1.4 The Completeness Axiom

For each of the following sets give its infimum if it is bounded below, otherwise write NOT BOUNDED BELOW.

(b) \((0, 1)\)  
(c) \(\{2, 7\}\)  
(f) \(\{0\}\)  
(h) \(\bigcup_{n=1}^{\infty}[2n, 2n + 1]\)  
(j) \(\{1 - \frac{1}{n} : n \in \mathbb{N}\}\)  
(l) \(\{r \in \mathbb{Q} : r^2 < 2\}\)  
(r) \(\bigcup_{n=1}^{\infty}(1 - \frac{1}{n}, 1 + \frac{1}{n})\)  
(s) \(\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}\)  
(v) \(\{\cos\left(\frac{n\pi}{3}\right) : n \in \mathbb{N}\}\)

6 * Let \(S\) be a non-empty subset of \(\mathbb{R}\).

(a) Prove that \(\inf S \leq \sup S\). (Proof should be short...)
(b) What can you say about \(S\) if \(\inf S = \sup S\)?

8 * Let \(S\) and \(T\) be non-empty subsets of \(\mathbb{R}\) with the following property: \(s \leq t\) for all \(s \in S\) and \(t \in T\).

(a) Prove that \(S\) is bounded above and \(T\) bounded below.
(b) Prove that \(\sup S \leq \inf T\).
(c) Give an example of such sets \(S, T\) where \(S \cap T\) is non-empty.
(d) Give an example of such sets \(S, T\) where \(S \cap T\) is empty, and \(\sup S = \inf T\).

10 * Prove that if \(a > 0\) then there exists \(n \in \mathbb{N}\) such that \(\frac{1}{n} < a < n\).

12 * Let \(I = \mathbb{R} \setminus \mathbb{Q}\) be the set of real numbers which are not rational (the irrational numbers). Prove that if \(a < b\), then there exists \(x \in I\) such that \(a < x < b\). \textit{Hint:} First show \(\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq I\).

14 Let \(A, B\) be non-empty bounded subsets of \(\mathbb{R}\), and let \(S\) be the set of all sums \(a + b\) where \(a \in A, b \in B\).

(a) Prove that \(\sup S = \sup A + \sup B\).
(b) Prove that \(\inf S = \inf A + \inf B\).

16 * Show that \(\sup\{r \in \mathbb{Q} : r < a\} = a\) for each \(a \in \mathbb{R}\).

1.5 The Symbols \(\pm \infty\)

Give the infimum and supremum of each of the following sets:

(a) \(\{x \in \mathbb{R} : x < 0\}\)  
(b) \(\{x \in \mathbb{R} : x^3 \leq 8\}\)  
(c) \(\{x^2 : x \in \mathbb{R}\}\)  
(d) \(\{x \in \mathbb{R} : x^2 < 8\}\)  

4 * Let \(S \subseteq \mathbb{R}\) be non-empty, and let \(-S = \{-s : s \in S\}\). Prove that \(\inf S = -\sup(-S)\).

6 * Let \(S, T \subseteq \mathbb{R}\) be non-empty such that \(S \subseteq T\). Prove that \(\inf T \leq \inf S \leq \sup S \leq \sup T\).
1.6 A Construction of \( \mathbb{R} \)

2. Show that if \( \alpha, \beta \) are Dedekind cuts, then so is

\[
\alpha + \beta = \{ r_1 + r_2 : r_1 \in \alpha, r_2 \in \beta \}
\]

4. * Let \( \alpha, \beta \) be Dedekind cuts and define the “product”:

\[
\alpha \cdot \beta = \{ r_1 r_2 : r_1 \in \alpha, r_2 \in \beta \}
\]

(a) Calculate some “products” of Dedekind cuts using the cuts \( 0^*, 1^* \) and \( (-1)^* \).
(b) Discuss why this definition of “product” is totally unsatisfactory for defining multiplication in \( \mathbb{R} \).
1.4 The Completeness Axiom

4 (b) \( \inf(0, 1) = 0 \)  (c) \( \inf\{2, 7\} = 2 \)  (f) \( \inf\{0\} = 0 \)
(h) \( \inf\bigcup_{n=1}^{\infty} [2n, 2n+1] = 2: \) set is \([2, 3] \cup [4, 5] \cup [6, 7] \cup \ldots\).
(j) \( \inf\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1 - \frac{1}{\infty} = \frac{2}{3}, \) also the minimum of the set.
(l) \( \inf\{r \in \mathbb{Q} : \sqrt{r} < 2\} = -\sqrt{2}. \)
(r) \( \inf\bigcup_{n=1}^{\infty} \left\{1 - \frac{1}{n}, 1 + \frac{1}{n}\right\} = 0: \) set is simply \((0, 2).\)
(s) \( \inf\left\{\frac{1}{n} : n \in \mathbb{N}\text{ and } n \text{ is prime}\right\} = 0, \) since there is no largest prime.
(v) \( \inf\{\cos\left(\frac{n\pi}{2}\right) : n \in \mathbb{N}\} = -1: \) set simply \(\{-1, -\frac{1}{2}, \frac{1}{2}, 1\}\).

