \leq s_n$ for all n. Theorem 2.21.1 shows that $\lim s_n = \infty = s$.
- If $s = -\infty$, instead use $s_n \leq v_n$.

($\Leftarrow$) We could prove this now, but it will come almost for free a little later... ∎

Cauchy Sequences

We now come to a class of sequences whose analogues will dominate your future studies.

Definition 2.32. A sequence (s_n) is *Cauchy*[19] if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon$$

A sequence is Cauchy when terms in the tails of the sequence are constrained to stay close to one another. As we'll see shortly, this will provide an alternative way to detect and describe *convergence*.

Examples 2.33. 1. Let $s_n = \frac{1}{n}$. Let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then

$$m > n > N \implies |s_m - s_n| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon \qquad \text{(WLOG } m > n\text{)}$$

Thus (s_n) is Cauchy. A similar argument works for any $s_n = \frac{1}{n^k}$ for positive k.

2. Suppose $s_1 = 5$ and $s_{n+1} = s_n + \frac{1}{n(n+1)}$. As before, let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then,

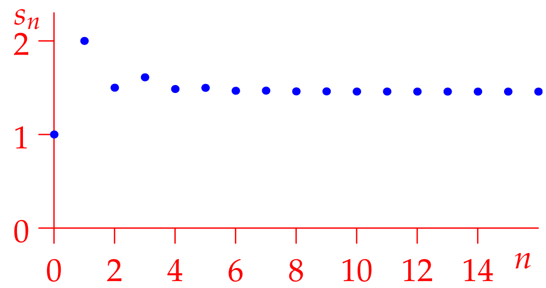
$$|s_{n+1} - s_n| = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\implies |s_m - s_n| \overset{\triangle}{\leq} |s_{n+1} - s_n| + \cdots + |s_m - s_{m-1}| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon$$

Again we have a Cauchy sequence.

[19] Augustin-Louis Cauchy (1789–1857) was a French mathematician, responsible (in part) for the ϵ-N definition of limit.

3. Define $(s_n)_{n=0}^{\infty}$ inductively:



$$s_0 = 1, \quad s_{n+1} = \begin{cases} s_n + 3^{-n} & \text{if } n \text{ even} \\ s_n - 2^{-n} & \text{if } n \text{ odd} \end{cases}$$

$$(s_n) = \left(1, 2, \frac{3}{2}, \frac{29}{18}, \frac{107}{72}, \ldots\right)$$

Since $|s_{n+1} - s_n| \le 2^{-n}$, we see that

$$m > n \implies |s_m - s_n| \overset{\triangle}{\le} |s_{n+1} - s_n| + \cdots + |s_m - s_{m-1}| = \sum_{k=n}^{m-1} |s_{k+1} - s_k|$$
$$\le \sum_{k=n}^{m-1} 2^{-k} = \frac{2^{-n} - 2^{-m}}{1 - 2^{-1}} < 2^{1-n}$$

where we used the familiar geometric sum formula from calculus: $\sum\limits_{k=a}^{b-1} r^k = \frac{r^a - r^b}{1-r}$.
Suppose $\epsilon > 0$ is given, and let $N = 1 - \log_2 \epsilon = \log_2 \frac{2}{\epsilon}$. Then

$$m > n > N \implies |s_m - s_n| < 2^{1-n} < 2^{1-N} = \epsilon$$

We conclude that (s_n) is Cauchy.

The last picture illustrates the essential point of Cauchy sequences: (s_n) appears to converge...

Theorem 2.34 (Cauchy Completeness). *A sequence of real numbers is convergent if and only if it is Cauchy.*

Proof. $(\Rightarrow)$ Suppose $\lim s_n = s$ (is finite). Given $\epsilon > 0$, we may choose N such that

$$m, n > N \implies |s_n - s| < \frac{\epsilon}{2} \quad \text{and} \quad |s_m - s| < \frac{\epsilon}{2}$$
$$\implies |s_n - s_m| = |s_n - s + s - s_m| \overset{\triangle}{\le} |s_n - s| + |s - s_m| < \epsilon$$

Otherwise said, (s_n) is Cauchy.

$(\Leftarrow)$ To discuss the convergence of (s_n) we need a potential limit. In view of Theorem 2.31, the obvious candidates are $\limsup s_n$ and $\liminf s_n$. We have two goals: show that (s_n) is bounded whence the limits superior and inferior are *finite*; then show that these are *equal*.

