# 3 Series

## 3.14 Infinite Series and the Series Tests

For millennia, certainly since Zeno's paradoxes of c. 430 BC, mathematicians have been interested in the meaning and evaluation of infinite sums such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The standard approach in modern mathematics is to outsource the definition to that of *limits.*

**Definition 3.1.** The $n^{th}$ *partial sum* $s_n$ of a sequence $(a_n)_{n=m}^{\infty}$ is the **finite sum**

$$s_n := \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

- The *(infinite) series*[23] $\sum\limits_{n=m}^{\infty} a_n$ is the limit $\lim s_n$ of the sequence $(s_n)$ of partial sums.
- A series *converges*, *s to* $\pm\infty$ or *diverges by oscillation* as does the sequence $(s_n)$.
- $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.
- $\sum a_n$ *converges conditionally* if it converges but not absolutely ($\sum |a_n|$ diverges to $\infty$).

We don't (yet) know whether our motivating example converges, but at least we have a meaning:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim s_n \quad \text{where} \quad s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2}$$

**Theorem 3.2 (Basic Series Laws).** *Infinite series behave nicely with respect to addition and scalar multiplication. For instance:*

1. *If $\sum a_n$ is convergent and $k$ is constant, then $\sum ka_n = k \sum a_n$ is convergent.*
2. *If $\sum a_n$ and $\sum b_n$ are convergent, then $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$ are also convergent.*
3. *If $\sum a_n = \infty$ and $k > 0$, then $\sum ka_n = \infty$.*
4. *If $\sum a_n = \infty$ and $\sum b_n$ converges, then $\sum (a_n + b_n) = \infty$.*

*Proof.* Simply apply the limit/divergence laws to the sequence of partial sums. E.g. for 1,

$$\sum ka_n = \lim_{n\to\infty} \sum_{j=m}^{n} ka_j \overset{\substack{\text{finite}\\ \text{sum}}}{=} \lim_{n\to\infty} k \sum_{j=m}^{n} a_j \overset{\substack{\text{limit}\\ \text{laws}}}{=} k \lim_{n\to\infty} \sum_{j=m}^{n} a_j = k \sum a_n$$

The others may be proved similarly. ∎

Series **do not** behave nicely with respect to multiplication (see also Exercise 3):

$$a_1b_1 + a_2b_2 + \cdots = \sum a_n b_n \neq \left(\sum a_n\right)\left(\sum b_n\right) = \left(a_1 + a_2 + \cdots\right)\left(b_1 + b_2 + \cdots\right)$$

[23] If the initial term is understood or is irrelevant to the situation, it is common to write $\sum a_n$.

### Series which may be evaluated exactly

Given a series $\sum a_n$, our primary goal is to answer a simple question: "Does it converge?" Even when the answer is *yes*, a precise computation of the limit will usually be beyond us. We instead develop techniques (the upcoming *series tests*) which typically rely on comparing $\sum a_n$ to some 'standard' series whose properties are completely understood: in particular...

**Definition 3.3 (Geometric series).** A sequence $(a_n)$ is *geometric* if the ratio of successive terms is constant: $a_n = ba^n$ for some constants $a, b$. A *geometric series* is the sum of a geometric sequence.

The computation of the sequence of partial sums should be familiar (for simplicity assume $b = 1$)

$$(1-a)s_n = \left(a^m + a^{m+1} + \cdots + a^n\right) - \left(a^{m+1} + a^{m+2} + \cdots + a^n + a^{n+1}\right) = a^m - a^{n+1}$$

from which we quickly conclude:

**Theorem 3.4.** *Suppose $a$ is constant. Then*

$$s_n = \sum_{k=m}^{n} a^k = \begin{cases} \dfrac{a^m - a^{n+1}}{1-a} & \text{if } a \neq 1 \\ n+1-m & \text{if } a = 1 \end{cases} \implies \sum_{n=m}^{\infty} a^n \begin{cases} \text{converges to } \dfrac{a^m}{1-a} & \text{if } |a| < 1 \\ \text{diverges to } \infty & \text{if } a \geq 1 \\ \text{diverges by oscillation} & \text{if } a \leq -1 \end{cases}$$

*In particular, $\sum a^n$ converges absolutely if $|a| < 1$ and diverges otherwise.*

**Examples 3.5.** 1. $\displaystyle\sum_{n=-1}^{\infty} 2\left(-\frac{4}{5}\right)^n = 2\frac{\left(-\frac{4}{5}\right)^{-1}}{1+\frac{4}{5}} = -\frac{5}{2}\cdot\frac{5}{9} = -\frac{25}{18}$

2. Consider the series $\sum a_n = \sum\limits_{n=3}^{\infty} \left(\frac{2}{5}\right)^n + 2^n$. If this were convergent, then

$$\sum 2^n = \sum a_n - \sum \left(\frac{2}{5}\right)^n$$

would converge (Theorem 3.2); a contradiction.