6 (a) \( \inf S \) is a lower bound, and \( \sup S \) is an upper bound. Thus \( \inf S \leq s \leq \sup S \) for all \( s \in S. \)

(b) Since \( \inf S \leq s \leq \sup S \) for all \( s \in S, \) it follows that \( \inf S = \sup S \) \( \implies S \) has only one element: \( S = \{s\}. \)

8 Let \( S \) and \( T \) be non-empty subsets of \( \mathbb{R} \) with the following property: \( s \leq t \) for all \( s \in S \) and \( t \in T. \)

(a) Let \( \ell \in T \) and \( \delta \in S. \) Then \( s \leq \ell \) for all \( s \in S, \) so \( S \) is bounded above. Moreover, \( \delta \leq t \) for all \( t \in T, \) so \( T \) is bounded below.

(b) \( S \) bounded above and \( T \) below imply that \( \sup S \) and \( \inf T \) exist. Now suppose that \( \inf T < \sup S \) and consider \( x = \frac{\inf T + \sup S}{2}; \) clearly \( \inf T < x < \sup S. \)

Claim: \( \exists \ell \in T \cap (\inf T, x). \) If not, then \( x \) is a lower bound for \( T, \) which contradicts definition of \( \inf T \) as the greatest such.

Similarly, \( \exists \delta \in S \cap (x, \sup S). \)

But now we have \( \ell < \delta, \) which is a contradiction. Hence we are forced to conclude that \( \sup S \leq \inf T. \)

(c) \( S = [0, 1], T = [1, 2] \) has \( S \cap T = \{1\}. \)

(d) \( S = [0, 1], T = (1, 2] \) has \( S \cap T = \emptyset \) and \( \sup S = \inf T = 1. \)

10 By the Archimedean principle applied to \( a \) and \( 1, \) there exist integers \( p, q \) such that

\[
p > a \quad \text{and} \quad qa > 1.
\]

Therefore \( \frac{1}{q} < a < p. \) Now let \( n = \max\{p, q\}, \) whence

\[
\frac{1}{n} < \frac{1}{q} < a < p \leq n,
\]

as required.

12 Claim: if \( r \in \mathbb{Q}, \) then \( r + \sqrt{2} \notin \mathbb{Q}. \) Suppose \( r + \sqrt{2} = x \in \mathbb{Q}. \) Then \( x - r \in \mathbb{Q} \) since \( \mathbb{Q} \) is a field.

Thus \( \sqrt{2} \in \mathbb{Q}, \) which is a contradiction.

How \( a - \sqrt{2} < b - \sqrt{2}, \) so there exists a rational number \( r \) between them. Therefore

\[
a < r + \sqrt{2} < b,
\]

as required.
Certainly $a$ is an upper bound for $\{r \in \mathbb{Q} : r < a\}$. Now suppose that $\sup \{r \in \mathbb{Q} : r < a\} < a$. Then there exists a rational number $q$ such that $\sup \{r \in \mathbb{Q} : r < a\} < q < a$. Since $q < a$ we conclude that $q \in \{r \in \mathbb{Q} : r < a\}$. Contradiction.

### 1.5 The Symbols $\pm \infty$

- (a) $\sup \{x \in \mathbb{R} : x < 0\} = 0$, $\inf \{x \in \mathbb{R} : x < 0\} = -\infty$.
- (b) $\sup \{x \in \mathbb{R} : x^3 \leq 8\} = 2$, $\inf \{x \in \mathbb{R} : x^3 \leq 8\} = -\infty$: set is $(-\infty, 2]$.
- (c) $\sup \{x^2 : x \in \mathbb{R}\} = \infty$, $\inf \{x^2 : x \in \mathbb{R}\} = 0$.
- (d) $\sup \{x \in \mathbb{R} : x^2 < 8\} = \sqrt{8}$, $\inf \{x \in \mathbb{R} : x^2 < 8\} = -\sqrt{8}$.

There are two cases:

1. Suppose $\inf S > -\infty$. Then $s \geq \inf S$ for all $s \in S$, whence $-s \leq -\inf S$ for all $-s \leq -S$. Thus $-\inf S$ is an upper bound for $-S$.