(Boundedness of (s_n)) Take $\epsilon = 1$ in Definition 2.32:

$$\exists N \text{ such that } m, n > N \implies |s_n - s_m| < 1$$

It follows that

$$n > N \implies |s_n - s_{N+1}| < 1 \implies s_{N+1} - 1 < s_n < s_{N+1} + 1$$

whence (s_n) is bounded. It follows that $\limsup s_n$ and $\liminf s_n$ are both *finite*.

(lim sup $s_n = \liminf s_n$) Suppose $\epsilon > 0$ is given. Since (s_n) is Cauchy,

$$\exists N \in \mathbb{N} \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon \implies s_n < s_m + \epsilon$$

Take the supremum over all $n > N$: since $v_{N+1} = \sup\{s_n : n \geq N+1\}$, we see that

$$m > N \implies v_{N+1} \leq s_m + \epsilon$$

Now take the infimum of the right hand side over all $m > N$ to obtain

$$v_{N+1} \leq u_{N+1} + \epsilon \qquad \text{(since } u_{N+1} = \inf\{s_m : m \geq N+1\}\text{)}$$

Since (v_{N+1}) is monotone-down and (u_{N+1}) monotone-up, we see that

$$\limsup s_n \leq v_{N+1} \leq u_{N+1} + \epsilon \leq \liminf s_n + \epsilon \implies \limsup s_n \leq \liminf s_n + \epsilon$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\limsup s_n \leq \liminf s_n$. By Lemma 2.29 we have equality.

By Theorem 2.31, we conclude that (s_n) converges to $\limsup s_n = \liminf s_n$. ■

By the Theorem, Examples 2.33 all converge. All three limits can be found precisely (for instance, see Exercise 7). With a small modification to the second example, however, we obtain something genuinely new:

Example (2.33.2 cont). Let $s_1 = 5$ and, for each n, define $s_{n+1} := s_n + \frac{\sin n}{n(n+1)}$. Since $|\sin n| \leq 1$, the computation proceeds almost the same as before:

$$|s_{n+1} - s_n| = \frac{|\sin n|}{n(n+1)} \leq \frac{1}{n(n+1)} = \cdots$$

The new sequence is Cauchy and thus convergent, though good luck explicitly finding its limit!

The main point is easy to miss: the Cauchy condition is a powerful tool for determining whether a sequence converges *without first guessing a limit.* While the proof depends on monotone convergence (via limit superior/inferior), Cauchy completeness is more powerful in that it applies even to non-monotone sequences.

An Alternative Definition of $\mathbb{R}$ Cauchy sequences suggest a *definition* of the real numbers which does not rely on Dedekind cuts (Section 1.6).

Define an equivalence relation $\sim$ on the collection $\mathcal{C}$ of all Cauchy sequences of rational numbers:[20]

$$(s_n) \sim (t_n) \iff \lim(s_n - t_n) = 0$$

Now define $\mathbb{R} := {}^{\mathcal{C}}/_{\sim}$ to be the set of equivalence classes. All this is done without reference to Cauchy completeness, though it certainly informs our intuition that (s_n) and (t_n) have the same limit (as real numbers). Significant work is still required to properly define $+, \cdot, \leq$, etc., and to verify the axioms of a complete ordered field—we won't pursue this.

[20]We don't need real numbers to define the limit of the *rational* sequence $(s_n - t_n)$: $\forall \epsilon \in \mathbb{Q}^+$ is enough...

Exercises 2.10. *Key concepts: Monotone sequences & Convergence, Cauchy sequences & completeness, Limits superior/inferior*

