Math 140A - Notes

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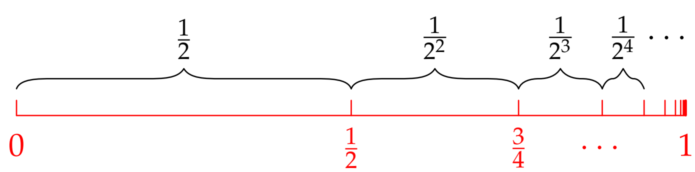
March 7, 2025

Introduction

Analysis is one of the major sub-disciplines of mathematics, concerned with continuity, limits, calculus, and accurate approximations.

Analytic ideas date back thousands of years. For instance, Archimedes (c. 287–212 BC) used limit-type approaches to approximate the circumference of a circle and to compute the area under a parabola.[1] Philosophical objections to such ideas are just as old: how can it make sense to sum infinitely many infinitesimally small quantities? This was part of a deeper debate among the ancient Greeks and other cultures: is the matter comprising the natural world *atomic* (consisting of minute, discrete, indivisible objects) or *continuous* (arbitrarily and infinitely divisible). Several of Zeno's famous paradoxes (5^{th} C. BC) grapple with such difficulties: *Achilles and the Tortoise* is essentially an argument that the infinite series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is meaningless.



As the picture suggests, with modern definitions it makes sense for this sum to evaluate to 1.

The development of calculus by Newton, Leibniz and others in the late 1600s permitted the easy application of infinitesimal ideas to important problems in the sciences, though they did not properly address the ancient philosophical concerns. The main subject of this course (and its sequel) is the rigorous logical development of the foundations of calculus: the triumph of 18^{th}–19^{th} century mathematics. The critical notions of limit and continuity only became settled in during the early 1800s (courtesy of Bolzano, Cauchy, Weierstrass and others), with another 50 years passing before Riemann's thorough description of the definite integral.

In this course we consider sequences, limits, continuity and infinite series, with power series, differentiation and integration relegated to the sequel. We begin with something more basic: to numerically measure continuous quantities, we need to familiarize ourselves with the *real numbers.* A concrete description is difficult, so we build up to it via the natural numbers and the rationals...

[1] Archimedes' circle is reminiscent of Riemann sums; his parabola requires evaluation of the infinite series $\sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{4}{3}$.

1 Completeness

1.1 The Set $\mathbb{N}$ of Natural Numbers

You've been using the natural numbers $\mathbb{N} = \{1, 2, 3, 4, 5, \dots\}$ since you first learned to count. In mathematics, these must be *axiomatically* described. Here is one approach.

Axioms 1.1 (Peano). The natural numbers are a set $\mathbb{N}$ satisfying the following properties:

1. (Non-emptiness) $\mathbb{N}$ is non-empty.
2. (Successor function) There exists a function $f : \mathbb{N} \to \mathbb{N}$. This is usually denoted '+1' so that we may write,
$$n \in \mathbb{N} \implies n + 1 \in \mathbb{N}$$
3. (Initial element) The successor function f is *not surjective.* Otherwise said, there is an element $1 \notin \operatorname{range} f$ which is not the successor of any element.[2]
4. (Unique predecessor/order) f is *injective.* Otherwise said,
$$m + 1 = n + 1 \implies m = n$$
5. (Induction) Suppose $A \subseteq \mathbb{N}$ is a subset satisfying

   (a) $1 \in A$, (b) $n \in A \Longrightarrow n + 1 \in A$.

   Then $A = \mathbb{N}$.

Axioms 1–4 state that $\mathbb{N}$ is defined by repeatedly adding 1 to the initial element; for instance

$$3 := f\big(f(1)\big) = f(1+1) = (1+1) + 1$$

Parts (a) and (b) of axiom 5 are the familiar *base case* and *induction step* a standard induction: let P_n be the proposition '$n \in A$' to recover the usual form of the *Principle of Mathematical Induction.*

Example 1.2. Prove that $7^n - 4^n$ is divisible by 3 for all $n \in \mathbb{N}$.
Let A be the set of natural numbers for which $7^n - 4^n$ is divisible by 3. It is required to prove that $A = \mathbb{N}$.