**Telescoping series** A rarer type of series can be evaluated using the algebra of partial fractions.

**Example 3.6.** To compute $\sum\limits_{n=1}^{\infty} \frac{1}{n(n+1)}$, first observe that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim\left(1 - \frac{1}{n+1}\right) = 1$$

Similar arguments can be made for other series such as $\sum \frac{1}{n(n+2)}$.

## The Cauchy Criterion

The starting point for our general series tests uses Cauchy completeness.

**Example 3.7.** Consider again the series $\sum \frac{1}{n^2}$. We show that the sequence of partial sums $(s_n)$ is Cauchy. Suppose $\epsilon > 0$ is given and let $N = \frac{1}{\epsilon}$. Then,

$$\begin{aligned} m > n > N \implies |s_m - s_n| &= \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} \\ &= \frac{1}{n} - \frac{1}{m} < \frac{1}{N} = \epsilon \end{aligned}$$

where most terms cancel analogous to the telescoping series approach. By Cauchy completeness (Theorem 2.34), $(s_n)$ converges and we conclude

$$\boxed{\sum \frac{1}{n^2} \text{ converges}}$$

Computing the value of this series is significantly harder: a sketch argument for why $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is in Exercise 10.

**Theorem 3.8 (Cauchy criterion for series).** *A series $\sum a_n$ converges precisely when*

$$\forall \epsilon > 0,\ \exists N \text{ such that } m > n > N \implies |s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$

The previous example essentially verified the Cauchy criterion for $\sum \frac{1}{n^2}$.

*Proof.* Let $(s_n)$ be the sequence of partial sums. Then

$$\begin{aligned} \sum a_n \text{ converges} &\iff (s_n) \text{ converges} \\ &\iff (s_n) \text{ is a Cauchy sequence} && \text{(Theorem 2.34)} \\ &\iff \Big( \forall \epsilon > 0,\ \exists N \text{ such that } m > n > N \implies |s_m - s_n| < \epsilon \Big) \end{aligned}$$

■

**Example 3.9.** For contradiction, suppose that the *harmonic series* $\sum \frac{1}{n}$ converges. Take $\epsilon = \frac{1}{2}$ in the Cauchy criterion to observe that

$$\exists N \text{ such that } m > n > N \implies \left| \sum_{k=n+1}^{m} \frac{1}{k} \right| < \frac{1}{2}$$

However, taking $m = 2n$ (plainly $m > n$ since $n > N \geq 1$) results in a contradiction:

$$\frac{1}{2} > \left| \sum_{k=n+1}^{m} \frac{1}{k} \right| = \left| \frac{1}{n+1} + \cdots + \frac{1}{m} \right| \geq \frac{m-n}{m} = 1 - \frac{n}{m} = \frac{1}{2}$$

We conclude that the harmonic series diverges to $\infty$.

### The Series Tests

For the remainder of this section we develop tests for the convergence/divergence of an infinite series: the $n^{\text{th}}$-term, comparison, root and ratio tests. The first follows quickly from the Cauchy criterion.

**Theorem 3.10 (Divergence/$n^{\text{th}}$-term test).** *If $\lim a_n \neq 0$ then $\sum a_n$ is divergent.*

*Proof.* We prove the contrapositive. Suppose $\sum a_n$ is convergent, and that $\epsilon > 0$ is given. Take $m = n+1$ in the Cauchy criterion. Then

$$\exists \tilde{N} \text{ such that } m > \tilde{N} \implies |a_m| < \epsilon \qquad (\text{let } \tilde{N} = N+1)$$

Otherwise said, $\lim a_n = 0$. ∎

**Examples 3.11.** 1. The series $\sum \sin(\frac{n\pi}{9})$ diverges.

2. The $n^{\text{th}}$-term test tells us that the geometric series $\sum a^n$ diverges whenever $|a| \geq 1$. We still need our earlier analysis (Theorem 3.4) for when $|a| < 1$.

3. The **converse** of the $n^{\text{th}}$-term test is **false!** Example 3.9 provides the canonical example: the **divergent** harmonic series $\sum \frac{1}{n}$ also satisfies $\lim \frac{1}{n} = 0$.

**Theorem 3.12 (Comparison test).** 1. *Let $\sum b_n$ be a convergent series of non-negative terms and assume $|a_n| \leq b_n$ for all (large) $n$. Then both $\sum a_n$ and $\sum |a_n|$ are convergent.*

2. *If $\sum a_n = \infty$ and $a_n \leq b_n$ for all (large) $n$, then $\sum b_n = \infty$.*

*Proof.* Suppose "large $n$" means $n > M$ for some fixed $M$.

