2.9 Limits Theorems for Sequences

2. Suppose that \( \lim x_n = 3 \), \( \lim y_n = 7 \) and that all \( y_n \) are non-zero. Determine the following limits:
   (a) \( \lim (x_n + y_n) \)
   (b) \( \lim \frac{3y_n - x_n}{y_n} \)

4. Let \( s_1 = 1 \) and for \( n \geq 1 \) let \( s_{n+1} = \sqrt{s_n + 1} \).
   (a) List the first four terms of \( (s_n) \).
   (b) It turns out that \( (s_n) \) converges. Assume this fact and prove that the limit is \( \frac{1}{2}(1 + \sqrt{5}) \).

6. Let \( x_1 = 1 \) and \( x_{n+1} = 3x_n^2 \) for \( n \geq 1 \).
   (a) Show that if \( a = \lim x_n \), then \( a = \frac{1}{3} \) or \( a = 0 \).
   (b) Does \( \lim x_n \) exist? Explain.
   (c) Discuss the apparent contradiction between parts (a) and (b).

12. Assume all \( s_n \neq 0 \) and that the limit \( L = \lim \left| \frac{s_{n+1}}{s_n} \right| \) exists.
   (a) Show that if \( L < 1 \), then \( \lim s_n = 0 \). Hint: Select \( a \) so that \( L < a < 1 \) and obtain \( N \) so that \( |s_{n+1}| < a |s_n| \) for \( n \geq N \). Then show that \( |s_n| < a^{n-N} |s_N| \) for \( n > N \).
   (b) Show that if \( L > 1 \), then \( \lim |s_n| = +\infty \). Hint: Apply (a) to the sequence \( t_n = \frac{1}{|s_n|} \).

14. Let \( p > 0 \). Use question 12 to show that
   \[
   \lim_{n \to \infty} \frac{a^n}{n^p} = \begin{cases} 
   0 & \text{if } |a| \leq 1 \\
   +\infty & \text{if } a > 1 \\
   \text{does not exist} & \text{if } a < -1
   \end{cases}
   \]

2.10 Monotone Sequences and Cauchy Sequences

6. (a) Let \( (s_n) \) be a sequence such that
   \[ |s_{n+1} - s_n| < 2^{-n} \quad \forall n \in \mathbb{N}. \]
   Prove that \( (s_n) \) is a Cauchy sequence and hence a convergent sequence.
   (b) Is the result in (a) true if we only assume that \( |s_{n+1} - s_n| < \frac{1}{n} \) for all \( n \in \mathbb{N} \).

7. Let \( S \) be a bounded nonempty subset of \( \mathbb{R} \) and suppose \( \sup S \notin S \). Prove that there is a nondecreasing sequence \( (s_n) \) of points in \( S \) such that \( \lim s_n = \sup S \).

8. Let \( (s_n) \) be a nondecreasing sequence of positive numbers and define \( \sigma_n = \frac{1}{n} (s_1 + s_2 + \cdots + s_n) \). Prove that \( (\sigma_n) \) is a nondecreasing sequence.

9. Let \( s_1 = 1 \) and \( s_{n+1} = \frac{n}{n+1} s_n^2 \) for \( n \geq 1 \).
   (a) Find \( s_2, s_3 \) and \( s_4 \).
(b) Show that \( \lim s_n \) exists.
(c) Prove that \( \lim s_n = 0 \).

10 Let \( s_1 = 1 \) and \( s_{n+1} = \frac{1}{3}(s_n + 1) \) for \( n \geq 1 \).

(a) Find \( s_2, s_3 \) and \( s_4 \).
(b) Use induction to show that \( s_n > \frac{1}{2} \) for all \( n \)
(c) Show that \( (s_n) \) is a nonincreasing sequence.
(d) Show that \( \lim s_n \) exists and find \( \lim s_n \).

