Decimal Expansions of Real Numbers

We are typically introduced to decimals in elementary mathematics; for many in grade-school they become a working *definition* of the real numbers. But what are they?

Definition. A *non-negative decimal* $d_0.d_1d_2d_3\cdots$ is an infinite series of the form

$$d_0 + \sum_{n=1}^{\infty} d_n\, 10^{-n} \quad \text{where } d_n \in \mathbb{N}_0 \text{ and } \forall n \geq 1 \implies d_n \leq 9$$

Let x be a non-negative real number. Its *decimal expansion* $D(x)$ is the decimal series arising from inductively defined sequences $(d_n)_{n=0}^{\infty}$ and $(R_n)_{n=0}^{\infty}$:

$$\begin{cases} d_0 = \lfloor x \rfloor, & d_{n+1} = \lfloor 10R_n \rfloor \\ R_0 = x - d_0, & R_{n+1} = 10R_n - d_{n+1} \end{cases}$$

where we use the *floor* function $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$.
The decimal expansion of $x < 0$ is negative that of $|x| = -x$.

Examples. 1. If $x = \frac{27}{20}$, then,

n	0	1	2	3	4	$\cdots$
d_n	$\lfloor \frac{27}{20} \rfloor = 1$	$\lfloor \frac{70}{20} \rfloor = 3$	$\lfloor \frac{10}{2} \rfloor = 5$	$\lfloor 0 \rfloor = 0$	0	$\cdots$
R_n	$\frac{7}{20}$	$\frac{1}{2}$	0	0	0	$\cdots$

Both sequences continue with zeros and we obtain the terminating decimal $D(\frac{27}{20}) = 1.35$.

2. If $x = \frac{1}{3}$, then,

n	0	1	2	3	$\cdots$
d_n	$\lfloor \frac{1}{3} \rfloor = 0$	$\lfloor \frac{10}{3} \rfloor = 3$	$\lfloor \frac{10}{3} \rfloor = 3$	$\lfloor \frac{10}{3} \rfloor = 3$	$\cdots$
R_n	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\cdots$

By induction, all $R_n = \frac{1}{3}$ and we recover the periodic decimal $D(\frac{1}{3}) = 0.33333\cdots$.

3. If $x = \frac{1}{7}$, then,

n	0	1	2	3	4	5	6	7	$\cdots$
d_n	0	$\lfloor \frac{10}{7} \rfloor = 1$	$\lfloor \frac{30}{7} \rfloor = 4$	$\lfloor \frac{20}{7} \rfloor = 2$	$\lfloor \frac{60}{7} \rfloor = 8$	$\lfloor \frac{40}{7} \rfloor = 5$	$\lfloor \frac{50}{7} \rfloor = 7$	1	$\cdots$
R_n	$\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$\frac{6}{7}$	$\frac{4}{7}$	$\frac{5}{7}$	$\frac{1}{7}$	$\frac{3}{7}$	$\cdots$

Since $R_6 = R_0$, both sequences will repeat: $R_{n+6} = R_n$ and $d_{n+6} = d_n$. We recover the *period-six* decimal $D(\frac{1}{7}) = 0.142857142857\cdots$.

In the main result, we check that the decimal expansion is well-defined and that it behaves as expected. We also give two well-known properties of decimal representations.

Theorem. *Let $x \in \mathbb{R}_0^+$ have decimal expansion $D(x) = \sum_{n=0}^{\infty} d_n 10^{-n}$. Then:*

(a) $D(x)$ is a decimal: each $d_n \in \{0, 1, 2, \ldots, 9\}$ whenever $n \geq 1$.

(b) $D(x)$ converges to x.

(c) The sequence (d_n) is eventually periodic *if and only if $x \in \mathbb{Q}_0^+$.*

(d) x equals a unique decimal series, except when $D(x) = d_0.d_1 \cdots d_m$ terminates ($d_m \neq 0$). In such a case there is a second decimal representation:

$$x = D(x) = d_0.d_1 \cdots d_m = d_0.d_1 \cdots d_{m-1}\hat{d_m}99999 \cdots$$

where $\hat{d_m} = d_m - 1$. Otherwise said, we subtract 1 from the final non-zero term and insert an infinite string of 9's.

Examples. 1. Part (c) explains why so many people enjoy the challenge of memorizing the digits of π: since π is irrational, the pattern never repeats.

2. Also referencing part (c), we explicitly evaluate a *period-three* decimal using geometric series:

$$\begin{aligned} 3.1279279279279 \cdots &= \frac{31}{10} + \frac{279}{10000} \sum_{n=0}^{\infty} 1000^{-n} = \frac{31}{10} + \frac{279}{10000} \cdot \frac{1}{1 - \frac{1}{1000}} \\ &= \frac{31}{10} + \frac{279}{9990} = \frac{1736}{555} \end{aligned}$$

3. Here are two examples of part (d):

$$1 = 0.99999 \cdots \qquad 27.164 = 27.1639999 \cdots$$

Exercises 1. Compute the decimal expansion of $\frac{32}{13}$.

2. Prove all parts of the Theorem. Here are some hints:

(a) Let $E_n = x - \sum_{k=0}^{n} d_k 10^{-k}$. Prove by induction that $R_n = 10^n E_n$ and conclude $\lim E_n = 0$.

(c) A decimal is eventually periodic with period r if

$$d_0.d_1 \cdots d_m d_{m+1} \cdots d_{m+r} d_{m+1} \cdots d_{m+r} \cdots = \sum_{k=0}^{m} d_k 10^{-k} + \left(\sum_{j=1}^{r} d_{m+j} 10^{-m-j} \right) \sum_{l=0}^{\infty} 10^{-rl}$$

Convince yourself that this is rational. For the converse, is $x = \frac{p}{q}$ is rational number observe that there are only *finitely many* possible values for the remainders $R_n = \frac{a}{q}$.

(d) If $d_0.d_1 d_2 \cdots = c_0.c_1 c_2 \cdots$, let m be minimal such that $c_m < d_m \ldots$

3. (a) Can you find a simple way to describe all the real numbers x for which $D(x)$ is terminating? Prove your assertion.

   (*Hint: What form can the denominator of R_n take if $x = \frac{p}{q}$?*)

   (b) Given a rational number $x = \frac{p}{q}$ in lowest terms with $q \in \mathbb{N}$, what is the *largest* possible eventual period of $D(x)$? Explain.

4. Similar analyses can be done for other representations of real numbers. For instance, by replacing 10 with 3 in the definition, we obtain the *ternary* (base-3) expansion of a real number

$$T(x) = [t_0.t_1t_2 \cdots]_3 = \sum_{n=0}^{\infty} t_n 3^{-n} \quad \text{where} \quad t_n \in \{0, 1, 2\} \quad \text{whenever} \quad n \geq 1$$

   For example, $[0.1]_3 = \frac{1}{3}$ and $[0.12]_3 = \frac{1}{3} + \frac{2}{3^2} = \frac{5}{9}$.

   (a) Compute $[0.02020202 \cdots]_3$ and find the ternary representations of $\frac{1}{2}$ and $\frac{1}{5}$.

   (b) Read over the Theorem. How can we modify its claims for ternary representations?

   (c) Describe all real numbers x whose ternary representation terminates. More generally, describe all real numbers x whose n-ary (base-n) representation terminates.

   (*The major take-away is that there is nothing special about base-10. Computers typically use base-2, 8 or 16; the ancient Babylonians used base-60. Modern humans likely settled on decimals because because we're blessed with 10 fingers…*)