1. Use Definition 2.32 to show that the sequence with $s_n = \frac{1}{n^2}$ is Cauchy. Repeat for $t_n = \frac{1}{n(n-2)}$.
2. Let $s_1 = 1$ and $s_{n+1} = \frac{n}{n+1}s_n^2$ for $n \geq 1$.
   (a) Find s_2, s_3 and s_4.
   (b) Show that $\lim s_n$ exists and hence prove that $\lim s_n = 0$.
3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.
   (a) Find s_2, s_3 and s_4.
   (b) Use induction to show that $s_n > \frac{1}{2}$ for all n, and conclude that (s_n) is monotone-down.
   (c) Show that $\lim s_n$ exists and find $\lim s_n$.
4. (a) Let (s_n) be a sequence such that $\forall n,\ |s_{n+1} - s_n| \leq 3^{-n}$. Prove that (s_n) is Cauchy.
   (b) Let $s_1 = 10$ and, for each n, let $s_{n+1} = s_n + \frac{\cos n}{3^n}$. Explain why (s_n) is convergent.
   (c) Is the result in (a) true if we only assume that $|s_{n+1} - s_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$?
5. Suppose (s_n) is *unbounded* and monotone-up. Prove that $\lim s_n = \infty$.
   (*Thus* $\lim s_n = \sup\{s_n\}$ *for any monotone-up sequence*)
6. Let $s_n = \frac{(-1)^n}{n}$. Find the sequences $(u_N), (v_N)$ and explicitly compute $\limsup s_n$ and $\liminf s_n$.
7. Consider the sequence in Example 2.33.3. Explain why $s_{2n} = s_{2n-2} - \frac{2}{4^n} + \frac{9}{9^n}$.
   Now use the geometric sum formula to evaluate $\lim s_{2n}$.
   (*Since* (s_n) *converges, this means the original sequence has the same limit*)
8. Let S be a bounded nonempty set for which $\sup S \notin S$. Prove that there exists a monotone-up sequence (s_n) of points in S such that $\lim s_n = \sup S$.
   (*Hint: for each* n*, use* $\sup S - \frac{1}{n}$ *to build* s_n)
9. Let (s_n) be a monotone-up sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove that (σ_n) is monotone-up.
10. (Hard!) We prove that the sequence defined by $s_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.
    (a) Show that
    $$\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}} = 1 - \frac{1}{(n+1)^2} \qquad \text{and} \qquad \frac{1 + \frac{1}{n}}{1 + \frac{1}{n+1}} = 1 + \frac{1}{n(n+2)}$$
    (b) Prove *Bernoulli's inequality* by induction:
    For all real $x > -1$ and $n \in \mathbb{N}_0$ we have $(1 + x)^n \geq 1 + nx$.
    (c) By considering $\frac{s_{n+1}}{s_n}$, use parts (a) and (b) to prove that (s_n) is monotone-up.
    (d) Similarly, show that $t_n := \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right) s_n$ defines a monotone-down sequence.
    (e) Prove that (s_n) and (t_n) converge, and to the *same* limit (this is Bernoulli's definition of e).
    (f) Prove that $\lim(1 - \frac{1}{n})^n = e^{-1}$.

2.11 Subsequences

The overall behavior of a sequence is often hard to describe, but if we delete some of its terms we might obtain a *subsequence* with much simpler behavior.

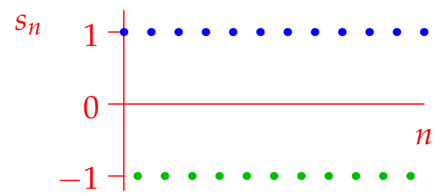
Definition 2.35. Let (s_n) be a sequence. A *subsequence* (s_{n_k}) is a subset $(s_{n_k}) \subseteq (s_n)$, where

$$n_1 < n_2 < n_3 < \cdots$$

A subsequence is simply an infinite subset whose order is inherited from the original sequence.

Example 2.36. Take $s_n = (-1)^n$ (recall Example 2.10.2) and let $n_k = 2k$. Then $s_{n_k} = 1$ for all k. Note two important facts:

- The subsequence $(s_{n_k})_{k=0}^{\infty}$ is indexed by k, not n.
- The subsequence is constant and thus *convergent*.



Our main goal in this section is to prove the famous Bolzano–Weierstraß theorem (illustrated in the example): that every bounded sequence has a convergent subsequence.

Lemma 2.37. *If $\lim_{n\to\infty} s_n = s$, then every subsequence (s_{n_k}) satisfies $\lim_{k\to\infty} s_{n_k} = s$.*

Proof. Suppose s is finite and suppose $\epsilon > 0$ is given. Then $\exists N$ such that $n > N \implies |s_n - s| < \epsilon$. Since $n_k \geq k$ for all k, we see that

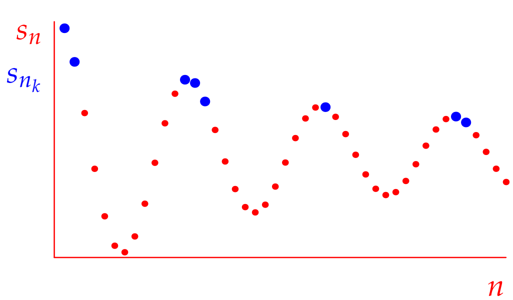
$$k > N \implies n_k > N \implies |s_{n_k} - s| < \epsilon$$