1. Let $\epsilon > 0$ be given. Since $\sum b_n$ converges, $\exists N \geq M$ such that

$$m > n > N \implies \left| \sum_{k=n+1}^{m} a_k \right| \overset{\triangle}{\leq} \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} b_k < \epsilon$$

2. The $n^{\text{th}}$ partial sum (post $M$) of $\sum b_n$ is

$$\sum_{k=M+1}^{n} b_k \geq \sum_{k=M+1}^{n} a_k \to \infty$$

∎

**Corollary 3.13.** 1. *(Absolute convergence implies convergence) Take $|a_n| = b_n$ in part 1 to see that $\sum |a_n|$ convergent $\implies \sum a_n$ convergent.*

2. *(Estimation of series) Suppose $\sum b_n$ is a convergent series of non-negative terms and that $|a_n| \leq b_n$ for **all** $n$. Then*

$$\sum a_n \leq \sum |a_n| \leq \sum b_n$$

**Examples 3.14.** 1. Since the geometric series $\sum \frac{2}{3^n}$ converges, $\frac{2n+1}{(n+2)3^n} \le \frac{2}{3^n}$, we see that

$$\sum_{n=0}^{\infty} \frac{2n+1}{(n+2)3^n} \le 2\sum_{n=0}^{\infty} 3^{-n} = \frac{2}{1-\frac{1}{3}} = 3$$

That is, the first series converges (absolutely) to some value $\le 3$.

2. One can sometimes find a sensible comparison series by considering how $a_n$ behaves for large $n$. For instance, when $n$ is large, $a_n = \frac{(n^2+1)^{1/2}}{(1+\sqrt{n})^4}$ behaves like $\frac{n}{n^2} = \frac{1}{n}$. Indeed, when $n \ge 2$,

$$a_n > \frac{n}{(1+\sqrt{n})^4} > \frac{n}{(2\sqrt{n})^4} = \frac{1}{16n}$$

Comparison with the divergent series $\frac{1}{16}\sum \frac{1}{n}$ shows that $\sum a_n$ also diverges to $\infty$.

3. Since $\ln n < n \implies \frac{1}{\ln n} > \frac{1}{n}$, we see that $\sum \frac{1}{\ln n}$ diverges to $\infty$ by comparison with $\sum \frac{1}{n}$.

4. $\sum \frac{\sin n}{n^2}$ converges absolutely by comparison to $\sum \frac{1}{n^2}$ (Example 3.7). Corollary 3.13 estimates its value ($\le \frac{\pi^2}{6}$):

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \le \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \le \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(approximately } 1.014 \le 1.280 \le 1.645\text{)}$$

5. The *alternating harmonic series* $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges via a sneaky comparison.

The series $t = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$ converges by comparison with $\sum \frac{1}{4(n-1)^2}$. Its $n^{\text{th}}$ partial sum

$$t_n = \sum_{k=1}^{n} \frac{1}{2k(2k-1)} = \sum_{k=1}^{n} \left[\frac{1}{2k-1} - \frac{1}{2k}\right]$$

is precisely the *even* partial sum of the alternating harmonic series $s_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$. Plainly $\lim s_{2n} = t$. Moreover $s_{2n+1} = s_{2n} + \frac{1}{2n+1} \implies \lim s_{2n+1} = t \implies \lim s_n = t$. Since the harmonic series $\sum \frac{1}{n}$ diverges (Example 3.9), we conclude that the alternating harmonic series **converges conditionally**. We'll revisit this discussion in the next section.

6. $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges by comparison with $\sum 2^{-n}$. To see this, recall Exercise 2.10.10:

$$\left(\frac{n}{n+1}\right)^n = \frac{n+1}{n}\left(1 - \frac{1}{n+1}\right)^{n+1} \xrightarrow[n\to\infty]{} e^{-1}$$

Plainly $e^{-1} < \frac{1}{2}$, whence for large $n$,

$$\left(\frac{n}{n+1}\right)^n \le \frac{1}{2} \implies \left(\frac{n}{n+1}\right)^{n^2} \le 2^{-n}$$

In fact $\left(\frac{n}{n+1}\right)^n$ is monotone-down, whence $e^{-1} \le \left(\frac{n}{n+1}\right)^n \le \frac{1}{2}$ **for all** $n$, and so

$$0.58198 \approx \frac{e^{-1}}{1-e^{-1}} = \sum_{n=1}^{\infty} e^{-n} \le \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \le \sum_{n=1}^{\infty} 2^{-n} = \frac{1/2}{1-1/2} = 1$$

A computer estimate yields $\sum\limits_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \approx 0.8174$.