(a) If $n = 1$, then $7^1 - 4^1 = 3$, whence $1 \in A$.

(b) Suppose $n \in A$. Then $7^n - 4^n = 3\lambda$ for some $\lambda \in \mathbb{N}$. But then
$$\begin{aligned} 7^{n+1} - 4^{n+1} &= 7 \cdot 7^n - 4^{n+1} = 7(3\lambda + 4^n) - 4^{n+1} = 3 \cdot 7\lambda + (7-4) \cdot 4^n \\ &= 3\big(7\lambda + 4^n\big) \end{aligned}$$
is divisible by 3. It follows that $n + 1 \in A$.

Appealing to axiom 5, we see that $A = \mathbb{N}$, hence result.

[2]By convention, the first natural number is 1; we could use 0, x, α, or any symbol you wish!

What about the integers? The integers satisfy only axioms 1, 2 and 4. For instance:

$\not{3}$. The function $f : \mathbb{Z} \to \mathbb{Z} : n \mapsto n+1$ is surjective (indeed bijective/invertible). The 'initial element' $1 \in \mathbb{N}$ is the successor of $0 \in \mathbb{Z}$.

Reversing this observation provides an explicit construction of $\mathbb{Z}$ from $\mathbb{N}$: simply extend the successor function f so that every element has a unique predecessor: 0 is the unique predecessor of 1, -1 the unique predecessor of 0, etc. In essence we are forcing $f(n) = n+1$ to be bijective!

Exercises 1.1. *Key concepts/results: Peano's Axioms, Induction*

Most of these exercises are to refresh your memory of induction. Use either the language of Peano's axiom 5, or the (possibly) more familiar base-case/induction-step formulation.

1. Prove that $1^2 + 2^2 + \cdots + n^2 = \frac{1}{6}n(n+1)(2n+1)$ for all natural numbers n.
2. Prove that $3 + 11 + \cdots + (8n-5) = 4n^2 - n$ for all $n \in \mathbb{N}$.
3. (a) Guess a formula for $1 + 3 + \cdots + (2n-1)$ by evaluating the sum for $n = 1, 2, 3$, and 4. (*For $n = 1$ the sum is simply 1*)
   (b) Prove your formula using mathematical induction.
4. Prove that $11^n - 4^n$ is divisible by 7 for all $n \in \mathbb{N}$.
5. The principle of mathematical induction can be extended as follows. A list $P_m, P_{m+1}, \ldots$ of propositions is true provided (i) P_m is true, (ii) P_{n+1} is true whenever P_n is true and $n \geq m$.
   (a) Prove that $n^2 > n+1$ for all integers $n \geq 2$.
   (b) Prove that $n! > n^2$ for all integers $n \geq 4$. (*recall that* $n! = n(n-1)\cdots 2 \cdot 1$)
6. Prove $(2n+1) + (2n+3) + (2n+5) + \cdots + (4n-1) = 3n^2$ for all $n \in \mathbb{N}$.
7. For each $n \in \mathbb{N}$, let P_n denote the assertion "$n^2 + 5n + 1$ is an even integer".
   (a) Prove that P_{n+1} is true whenever P_n is true.
   (b) For which n is P_n actually true? What is the moral of this exercise?
8. For $n \in \mathbb{N}$, let $n!$ denote the factorial function ($0! = 1$) and define the binomial coefficient

   $$\binom{n}{k} = \frac{n!}{k!(n-k)!} \quad \text{for} \quad k = 0, 1, \ldots, n$$

   The *binomial theorem* asserts that, for all $n \in \mathbb{N}$,

   $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \cdots + nab^{n-1} + b^n$$

   (a) Show that $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ for $k = 1, 2, \ldots, n$.
   (b) Prove the binomial theorem by induction.
9. Show that Peano's induction axiom is *false* for the set of integers $\mathbb{Z}$ by exhibiting a *proper subset* $A \subset \mathbb{Z}$ which satisfies conditions (a) and (b).
10. Consider $\mathbb{Z}_3 = \{0, 1, 2\}$ under addition modulo 3. That is,

    $$0+1 = 1, \quad 1+1 = 2, \quad 2+1 = 0$$

    Which of Peano's axioms are satisfied?