   Suppose there exists a smaller upper bound for $-S$: $a < -\inf S$. Then $-s \leq a < -\inf S$ for all $-s \in -S$, whence $\inf S < -a \leq s$ for all $s \in S$: a contradiction. Thus $-\inf S$ is the least upper bound for $-S$.

2. Suppose $\inf S = -\infty$. Then $S$ is not bounded below. Then $-S$ is not bounded above, whence $\sup(-S) = \infty$.

First prove $\sup S \leq \sup T$: two cases.

1. First suppose $\sup T < \infty$. Suppose, for contradiction, that $\sup T < \sup S$. Then $\exists s \in S$ such that $s > \sup T$. But then $s \not\in T$: contradiction. Hence $\sup S \leq \sup T$.

2. Now suppose $\sup T = \infty$. Then $\sup S \leq \sup T$ is trivial.

\[\text{inf } T \leq \text{inf } S \text{ is almost identical: details unnecessary.}\]
\[\text{inf } S \leq \sup S \text{ is by definition.}\]
1.6 A Construction of $\mathbb{R}$

Recall the definition of Dedekind cut: we must show three things:

(a) $\alpha + \beta \neq \mathbb{Q}$ and non-empty. Trivial since $\alpha, \beta$ themselves non-empty, and both are bounded above by elements of $\mathbb{Q}$ (thus $\alpha + \beta$ bounded above by their sum).

(b) If $r \in \alpha + \beta$, $s \in \mathbb{Q}$, and $s < r$, then $s \in \alpha + \beta$.

(c) $\alpha + \beta$ contains no largest rational.

For (a), $\alpha, \beta$ are themselves non-empty, hence so is $\alpha + \beta$. Both $\alpha, \beta$ are bounded above by elements $a, b \in \mathbb{Q}$, whence $\alpha + \beta$ is bounded above by their sum $a + b$. Thus $\alpha + \beta \neq \mathbb{Q}$.

For (b), let $r \in \alpha + \beta$. Then $r = r_1 + r_2$ where $r_1 \in \alpha, r_2 \in \beta$. Suppose $s \in \mathbb{Q}$ with $s < r$. Then $s - r_1 \in \mathbb{Q}$, whence,

$$s - r_1 < r_2 \implies s - r_1 \in \beta \implies s = r_1 + (s - r_1) \in \alpha + \beta.$$  

For (c), suppose $\max(\alpha + \beta) = R_1 + R_2$ exists, where $R_1 \in \alpha, R_2 \in \beta$. Now let $r_2 \in \beta$. Clearly $R_1 + r_2 \in \alpha + \beta$, whence $R_1 + r_2 \leq R_1 + R_2 \implies r_2 \leq R_2$. Therefore $R_2 = \max \beta$: a contradiction. Therefore $\alpha + \beta$ is a Dedekind cut.

(a) Recall that $x^* = \{r \in \mathbb{Q} : r < x\}$

$$0^* \cdot 0^* = \{r_1 r_2 : r_1, r_2 \in 0^*\} = \{r_1 r_2 : r_1, r_2 \in \mathbb{Q}, r_1, r_2 < 0\} = \{r \in \mathbb{Q} : r > 0\}$$

just take $r_1 = -1$ and $r_2$ to be any -ve rational.

$$1^* \cdot 1^* = \{r_1 r_2 : r_1, r_2 \in 1^*\} = \{r_1 r_2 : r_1, r_2 \in \mathbb{Q}, r_1, r_2 < 1\} = \mathbb{Q}$$

since can take $r_1$ to be any -ve integer $-m$, and $r_2$ to be any integer reciprocal $\pm \frac{1}{n}$. Similar analysis shows:

$$(-1)^* \cdot (-1)^* = \{r_1 r_2 : r_1, r_2 \in \mathbb{Q}, r_1, r_2 < -1\} = \{r \in \mathbb{Q} : r > 1\}$$

$$0^* \cdot 1^* = \{r_1 r_2 : r_1, r_2 \in \mathbb{Q}, r_1 < 0, r_2 < 1\} = \mathbb{Q}$$

$$0^* \cdot (-1)^* = \{r_1 r_2 : r_1, r_2 \in \mathbb{Q}, r_1 < 0, r_2 < -1\} = \{r \in \mathbb{Q} : r > 0\}$$

$$1^* \cdot (-1)^* = \{r_1 r_2 : r_1, r_2 \in \mathbb{Q}, r_1 < 1, r_2 < -1\} = \mathbb{Q}$$

(b) None of the “products” give the Dedekind cut required if this is to model multiplication in $\mathbb{R}$. E.g. would require $0^* \cdot 0^* = 0^*$, etc.