Our last two tests in this section are less powerful but often easier to use.

**Theorem 3.15 (Root test).** *Suppose* $\limsup |a_n|^{1/n} = L$*.*

1. *If* $L < 1$*, then* $\sum a_n$ *converges absolutely.*
2. *If* $L > 1$*, then* $\sum a_n$ *diverges.*

*If* $L = 1$*, then no conclusion can be drawn.*

We defer the proof until after some examples. By combining with the inequalities of Theorem 2.51

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

we obtain a second familiar test.

**Corollary 3.16 (Ratio test).** *Suppose* $(a_n)$ *is a sequence of non-zero terms.*

1. *If* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$*, then* $\sum a_n$ *converges absolutely.*
2. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$*, then* $\sum a_n$ *diverges.*

In elementary calculus you likely saw the special cases when

$$L = \lim |a_n|^{1/n} = \lim \left|\frac{a_{n+1}}{a_n}\right|$$

Our versions are more general since these limits *might not exist.*

**Examples 3.17.** 1. The ratio test is particularly useful for series involving *factorials* and *exponentials.*

   (a) $\sum \frac{n^4}{2^n}$ converges, since $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)^4}{2n^4} = \frac{1}{2} < 1$.

   (b) $\sum \frac{n!}{2^n}$ diverges, since $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)!}{2n!} = \lim \frac{n+1}{2} = \infty$.

2. Both tests are inconclusive for rational sequences: if $a_n = \frac{b_n}{c_n}$ where $b_n, c_n$ are polynomials, then

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = 1 = \lim |a_n|^{1/n}$$

   For example, attempting to apply the ratio test to $\sum \frac{n+5}{n^2}$ results in

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+6)n^2}{(n+5)(n+1)^2} = 1$$

   This series is divergent by comparison with the harmonic series $\sum \frac{1}{n}$.

3. In Example 3.14.6, our use of the comparison test was really the root test in disguise:

$$a_n = \left(\frac{n}{n+1}\right)^{n^2} \implies \lim |a_n|^{1/n} = \lim \left(\frac{n}{n+1}\right)^n = e^{-1} < 1 \implies \sum a_n \text{ converges}$$

   In this case the root test was much easier to apply.

4. The ratio test is the weakest test thus far; certainly it does not apply if any of the terms $a_n$ are zero! For a more subtle example of its failure, consider

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even} \\ 3^{-n} & \text{if } n \text{ is odd} \end{cases} \implies \frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^n & \text{if } n \text{ is even} \\ \frac{1}{2}\left(\frac{3}{2}\right)^n & \text{if } n \text{ is odd} \end{cases}$$
$$\implies \liminf \left|\frac{a_{n+1}}{a_n}\right| = 0, \quad \limsup \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

The ratio test is therefore inconclusive. However, applying the root test it almost trivial!

$$|a_n|^{1/n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{3} & \text{if } n \text{ is odd} \end{cases} \implies \limsup |a_n|^{1/n} = \frac{1}{2} < 1 \implies \sum a_n \text{ converges}$$

We need not even have used the root test: $\sum a_n$ plainly converges by comparison with $\sum 2^{-n}$!

For a precise value, note that the sequence of $n^{\text{th}}$ partial sums converges monotone up to the sum of two geometric series,

$$\sum_{n=0}^{\infty} a_n = \sum_{k=0}^{\infty} 2^{-2k} + 3^{-2k-1} = \frac{1}{1-1/4} + \frac{1/3}{1-1/9} = \frac{11}{8}$$

*Proof of the Root Test.* 1. Suppose $L < 1$. Choose any $\epsilon > 0$ such that $L + \epsilon < 1$ (say $\epsilon = \frac{1-L}{2}$). Since $v_N = \sup\{|a_n|^{1/n} : n \geq N\}$ defines a *monotone-down* sequence converging to $L$, we see that

$$\exists N \text{ such that } v_N - L < \epsilon$$

But then

$$n \geq N \implies |a_n|^{1/n} - L < \epsilon \implies |a_n| < (L+\epsilon)^n$$

$\sum |a_n|$ therefore converges by comparison with the geometric series $\sum (L+\epsilon)^n$.

2. If $L > 1$ then there exists some subsequence $(a_{n_k})$ such that $|a_{n_k}|^{1/n_k} \to L > 1$. In particular, infinitely many terms of this subsequence must be greater than 1 and so $(a_n)$ does not converge to zero. $\sum a_n$ thus diverges by the $n^{\text{th}}$-term test.