The case where $s = \pm\infty$ is an exercise. ∎

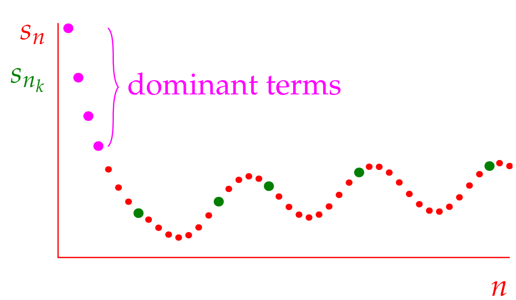
Lemma 2.38. *Every sequence has a monotonic subsequence.*

Proof. Given (s_n), we call the term s_n 'dominant' if $m > n \implies s_m < s_n$. There are two cases:

1. If there are infinitely many dominant terms, then the subsequence of such is monotone-down.
2. If there are finitely many dominant terms, choose s_{n_1} after all such. Since s_{n_1} is not dominant, $\exists n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Induct to obtain a monotone-up subsequence.



Case 1: monotone-down subsequence



Case 2: monotone-up subsequence ∎

Theorem 2.39. *Given a sequence (s_n), there exist subsequences (s_{n_k}) and (s_{n_l}) such that*

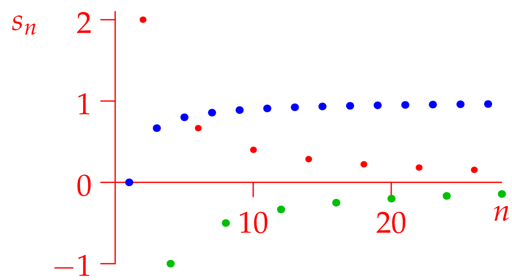
$$\lim s_{n_k} = \limsup s_n \quad \text{and} \quad \lim s_{n_l} = \liminf s_n$$

By the lemmas, we may moreover assume that these subsequences are monotonic.

Example 2.40. The picture shows the sequence with n^{th} term

$$s_n = \begin{cases} \frac{4}{n}(-1)^{\frac{n}{2}+1} & \text{when } n \text{ is even} \\ 1 - \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$$

Monotonic subsequences with limits $\limsup s_n = 1$ and $\liminf s_n = 0$ are indicated.



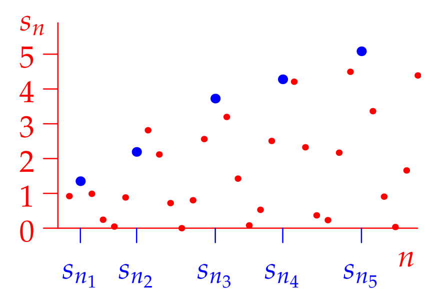
Proof. We prove only the lim sup claim, since the other is similar. There are three cases to consider; visualizing the third is particularly difficult and may take several readings.

($\limsup s_n = \infty$) Since (s_n) is unbounded above, for any $k > 0$ there exist *infinitely many* terms $s_n > k$. We may therefore inductively choose a subsequence (s_{n_k}) via

$$n_1 = \min\{n \in \mathbb{N} : s_{n_1} > 1\}$$
$$n_k = \min\{n \in \mathbb{N} : n_k > n_{k-1}, \text{ and } s_{n_k} > k\}$$

Choosing the minimum isn't necessary, though it keeps the subsequence explicit. Clearly

$$s_{n_k} > k \implies \lim_{k\to\infty} s_{n_k} = \infty = \limsup s_n$$



Example: $\limsup \frac{\sqrt{n}}{2}(1 + \sin n) = \infty$

($\limsup s_n = -\infty$) Since $\liminf s_n \le \limsup s_n = -\infty$, Lemma 2.31 says that $\lim s_n = -\infty$, whence (s_n) itself is a suitable subsequence.