2.11 Subsequences

2 Consider the sequences defined as follows:
\[
\begin{align*}
  a_n &= (-1)^n \\
  b_n &= \frac{1}{n} \\
  c_n &= n^2 \\
  d_n &= \frac{6n + 4}{7n - 3}
\end{align*}
\]

(a) For each sequence, give an example of a monotone subsequence.
(b) For each sequence, give its set of subsequential limits.
(c) For each sequence, give its \( \text{lim sup} \) and \( \text{lim inf} \).
(d) Which of the sequences converges? diverges to \( +\infty \)? diverges to \( -\infty \)?
(e) Which of the sequences is bounded?

6 Show that every subsequence of a subsequence of a given sequence is itself a subsequence of the given sequence.

7 Let \( (r_n) \) be an enumeration of the set \( \mathbb{Q} \) of all rational numbers. Show that there exists a subsequence \( (r_{n_k}) \) such that \( \lim_{k \to \infty} r_{n_k} = +\infty \).

9 (a) Show that the closed interval \([a, b]\) is a closed set. Hint: a set is closed if it contains all subsequential limits of all sequences in itself.
(b) Is there a sequence \( (s_n) \) such that \((0, 1)\) is its set of subsequential limits?

10 Let \( (s_n) \) be the sequence of numbers in the following figure, listed in the indicated order.

(a) Find the set \( S \) of subsequential limits of \( (s_n) \).
(b) Determine \( \limsup s_n \) and \( \liminf s_n \).
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2.9 Limits Theorems for Sequences

2. (a) \(\lim(x_n + y_n) = \lim x_n + \lim y_n = 3 + 7 = 10.\)

(b) \(\lim \frac{3y_n - x_n}{y_n^2} = \lim \frac{3\lim y_n - \lim x_n}{(\lim y_n)^2} = \frac{3 \cdot 7 - 3}{7^2} = \frac{18}{49}.\)

4. Let \(s_1 = 1\) and for \(n \geq 1\) let \(s_{n+1} = \sqrt{s_n + 1}.\)

(a) \((s_n) = (1, \sqrt{2}, \sqrt{\sqrt{2} + 1}, \sqrt{\sqrt{\sqrt{2} + 1} + 1}, \ldots).\)

(b) If \(\lim s_n = s\) then taking the limit of both sides of \(s_{n+1} = \sqrt{s_n + 1}\) results in

\[s = \sqrt{s + 1} \implies s^2 = s + 1 \implies s = \frac{1}{2}(1 \pm \sqrt{5})\]

Given the \(s_n\) is clearly a positive sequence, it follows that \(s \geq 0\) and so \(s = \frac{1}{2}(1 + \sqrt{5}).\)

6. Let \(x_1 = 1\) and \(x_{n+1} = 3x_n^2\) for \(n \geq 1.\)

(a) If \(a = \lim x_n\), then taking limits of \(x_{n+1} = 3x_n^2\) results in \(a = 3a^2\), which has solutions \(a = \frac{1}{3}\) and \(a = 0.\)

(b) Given that \(x_1 = 1\) and \(x_{n+1} = 3x_n^2\), we see that \(x_n\) is always \(\geq 1.\) It is therefore impossible for \(\lim x_n\) to equal either \(\frac{1}{3}\) or 0. \(\lim x_n\) does not exist: indeed \(x_n \to +\infty.\) For a proof, we use induction to see that \(x_n \geq n\) for all \(n:\)

Clearly true for \(n = 1\) since \(x_1 = 1.\)

Suppose \(x_n \geq n\) for some \(n.\) Then \(x_{n+1} = 3x_n^2 \geq 3n^2 \geq n + 1.\) Hence result.

(c) There is no contradiction. We needed to assume the existence of a limit in (a) to then calculate what it would be. If we don’t assume there is a limit, (a) is meaningless.