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**Summary** The logical flow of the tests in this section is as follows:

$$\begin{array}{lccccc}
\text{(divergence testing)} & \text{Comparison} & & n^{\text{th}}\text{-term} \Longrightarrow \text{Root} \Longrightarrow \text{Ratio} \\
& \Uparrow & & \Uparrow \\
& \text{Definition of } \sum a_n & \Longleftrightarrow & \text{Cauchy criterion} \\
& & & \Downarrow \\
\text{(convergence testing)} & & & \text{Comparison} \Longrightarrow \text{Root} \Longrightarrow \text{Ratio}
\end{array}$$

The ratio test is typically the easiest to use, but the least powerful. Every series which converges by the ratio test can be seen to converge by the root and comparison tests, and the Cauchy criterion. If you find that a series diverges by the ratio test, you could have just used the $n^{\text{th}}$-term test!

**Exercises 3.14.** *Key concepts: Infinite series, Cauchy criterion, Comparison/root/ratio tests*

1. Determine which of the following series converge. Justify your answers.

   (a) $\sum \frac{n-1}{n^2}$ (b) $\sum(-1)^n$ (c) $\sum \frac{3^n}{n^3}$ (d) $\sum \frac{n^3}{3^n}$ (e) $\sum \frac{n^2}{n!}$

   (f) $\sum \frac{1}{n^n}$ (g) $\sum \frac{n}{2^n}$ (h) $\sum \frac{n!}{n^n}$ (i) $\sum\limits_{n=2}^{\infty} \left[n+(-1)^n\right]^{-2}$ (j) $\sum\left[\sqrt{n+1}-\sqrt{n}\right]$

2. Let $\sum a_n$ and $\sum b_n$ be convergent series of non-negative terms. Prove that $\sum \sqrt{a_n b_n}$ converges. (*Hint: start by showing that* $\sqrt{a_n b_n} \leq a_n + b_n$)

3. (a) If $\sum a_n$ converges absolutely, prove that $\sum a_n^2$ converges.
   (b) More generally, if $\sum |a_n|$ converges and $(b_n)$ is a bounded sequence, prove that $\sum a_n b_n$ converges absolutely.

4. Find a series $\sum a_n$ which diverges by the root test but for which the ratio test is inconclusive.

5. Suppose $\liminf |a_n| = 0$. Prove that there is a subsequence $(a_{n_k})$ such that $\sum a_{n_k}$ converges. (*Hint: Try to construct a subsequence which converges to zero faster than* $\frac{1}{k^2}$.

6. Prove that the harmonic series $\sum \frac{1}{n}$ diverges by comparing with the series $\sum a_n$, where

$$(a_n) = \left(1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{32}, \ldots\right)$$

7. Suppose $b_n \leq a_n$ for all $n$ and that $\sum b_n$ and $\sum a_n$ converge. Prove that $\sum b_n \leq \sum a_n$. (*This also proves part 2 of Corollary 3.13*)

8. Given $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the values of $\sum \frac{1}{(2n)^2}$, $\sum \frac{1}{(2n+1)^2}$ and $\sum \frac{(-1)^{n+1}}{n^2}$.

9. The *limit comparison test* states:

   Suppose $\sum a_n, \sum b_n$ are series of positive terms and that $L = \lim \frac{a_n}{b_n} \in (0, \infty)$. Then the series have the same convergence status (both converge or both diverge to $\infty$).

   (a) Use the limit comparison test with $b_n = \frac{1}{n^2}$ to show that the series $\sum \frac{1}{n} \ln\left(1+\frac{1}{n}\right)$ converges. (*Hint: Recall that* $e = \lim\left(1+\frac{1}{n}\right)^n$)
   (b) Prove the limit comparison test. (*Hint: first show that* $\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$ *for large* $n$)
   (c) What can you say about the series $\sum a_n$ and $\sum b_n$ if $L = 0$ or $L = \infty$? Explain.

10. Euler asserted that the sine function, written as an infinite polynomial in the form of a Maclaurin series, could also be expressed as an infinite product,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

   By considering the solutions to $\sin x = 0$, give some weight to Euler's claim. By comparing coefficients in these expressions, deduce the fact $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$. (*As presented, this argument is non-rigorous!*)

## 3.15 The Integral and Alternating Series Tests

In this section we develop two further standalone series tests, both with narrower applications than our previous tests.