($\limsup s_n = v$ finite) Let $n_1 = 1$ and define s_{n_k} for $k \ge 2$ inductively:

- Since (v_N) is monotone-down and converges to v, take $\epsilon = \frac{1}{2k}$ to see that[21]

$$\exists N_k > n_{k-1} \text{ such that } v \le v_{N_k} < v + \frac{1}{2k}$$

- Since $v_{N_k} = \sup\{s_n : n \ge N_k\}$, Lemma 1.20 says

$$\exists n_k \ge N_k \text{ such that } s_{n_k} > v_{N_k} - \frac{1}{2k}$$

But then $|v - s_{n_k}| \overset{\triangle}{\le} |v - v_{N_k}| + |v_{N_k} - s_{n_k}| < \frac{1}{k}$. The squeeze theorem says that $\lim_{k\to\infty} s_{n_k} = v$. ■

[21] (v_N) being monotone-down is crucial: if N satisfies $v_N - v < \frac{1}{2k}$, so does $N_k := \max(N, 1 + n_{k-1})$.

Example (2.40 cont.). The example shows why the two-step construction is necessary. It may seem that we should simply be able to choose subsequences of (u_N) and (v_N). Indeed,

$$(u_N) = (\underbrace{-1,-1,-1,-1}_{s_4}, \underbrace{-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}}_{s_8}, \underbrace{-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}}_{s_{12}},\ldots)$$

contains a subsequence of (s_n) converging to $\liminf s_n = 0$. Unfortunately, $(v_N) = (2,2,1,1,1,\ldots)$ does not contain a subsequence of (s_n). Taking $n_k = 2k+1$ ($k \geq 2$) results in the displayed sequence:

$$s_{n_k} = 1 - \frac{1}{2k+1} > 1 - \frac{1}{2k} = v_{N_k} - \frac{1}{2k}$$

There are two immediate corollaries. The first (see Exercise 3) fully establishes Theorem 2.31

$$\limsup s_n = \liminf s_n \iff \lim s_n \text{ exists} \qquad \text{(could be } \pm\infty\text{)}$$

The second is the main goal of this section.

Theorem 2.41 (Bolzano–Weierstraß). *Every bounded sequence has a convergent subsequence.*

Proof 1. Lemma 2.38 says there exists a monotone subsequence. This is bounded and thus converges by the monotone convergence theorem. ■

Proof 2. By Theorem 2.39, there exists a subsequence converging to the *finite* value $\limsup s_n$. ■

For a third proof(!) we present the classic 'shrinking-interval' argument which has the benefit of easily generalizing to higher dimensions.

Proof 3. Suppose (s_n) is bounded by M. One of the intervals $[-M,0]$ or $[0,M]$ must contain infinitely many terms of the sequence (perhaps both do!). Call this interval E_0 and choose any $n_0 \in E_0$.

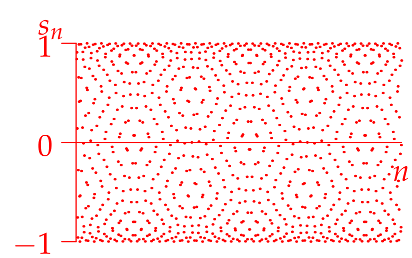
Split E_0 into left- and right half-intervals, one of which must contain infinitely many terms of the sequence for which $n > n_0$;[22] call this half-interval E_1 and choose any $s_{n_1} \in E_1$ with $n_1 > n_0$.

Repeat this *ad infinitum* to obtain a subsequence (s_{n_k}) and a family of nested intervals

$$[-M,M] \supset E_0 \supset E_1 \supset E_2 \supset \cdots \quad \text{of width} \quad |E_k| = \frac{M}{2^k} \quad \text{with} \quad s_{n_k} \in E_k$$

It remains only to see that (s_{n_k}) converges; we leave this to Exercise 5. ■

Example 2.42. $(s_n) = (\sin n)$ is bounded and therefore has a convergent subsequence! Its limit s must lie in the interval $[-1,1]$. The picture shows the first 1000 terms—remember that n is measured in *radians*. It is not at all clear from the picture what s or our mystery subsequence should be! There is a reason for this, as we'll see momentarily...



[22]Only *finitely many* terms in (s_n) could come before s_{n_0} ...