12. (a) If \(L < 1\) then let \(a = \frac{1 - L}{2}.\) Clearly \(L < a < 1.\) Now let \(\epsilon = \frac{1 - L}{2}.\) Since \(L = \lim \left| \frac{s_{n+1}}{s_n} \right|,\) there exists \(N \in \mathbb{N}\) such that

\[n \geq N \implies \left| \frac{s_{n+1}}{s_n} - L \right| < \epsilon \implies -\epsilon + L < \left| \frac{s_{n+1}}{s_n} \right| < \epsilon + L\]

\[\implies |s_{n+1}| < (\epsilon + L) |s_n| = \frac{1 + L}{2} |s_n| = a |s_n|\]

Iterating this we immediately obtain \(|s_n| < a^{n-N} |s_N|\). Since \(a^n \to 0\) for \(0 < a < 1,\) we conclude that \(\lim s_n = 0.\)

(b) Writing \(t_n = \frac{1}{|s_n|},\) we have \(L^{-1} = \lim t_{n+1} \) existing and \(0 < L^{-1} < 1.\) Applying part(a) to \((t_n)\) we obtain \(\lim t_n \to 0,\) whence \(\lim |s_n| = +\infty.\)

14. If \(s_n = \frac{a^n}{n^p},\) then \(\left| \frac{s_{n+1}}{s_n} \right| = \left| \frac{a^{n+1}p}{n^{p+1}} \right| = |a| \left( \frac{n}{n+1} \right)^p \to |a|.\) This is the limit \(L\) from question 12. We immediately have

\[\lim_{n \to \infty} \left| \frac{a^n}{n^p} \right| = \begin{cases} 0 & \text{if } |a| < 1 \\ +\infty & \text{if } |a| > 1 \end{cases}\]
Additionally, if \( |a| = 1 \), we have the sequence \( \frac{1}{n} \) or \( \frac{(-1)^n}{n^p} \) both of which converge to zero. If \( a > 1 \) then \( \frac{a^n}{m+p} = \left| \frac{a^n}{m+p} \right| \) which therefore diverges to \( +\infty \). If \( a < -1 \), then \( \frac{a^n}{m+p} = \frac{(-1)^n|a|^n}{n^p} \) which diverges by oscillation. Hence result.

### 2.10 Monotone Sequences and Cauchy Sequences

(a) Suppose \( m > n \), then

\[
|s_m - s_n| = |s_m - s_{m-1} + s_{m-1} - \cdots + s_{n+1} - s_n| \\
\triangleq |s_m - s_{m-1}| + \cdots + |s_{n+1} - s_n| \\
< 2^{m-1} + \cdots + 2^{-n} \leq 2^{-n} \sum_{j=0}^{\infty} 2^{-j} = 2^{1-n}
\]

Now let \( \epsilon > 0 \) be given and let \( N = 1 - \frac{\ln \varepsilon}{m^2} \). Then

\[
m > n > N \implies |s_m - s_n| < 2^{1-n} < 2^{1-N} = \epsilon.
\]

\((s_n)\) is therefore Cauchy.

(b) If we only assume that \( |s_{n+1} - s_n| < \frac{1}{n} \) for all \( n \in \mathbb{N} \), then the best we can do is to conclude that

\[
|s_m - s_n| < \frac{1}{m-1} + \cdots + \frac{1}{n} \leq \frac{m-n}{n}.
\]

We cannot conclude that this is bounded by \( \epsilon \). As a counterexample, consider \( s_n = \ln n \). By the Mean Value Theorem applied to \( f(x) = \ln x \) on the interval \([n,n+1]\),

\[
\frac{\ln(n+1) - \ln n}{n+1-n} = \frac{1}{x} \text{ for some } x \in (n,n+1)
\]

Thus \( |s_{n+1} - s_n| < \frac{1}{n} \). However \( s_n \to +\infty \), so \((s_n)\) is not Cauchy.

Let \( t_n = \sup S - \frac{1}{n} \). Since \( t_n < \sup S \), the set \( S \cap (t_n, \sup S) \) is non-empty. Define \( s_n \) to be the infimum of this set. This is a nondecreasing sequence (since \( S \cap (t_{n+1}, \sup S) \) is a subset of \( S \cap (t_n, \sup S) \) and \( A \subseteq B \implies \inf B \geq \inf A \)). Moreover,

\[
0 < \sup S - s_n \leq \sup S - t_n = \frac{1}{n} \to 0
\]

hence \( \lim s_n = \sup S \).