The first is a little out of place given that it requires (improper) integration.[24]

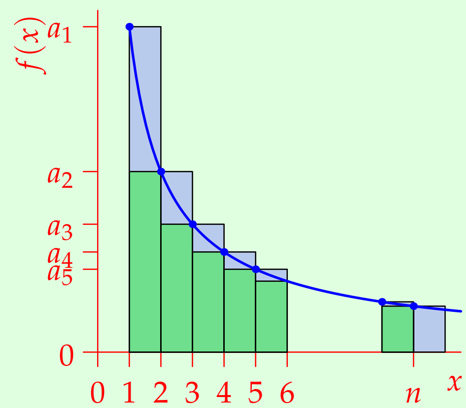
**Theorem 3.18 (Integral test).** *Let $a_n = f(n)$, where $f$ is non-negative, non-increasing, and integrable on $[1, \infty)$. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,\mathrm{d}x \text{ converges}$$

*In such a situation,*

$$\int_1^{\infty} f(x)\,\mathrm{d}x \le \sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,\mathrm{d}x$$

*The statement is easily modified if the initial term is not $a_1$.*




*Proof.* We need only interpret the picture as describing upper and lower Riemann sums:

$$\int_1^{n+1} f(x)\,\mathrm{d}x \le \sum_{k=1}^{n} a_k = s_n = a_1 + \sum_{k=2}^{n} a_k \le a_1 + \int_1^{n} f(x)\,\mathrm{d}x \tag{$*$}$$

Take limits as $n \to \infty$ for the result. ∎

Even for divergent sums, $(*)$ allows us to estimate the growth of $(s_n)$. For greater accuracy, we may evaluate the first few terms explicitly and modify the integral test to estimate the remainder.

An important application of the integral test is to provide a complete description of the convergence status of $p$-**series**: a useful family of series to which others may be compared.

**Corollary 3.19 ($p$-series).** *Let $p > 0$. The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.*

**Examples 3.20.** 1. $\sum \frac{1}{n^3}$ converges (it is a $p$-series with $p > 1$). Moreover,

$$\int_1^{\infty} \frac{1}{x^3}\,\mathrm{d}x = \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^b = \frac{1}{2} \implies \frac{1}{2} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le \frac{3}{2}$$

This is a poor estimate, particularly the lower bound. For a quick improvement, evaluate the first term and re-run the test starting at $n = 2$:

$$1 + \int_2^{\infty} \frac{1}{x^3}\,\mathrm{d}x \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1 + \frac{1}{8} + \int_2^{\infty} \frac{1}{x^3}\,\mathrm{d}x \implies 1 + \frac{1}{8} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1 + \frac{1}{4}$$

If greater accuracy is required, more terms can be explicitly evaluated.

[24] Which in turn requires limits of functions: $\int_1^{\infty} f(x)\,\mathrm{d}x := \lim_{b\to\infty} \int_1^b f(x)\,\mathrm{d}x$. While we haven't rigorously developed these concepts, the relevant computations should be familiar from elementary calculus.

2. In Example 3.9, we used the Cauchy criterion to show that the harmonic series diverges to $\infty$. The integral test makes this much easier and allows us to estimate how many terms are required for the partial sum to $s_n$ to reach a certain threshold: 10 say. Since

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \le s_n = \sum_{k=1}^{n} \frac{1}{k} \le 1 + \int_1^{n} \frac{1}{x}\,\mathrm{d}x = 1 + \ln n$$

we see that $s_n \approx 10$ requires

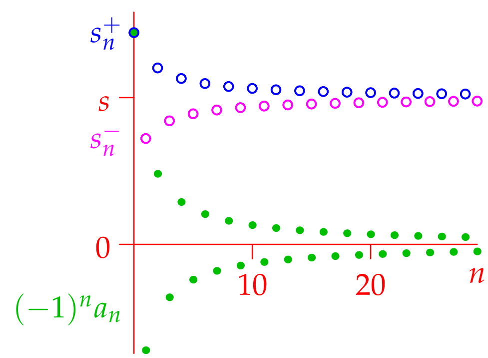
$$\ln(n+1) \le 10 \le 1 + \ln n \implies e^9 \le n \le e^{10} - 1 \implies 8104 \le n \le 22025$$

The harmonic series diverges to infinity, but it does so *very slowly.*

3. The integral test shows that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$. To exceed 10, somewhere between $10^{3223}$ and $10^{6631}$ terms are required. To exceed 100 requires 'roughly' $10^{6\times 10^{42}}$ terms: 1 followed by 1000 zeros for each water molecule in Lake Tahoe puts you at least in the right ballpark. . .

4. The series $\sum \frac{2n+1}{\sqrt{4n^3-1}}$ diverges to $\infty$ by comparison with the $p$-series $\sum \frac{1}{\sqrt{n}}$.