Subsequential Limits, Divergence by Oscillation & Closed Sets

Recall Definition 2.19: a sequence (s_n) *diverges by oscillation* if it neither converges nor diverges to $\pm\infty$. We can now give a positive statement of this idea.

$$\begin{aligned}(s_n) \text{ diverges by oscillation} &\overset{\text{Thm 2.31}}{\iff} \liminf s_n \neq \limsup s_n \\ &\overset{\text{Thm 2.39}}{\iff} (s_n) \text{ has subsequences tending to different limits}\end{aligned}$$

The word *oscillation* comes from the third interpretation: if $s_1 \neq s_2$ are limits of two subsequences, then any tail of the sequence $\{s_n : n > N\}$ contains infinitely many terms arbitrarily close to s_1 and infinitely many (other) terms arbitrarily close to s_2. The original sequence (s_n) therefore *oscillates* between neighborhoods of s_1 and s_2. Of course there could be many other *subsequential limits*…

Definition 2.43. We call $s \in \mathbb{R} \cup \{\pm\infty\}$ a *subsequential limit* of a sequence (s_n) if there exists a subsequence (s_{n_k}) such that $\lim\limits_{k\to\infty} s_{n_k} = s$.

Examples 2.44.
1. The sequence defined by $s_n = \frac{1}{n}$ has only one subsequential limit, namely zero. Recall Lemma 2.37: $\lim s_n = 0$ implies that every subsequence also converges to 0.
2. If $s_n = (-1)^n$, then the subsequential limits are ± 1.
3. The sequence $s_n = n^2(1 + (-1)^n)$ has subsequential limits 0 and ∞.
4. All positive even integers are subsequential limits of $(s_n) = (2, 4, 2, 4, 6, 2, 4, 6, 8, 2, 4, 6, 8, 10, \ldots)$.
5. (Hard!) Recall the countability of $\mathbb{Q}$ from a previous class: the standard argument enumerates the rationals by constructing a sequence

$$(r_n) = \Big(\frac{0}{1}, \underbrace{\frac{1}{1}, -\frac{1}{1}}_{|p|+q=2}, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}}_{|p|+q=3}, \underbrace{\frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}}_{|p|+q=4}, \underbrace{\frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1}}_{|p|+q=5}, \ldots\Big)$$

   We claim that the set of subsequential limits of (r_n) is in fact the set $\mathbb{R} \cup \{\pm\infty\}$!

   To see this, let $a \in \mathbb{R}$ be given and choose a subsequence (r_{n_k}) inductively:

   - By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Corollary 1.23), the set $S_n = \mathbb{Q} \cap (a - \frac{1}{n}, a + \frac{1}{n})$ contains infinitely many rational numbers and thus infinitely many terms of the sequence (r_n).
   - Choose any $r_{n_1} \in S_1$ and, for each $k \geq 2$, choose any

     $$r_{n_k} \in S_k \ \text{ such that } \ n_k > n_{k-1}$$

   - Since $|r_{n_k} - a| < \frac{1}{n_k} \leq \frac{1}{k}$, we conclude that $\lim\limits_{k\to\infty} r_{n_k} = a$.

   An argument for the subsequential limits $\pm\infty$ is in the Exercises. Somewhat amazingly, the *specific* sequence (r_n) is irrelevant: the conclusion is the same for *any* sequence enumerating $\mathbb{Q}$!
6. (Even harder—Example 2.42, cont.) We won't prove it, but the set of subsequential limits of $(s_n) = (\sin n)$ is the *entire interval* $[-1, 1]$! Otherwise said, for any $s \in [-1, 1]$ there exists a subsequence $(\sin n_k)$ such that $\lim\limits_{k\to\infty} \sin n_k = s$.

Theorem 2.45. *Let (s_n) be a sequence in $\mathbb{R}$ and let S be its set of subsequential limits. Then*

1. *S is non-empty (as a subset of $\mathbb{R} \cup \{\pm\infty\}$).*
2. *$\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.*
3. *$\lim s_n$ exists iff S has only one element: namely $\lim s_n$.*

Proof. 1. By Theorem 2.39, $\limsup s_n \in S$.

2. By part 1, $\limsup s_n \le \sup S$. For any convergent subsequence (s_{n_k}) we have $n_k \ge k$, whence

$$\forall N,\ \{s_{n_k} : k \ge N\} \subseteq \{s_n : n \ge N\} \implies \lim s_{n_k} = \limsup s_{n_k} \le \limsup s_n$$

Since this holds for *every* convergent subsequence, we have $\sup S \le \limsup s_n$ and therefore equality. The result for $\inf S$ is similar.

3. Applying Theorem 2.31, we see that $\lim s_n$ exists if and only if

$$\limsup s_n = \liminf s_n \iff \sup S = \inf S \iff S \text{ has only one element}$$

■

Closed Sets You should be comfortable with the notion of a *closed interval* (e.g. $[0,1]$) from elementary calculus. Sequences allow us to make a formal definition.