Consider \( \sigma_{n+1} - \sigma_n \). Since \( s_{n+1} \geq s_n \) for all \( n \), we have that

\[
\sigma_{n+1} - \sigma_n = \frac{1}{n+1}(s_1 + s_2 + \cdots + s_n + s_{n+1}) - \frac{1}{n}(s_1 + s_2 + \cdots + s_n)
\]

\[
= \frac{1}{n(n+1)}((ns_{n+1} + (n - (n + 1))(s_1 + s_2 + \cdots + s_n))
\]

\[
= \frac{1}{n(n+1)}((ns_{n+1} - (s_1 + s_2 + \cdots + s_n))
\]

4
\[ \frac{1}{n(n+1)}(ns_{n+1} - (s_{n+1} + s_{n+1} + \cdots + s_{n+1})) = 0 \]

Thus \((\sigma_n)\) is a nondecreasing sequence.

9 (a) \(s_2 = \frac{1}{2}, s_3 = \frac{2}{3} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{6}\) and \(s_4 = \frac{3}{4} \cdot \left(\frac{1}{6}\right)^2 = \frac{1}{48}\).

(b) We claim that \((s_n)\) is nonincreasing. Indeed,

\[
s_{n+1} - s_n = \frac{n}{n+1}s_n^2 - s_n = \frac{s_n}{n+1}(ns_n - n - 1).
\]

Since \(s_1 \leq 1\) it follows that \(s_2 - s_1 \leq 0 \implies s_2 \leq 1\). Continuing by induction, we see that \(s_{n+1} - s_n \leq 0\) for all \(n\).

\((s_n)\) is also a positive sequence and is thus bounded below. \((s_n)\) thus converges.

(c) By (b), we know that \(\lim s_n = s\) for some \(s\). Given this, \(s\) must satisfy

\[
s = \lim s_{n+1} = \lim \frac{n}{n+1}s_n^2 = s^2 \implies s = 0, 1.
\]

Since \(s_2 = \frac{1}{6} < 1\) and \((s_n)\) is non-increasing, it follows that \(s = 0\) is the only possible choice.

10 (a) \(s_2 = \frac{2}{3}, s_3 = \frac{5}{9}\) and \(s_4 = \frac{14}{27}\).

(b) Certainly \(s_1 = 1 > \frac{1}{2}\). Now suppose that \(s_n > \frac{1}{2}\) for some \(n\). Then

\[
s_{n+1} = \frac{1}{3}(s_n + 1) > \frac{1}{3}\left(\frac{1}{2} + 1\right) = \frac{1}{2}.
\]

Hence, by induction, \(s_n > \frac{1}{2}\) for all \(n\).

(c) \(s_{n+1} - s_n = \frac{1}{3}(1 - 2s_n) < \frac{1}{3}(1 - 2 \cdot \frac{1}{2}) = 0\), thus \((s_n)\) is nonincreasing.

(d) \((s_n)\) is nonincreasing and bounded below, thus \(\lim s_n\) exists. Call the limit \(s\). Clearly \(s\) satisfies

\[
s = \lim s_{n+1} = \lim \frac{1}{3}(s_n + 1) = \frac{1}{3}(s + 1) \implies \frac{2}{3}s = \frac{1}{3} \implies s = \frac{1}{2}.
\]

2.11 Subsequences

2 (a) \(a_n = 1\). Here \(n_k = 2k\).

\((b_n), (c_n)\) and \((d_n)\) are already monotone. For the last, observe that

\[
d_n = \frac{6}{7}(7n - 3) + \frac{18}{7} + 4 = \frac{6}{7} + \frac{46}{7(7n - 3)}
\]

is decreasing.

(b) \((a_n)\) has subsequential limit set \([-1, 1]\).