**Alternating Series and Conditional Convergence**




Our final test is unique in that it can detect *conditional convergence.* The canonical example is the *alternating harmonic series* (Example 3.14.5). With an eye on generalization, we re-index so that the first term is $a_0 = 1$:

$$\begin{aligned} s &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots \end{aligned}$$

The *alternating* $\pm$-signs give the series its name. Consider the behavior of the sequence of partial sums $(s_n)$, in particular two subsequences $(s_n^+) = (s_{2n})$ and $(s_n^-) = (s_{2n-1})$:

$$s_n^+ = \sum_{k=0}^{2n} (-1)^k a_k = 1 - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_1 - a_2} - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_3 - a_4} - \cdots - \underbrace{\left(\frac{1}{2n} - \frac{1}{2n+1}\right)}_{a_{2n-1} - a_{2n}} \qquad (n \ge 0)$$

$$s_n^- = \sum_{k=0}^{2n-1} (-1)^k a_k = \underbrace{\left(1 - \frac{1}{2}\right)}_{a_0 - a_1} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{a_2 - a_3} + \cdots + \underbrace{\left(\frac{1}{2n-1} - \frac{1}{2n}\right)}_{a_{2n-2} - a_{2n-1}} \qquad (n \ge 1)$$

Each bracketed term is non-negative, so $(s_n^+)$ is monotone-down and $(s_n^-)$ monotone-up. Moreover,

$$\frac{1}{2} = s_1^- \le s_n^- \le s_n^- + a_{2n} = s_n^+ \le s_0^+ = 1 \qquad (\dagger)$$

from which both subsequences are *bounded* and thus *convergent.* Not only this, but

$$\lim(s_n^+ - s_n^-) = \lim a_{2n} = 0$$

shows that the limits of both subsequences are *identical* (of course both are $s$).

The above discussion depends only on two simple properties of the sequence $(a_n)$. We've therefore proved a general statement.

**Theorem 3.21 (Alternating series test).** *Suppose $(a_n)$ is monotone-down and that $\lim a_n = 0$. Then:*

1. *The alternating series $s = \sum(-1)^n a_n$ converges.*
2. *If $(s_n)$ is the sequence of partial sums, then $|s - s_n| \le a_{n+1}$.*

Think about where the hypotheses regarding $(a_n)$ are used in the proof.
It can be shown that the alternating harmonic series converges to $\ln 2$, though the estimates provided by the alternating series test are very poor: to guarantee accuracy to two decimal place requires us to sum 100 terms of the series!

**Examples 3.22.** 1. Since $a_n = \frac{1}{n!}$ converges monotone-down to zero, the alternating series $\sum \frac{(-1)^n}{n!}$ converges. By evaluating $s_8$ and $s_9$ explicitly, we see that

$$0.3678791887\ldots \le \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \le 0.3678819444\ldots$$

yielding an estimate of 0.36788 to 5 decimal places (the exact value is in fact $e^{-1}$). The alternating series test is only needed for the estimate, since the series converges absolutely.

2. The series $\sum_{n=2}^{\infty} \frac{\sin \frac{\pi}{2} n}{\ln n}$ can be viewed as an alternating series since every *even* term is zero. Writing $m = 2n+1$, we obtain

$$\sum_{n=2}^{\infty} \frac{\sin \frac{\pi}{2} n}{\ln n} = \sum_{m=1}^{\infty} \frac{\sin(\pi m + \frac{\pi}{2})}{\ln(2m+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m}{\ln(2m+1)}$$

Since $\frac{1}{\ln(2m+1)}$ decreases to zero, the alternating series test demonstrates convergence.

### Rearranging Infinite Series

A *rearrangement* of an infinite series $\sum a_n$ is a series that results from changing the *order* of the terms of the sequence $(a_n)$ *before* computing the partial sums. The new series must still use every term of the original. Since the new sequence of partials sums is likely different, we shouldn't assume that the rearranged series has the same convergence properties as the old.

**Example 3.23.** We rearrange the alternating harmonic series by summing two positive terms before each negative term:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} + \cdots$$

Every term of the original sequence is used here, so this is a genuine rearrangement. It is perhaps surprising to discover that the new series converges, though its limit is *not the same* as the original alternating harmonic series! We leave the details to Exercise 11. This behavior is quite different to that of finite sums, where the order of summation makes no difference at all.

The general situation is summarized in a famous result of Riemann. The first part says that absolutely convergent series behave just like finite sums. Conditionally convergent series are much stranger.[25]

**Theorem 3.24 (Riemann rearrangement).**
1. *If a series converges absolutely, then all rearrangements converge to the same limit.*
2. *If a series converges conditionally and $s \in \mathbb{R} \cup \{\pm\infty\}$ is given, then there exists a rearrangement which tends to $s$.*

We omit the proofs since they are prohibitively lengthy. Instead we illustrate the rough idea of part 2 via an example.

**Example 3.25.** We show how to construct a rearrangement of the alternating harmonic series which converges to $s = \sqrt{2} = 1.41421\ldots$

First we convince ourselves that the sum of the positive terms $\sum a_n^+$ diverges to infinity. The comparison test makes this easy:

$$\frac{1}{2n-1} > \frac{1}{2n} \implies \sum a_n^+ = \sum \frac{1}{2n-1} > \frac{1}{2}\sum \frac{1}{n} = \infty$$

The negative terms also diverge: $\sum a_n^- = -\infty$. Construction of the rearrangement is inductive.