Definition 2.46. Let $A \subseteq \mathbb{R}$.

- We say that $s \in \mathbb{R}$ is a *limit point* of A if there exists a sequence $(s_n) \subseteq A$ converging to s.
- The *closure* $\overline{A}$ is the set of limit points of A. Plainly $A \subseteq \overline{A}$ for any set.
- A is *closed* if it equals its closure: $A = \overline{A}$.

Examples 2.47. 1. The interval $[0,1]$ is closed. If $(s_n) \subseteq [0,1]$ has $\lim s_n = s$, then

$$0 \le s_n \le 1 \overset{\text{Thm 2.11}}{\Longrightarrow} s \in [0,1]$$

More generally, every 'closed interval' $[a,b]$ is closed, as are *finite* unions of closed intervals, for instance $[1,5] \cup [7,11]$.

2. The 'half-open' interval $(0,1]$ is not closed: its closure is $\overline{(0,1]} = [0,1]$. In particular, the sequence $s_n = \frac{1}{n}$ lies in $(0,1]$, but $\lim s_n = 0 \notin (0,1]$.

3. Example 2.44.5 shows that the closure of the rational numbers is the reals: $\overline{\mathbb{Q}} = \mathbb{R}$.

Theorem 2.48. *If (s_n) is a sequence, then its set of (finite) subsequential limits is closed.*

We omit the proof since it involve unpleasantly many subscripts (subsequences of subsequences...).

Exercises 2.11. *Key concepts: Subsequences of a convergent sequence all have the same limit, Existence of (monotone) subsequences tending to* $\limsup s_n$ / $\liminf s_n$, *Subsequential Limits, Bolzano–Weierstraß: boundedness* $\Longrightarrow \exists$ *convergent subsequence*

1. Consider the sequences with the following n^{th} terms:

$$a_n = (-1)^n \qquad b_n = \frac{1}{n} \qquad c_n = n^2 \qquad d_n = \frac{6n+4}{7n-3}$$

For each sequence: state whether it converges, diverges to $\pm\infty$, or diverges by oscillation; give an example of a monotone subsequence; state the set of subsequential limits; state the limits superior and inferior.

2. Prove the case of Lemma 2.37 when $\lim s_n = \infty$

3. Suppose that $\lim s_n = s$ (could be $\pm\infty$). Use Theorem 2.39 and Lemma 2.37 to prove that $\limsup s_n = s = \liminf s_n$.

(*This completes the proof of Theorem 2.31*)

4. Suppose that $L = \lim s_n^2$ exists and is finite.

   (a) Given an example of such a sequence where (s_n) is *divergent.*

   (b) Prove that (s_n) contains a convergent *subsequence.* What are the possible limits of this subsequence? Why?
   (*Hint: use Bolzano–Weierstraß*)

5. Complete the third proof of Bolzano–Weierstraß (Theorem 2.41) by proving that the constructed subsequence (s_{n_k}) is Cauchy.

6. (a) Show that the closed interval $[a, b]$ is a closed set in the sense of Definition 2.46.

   (b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

7. By considering Example 2.47.2, or otherwise, show that an *infinite* union of closed intervals need not be closed.

8. Let (r_n) be any sequence enumerating of the set $\mathbb{Q}$ of rational numbers. Show that there exists a subsequence (r_{n_k}) such that $\lim\limits_{k\to\infty} r_{n_k} = +\infty$.

(*Hint: modify the argument in Example 2.44.5*)

9. (Hard) Let (s_n) be the sequence of numbers defined in the figure, listed in the indicated order.

   (a) Find the set S of subsequential limits of (s_n).

   (b) Determine $\limsup s_n$ and $\liminf s_n$.

$$\begin{matrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{matrix}$$

2.12 Lim sup and Lim inf

In this short section we collect a couple of useful results, mostly for later use. First we observe that the limit laws do not work as tightly for limits superior and inferior.

Theorem 2.49. *Let (s_n), (t_n) be bounded sequences. Then:*

1. $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$
2. *If, in addition, (s_n) is convergent to s, then we have equality*

$$\limsup(s_n + t_n) = s + \limsup t_n$$

Natural modifications can be made for infima and products of sequences (see Exercise 3).

Example 2.50. To convince yourself that equality is unlikely, consider $s_n = (-1)^n = -t_n$. Plainly

$$\limsup(s_n + t_n) = 0 < 2 = \limsup s_n + \limsup t_n$$

Proof.