\((b_n)\) has subsequential limit set \([0]\).

\((c_n)\) has subsequential limit set \([+\infty]\).

\((d_n)\) has subsequential limit set \([0]\).
(c) \( \limsup a_n = 1, \liminf a_n = -1. \)
\[ \limsup b_n = 0 = \liminf b_n. \]
\[ \limsup c_n = +\infty = \liminf c_n. \]
\[ \limsup d_n = \frac{1}{2} = \liminf a_n. \]

(d) \((b_n)\) and \((d_n)\) converge, \((c_n)\) diverges to \(+\infty\), and \((a_n)\) diverges by oscillation.

(e) All but \((c_n)\) are bounded.

6 Here are two possible arguments.
- Use the definition of the subsequence \((s_{n_k})\) as a function \( t = s \circ \sigma \), where \( \sigma(k) = n_k \) is an increasing function \( \sigma : \mathbb{N} \to \mathbb{N} \). Thus a subsequence of \((s_{n_k})\) is a function \( u = t \circ \mu \), where \( \mu(l) = k_l \) is an increasing function \( \mu : \mathbb{N} \to \mathbb{N} \). But then, by the associativity of functional composition, we have
  \[ u = (s \circ \sigma) \circ \mu = s \circ (\sigma \circ \mu), \]
which defines \((s_{n_{k_l}})\) as a subsequence of \((s_n)\). It should be clear that \( \sigma \circ \mu : \mathbb{N} \to \mathbb{N} \) is increasing.
- View a subsequence as an ordered subset of a sequence using the ordering \( s_m \preceq s_n \iff m \leq n \). If \((s_{n_{k_l}})\) is a subsequence of \((s_{n_k})\), which is a subsequence of \((s_n)\), then we have the inclusion
  \[ (s_{n_{k_l}}) \subseteq (s_{n_k}) \subseteq (s_n) \]
as ordered sets. Since the subset relation preserves ordering and set inclusion is transitive, it is immediate that \((s_{n_{k_l}})\) is a subsequence of \((s_n)\).

7 Let \((r_n)\) be an enumeration of the set \( Q \) of all rational numbers. As observed in lectures, every real number and \( \{+\infty, -\infty\} \) are subsequential limits if \((r_n)\). Thus \( \limsup r_n = +\infty \) whence, (Corollary 11.4) there exists a monotone subsequence \((r_{n_k})\) such that \( \lim_{k \to \infty} r_{n_k} = \limsup r_n = +\infty \).

9 (a) We must prove that the set of all limits of all sequences taking values in \([a, b]\) is itself \([a, b]\).
Firstly, trivially note that every \( t \in [a, b] \) is the limit of the sequence \((t, t, t, t, \ldots)\).
Now let \( a \leq s_n \leq b \). Then any subsequence of \( s_n \) with a limit \( s \) must satisfy \( a \leq s \leq b \).
Hence \( s \in [a, b] \), and so \([a, b]\) is closed.

(b) First a cheating answer: Any set of subsequential limits must be closed. \((0, 1)\) is open, hence there is no such sequence.
We need to do better than this, for we have not yet got a definition of ‘open’, and so cannot categorize it as somehow (at least for bounded sets), an opposite concept to ‘closed’.
Suppose there is a sequence \((s_n)\) such that \( S = (0, 1) \) is its set of subsequential limits. Then (Corollary 11.4 from book) there exists a monotone subsequence whose limit is \( \limsup s_n \).
However (Theorem 11.7) we know that \( \limsup s_n = \sup S = 1 \). Thus \( 1 \) is the limit of some subsequence of \((s_n)\). Contradiction.

10 (a) Since each term \( \frac{1}{n} \) is repeated infinitely often, the set \( S \) of subsequential limits of \((s_n)\) is precisely \( \{\frac{1}{n} : n \in \mathbb{N}\} \).
(b) \( \limsup s_n = 1 \) and \( \liminf s_n = 0 \) as these are the supremum and infimum of the above set \( S \).