1. Sum just enough positive terms $S_1 = a_1^+ + a_2^+ + \cdots + a_{n_1}^+$ *in order* until the partial sum exceeds $s$: plainly $S_1 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.53333$ will do here.
2. Add negative terms starting at the beginning of the sequence until the sum is less than $s$:
$$S_2 = S_1 + (a_1^- + a_2^- + \cdots + a_{m_1}^-) = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = 1.0333\cdots < s$$
3. Repeat: add positive terms until the sum just exceeds $s$, then add negative terms, etc.,
$$S_3 = S_2 + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} = 1.4551\ldots > s, \qquad S_4 = S_3 - \frac{1}{4} = 1.2051\ldots < s$$

Continuing the process ad infinitum, we claim that

$$s = \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{4} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} - \frac{1}{6} + \frac{1}{23} + \frac{1}{25} + \cdots$$

To see why, observe:

- Since $\sum a_n^+ = \infty$ and $\sum a_n^- = -\infty$, at each stage we need only add/subtract *finitely many terms.*
- All terms of the original sequence $(a_n)$ are eventually used since we add the positive (negative) terms *in order.* E.g., $a_{495} = \frac{1}{495}$ appears, *at the latest*, during the 495th positive-addition phase.
- $|S_n - s| \le |a_{m_n}|$, where $a_{m_n}$ is the final term used at the $n^{\text{th}}$ stage. The right hand side converges to zero ($n^{\text{th}}$-term test!), whence $\lim S_n = s$.

[25] Riemann's second result is in fact even stronger. Conditionally convergent series also have rearrangements whose sequence of partial sums diverges by oscillation to any given $\liminf s_n < \limsup s_n$!

**Exercises 3.15.** *Key concepts: Integral test and approximation, Alternating series and approximation*

1. Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.
2. Prove Corollary 3.19 regarding the convergence/divergence of $p$-series.
3. Let $s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Estimate how many terms are required before $s_n \geq 100$.
4. (Example 3.20.3) Verify the claim that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$ and the claim regarding the estimate.
5. (a) Use calculus to show that $a_n = \frac{\ln n}{n^2}$ is monotone-down whenever $n \geq 2$.
   (b) Show that $\lim a_n = 0$, and that the hypotheses of the integral test are therefore satisfied.
   (c) Determine whether the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.
6. (a) Give an example of a series $\sum a_n$ which converges, but for which $\sum a_n^2$ diverges.
   (*Exercise 3.14.3 really requires that $\sum a_n$ be absolutely convergent!*)
   (b) Give an example of a divergent series $\sum b_n$ for which $\sum b_n^2$ converges.
7. Suppose $(a_n)$ satisfies the hypotheses of the alternating series test except that $\lim a_n = a$ is **strictly positive**. What can you say about the sequences $(s_n^+)$ and $(s_n^-)$ and the series $\sum(-1)^n a_n$?
8. Let $a_n = \frac{1}{n}$ have partial sum $s_n = \sum_{k=1}^{n} a_n$, and define a new sequence $(t_n)$ by
   $$t_n = s_n - \ln n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$
   Prove that $(t_n)$ is a positive, monotone-down sequence, which therefore converges.[26]
   (*Hint: You'll need the mean value theorem from elementary calculus*)
9. Suppose $\sum a_n$ is conditionally convergent and let $\sum a_n^+$ be the series obtained by summing, in order, the *positive* terms of the sequence $(a_n)$. Prove that $\sum a_n^+ = \infty$.
10. (a) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is conditionally convergent to some real number $s$.
    (b) How many terms are required for the partial sum $s_n$ to approximate $s$ to within 0.01.
    (c) Following Example 3.25, use a calculator to state the first twelve terms in a rearrangement of the series in part (a) which converges to 0.
11. Recall the rearrangement of the alternating harmonic series in Example 3.23.
    (a) Verify that the *subsequence* of partial sums $(s_{3n})$ is monotone-up, by checking that
    $$b_n := \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0, \qquad \text{for all } n \in \mathbb{N}$$
    (b) Use the comparison test to show that $\sum b_n$ converges.
    (c) Prove that the rearranged series converges to some value $s > \frac{5}{6}$.
    (*Thus $s > \ln 2 \approx 0.69$, the limit of the original alternating harmonic series*)

[26]The limit $\gamma := \lim t_n \approx 0.5772$ is the *Euler–Mascheroni constant.* It appears in several mathematical identities, and yet very little about it is understood; it is not even known whether $\gamma$ is irrational!