1. For each N, the set $\{s_n + t_n : n \ge N\}$ is bounded above by

$$\sup\{s_n : n \ge N\} + \sup\{t_n : n \ge N\}$$

from which

$$\sup\{s_n + t_n : n \ge N\} \le \sup\{s_n : n \ge N\} + \sup\{t_n : n \ge N\}$$

Take limits as $N \to \infty$ for the first result.

2. By part 1, we know that

$$\limsup(s_n + t_n) \le s + \limsup t_n$$

For the other direction, rearrange and apply part 1 again:

$$\begin{aligned}\limsup t_n &= \limsup\big((s_n + t_n) - s_n\big) \le \limsup(s_n + t_n) + \limsup(-s_n)\\ &= \limsup(s_n + t_n) - s\end{aligned}$$

■

The next result will be critical when we study infinite series, particularly the ratio and root tests.

Theorem 2.51. *Let (s_n) be a non-zero sequence. Then*

$$\liminf\left|\frac{s_{n+1}}{s_n}\right| \le \liminf|s_n|^{1/n} \le \limsup|s_n|^{1/n} \le \limsup\left|\frac{s_{n+1}}{s_n}\right|$$

In particular, $\lim\left|\frac{s_{n+1}}{s_n}\right| = L \Longrightarrow \lim|s_n|^{1/n} = L.$ (†)

Examples 2.52. 1. Here is a quick proof that $\lim n^{1/n} = 1$ (recall Theorem 2.17). Let $s_n = n$, then

$$\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim \frac{n+1}{n} = 1 \implies \lim n^{1/n} = \lim |s_n|^{1/n} = 1$$

2. Apply the corollary to $s_n = n!$ to see that

$$\lim(n!)^{1/n} = \lim \left|\frac{s_{n+1}}{s_n}\right| = \lim(n+1) = \infty$$

Proof. We prove the third inequality. Assume $\limsup \left|\frac{s_{n+1}}{s_n}\right| = L \neq \infty$ (otherwise the inequality is trivial). Suppose $\epsilon > 0$ is given, and denote $a = L + \epsilon$. Then

$$L = \lim_{N\to\infty} \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \geq N\right\} < a \implies \exists N \text{ such that } \sup\left\{\left|\frac{s_{n+1}}{s_n}\right| : n \geq N\right\} < a$$

Now let $b = a^{-N}|s_N|$. For any $n \geq N$, we therefore have $\left|\frac{s_{n+1}}{s_n}\right| < a$, whence

$$\begin{aligned} n > N &\implies |s_n| < a^{n-N}|s_N| \implies |s_n|^{1/n} < a\left(a^{-N}|s_N|\right)^{1/n} = ab^{1/n} \\ &\implies \limsup |s_n|^{1/n} \leq a \lim b^{1/n} = a = L + \epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude the third inequality: $\limsup |s_n|^{1/n} \leq L$.
The second inequality is trivial and the first is similar to the third. ∎

Exercises 2.12. *Key concepts: "Limit laws" for* $\limsup$ *and* $\liminf$, $\lim \left|\frac{s_{n+1}}{s_n}\right| = \lim |s_n|^{1/n}$

1. Compute $\lim \frac{1}{n}(n!)^{1/n}$
   (*Hint: let* $s_n = \frac{n!}{n^n}$ *in Theorem 2.51 and recall that* $\lim\left(1+\frac{1}{n}\right)^n = e$)

2. Evaluate $\lim\left(\frac{(2n)!}{(n!)^2}\right)^{1/n}$

3. Let (s_n) and (t_n) be non-negative, bounded sequences.
   (a) Prove that $\limsup(s_n t_n) \leq (\limsup s_n)(\limsup t_n)$
   (b) Give an example which shows that we do not expect equality in part (a).
   (c) If, in addition, $\lim s_n = s$, prove that $\limsup(s_n t_n) = s \limsup t_n$.

4. Consider the sequence with $s_{2m} = s_{2m+1} = 2^{-m}$:

$$(s_n)_{n=0}^{\infty} = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots\right)$$

   Compute $|s_n|^{1/n}$ and $\left|\frac{s_{n+1}}{s_n}\right|$ when n is even and then when it is odd. Thus find all expressions in Theorem 2.51 and conclude that the converse of (†) is *false.*