1.2 The Set $\mathbb{Q}$ of Rational Numbers

The rational numbers may be defined in several ways. For instance, we could consider the set of relatively prime ordered pairs

$$\mathbb{Q} = \{(p,q) : p \in \mathbb{Z},\ q \in \mathbb{N},\ \gcd(p,q) = 1\} \subseteq \mathbb{Z} \times \mathbb{N}$$

Things seem more familiar if we write $\frac{p}{q}$ instead of (p,q) and adopt the convention that $\frac{\lambda p}{\lambda q} = \frac{p}{q}$ for any non-zero $\lambda \in \mathbb{Z}$. The usual operations $(+, \cdot$, etc.$)$ are easily defined, consistently with those for the integers (Exercise 6).

An alternative approach involves equations. Each *linear equation* $qx - p = 0$ where $p, q \in \mathbb{Z}$ and $q \neq 0$ corresponds to a rational number. For example

$$13x + 27 = 0 \leftrightsquigarrow x = -\frac{27}{13}$$

Of course the equation $26x + 54 = 0$ *also* corresponds to the same rational number!

Extending this process naturally leads us to consider higher degree polynomials.

Definition 1.3. A number x is *algebraic* if it satisfies an equation of the form[3]

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \qquad (*)$$

for some integers $a_0, \ldots, a_n$.

Examples 1.4. 1. $\sqrt{2}$ is algebraic since it satisfies the equation $x^2 - 2 = 0$.

2. $x = \sqrt[5]{7 + \sqrt{3}}$ is also algebraic:

$$x^5 - 7 = \sqrt{3} \implies (x^5 - 7)^2 = 3 \implies x^{10} - 14x^5 + 46 = 0$$

The next result is helpful for deciding whether a given number is rational and can assist with factorizing polynomials.

Theorem 1.5 (Rational Roots). *Suppose $a_0, \ldots, a_n \in \mathbb{Z}$ and that $x \in \mathbb{Q}$ satisfies $(*)$. If $x = \frac{p}{q}$ is rational, written in lowest terms, then $p \mid a_0$ and $q \mid a_n$.*

Proof. Substitute $x = \frac{p}{q}$ into the polynomial equation and multiply through by q^n to see that

$$a_n p^n + a_{n-1} p^{n-1} q + \cdots + a_1 p q^{n-1} + a_0 q^n = 0$$

This is an equation *in integers.* All terms except the last contain a factor of p, whence $p \mid a_0 q^n$. Since $\gcd(p,q) = 1$, it follows that $p \mid a_0$. The result for q is almost identical: all but the first term above has a factor of q. ∎

[3]You should be alarmed by this! We seem to have given up *constructing* new numbers and instead are merely *describing* their properties. No matter, a construction of the real numbers will come later.

Examples 1.6. 1. We prove that $\sqrt{2}$ is irrational. Plainly $x = \sqrt{2}$ satisfies the polynomial equation $x^2 - 2 = 0$. If $\sqrt{2} = \frac{p}{q}$ were rational in lowest terms, then the rational roots theorem forces

$$p \mid 2 \quad \text{and} \quad q \mid 1 \implies \sqrt{2} \in \{\pm 1, \pm 2\}$$

Since none of the values $\pm 1, \pm 2$ satisfy $x^2 - 2 = 0$, we have a contradiction.

2. $y = (\sqrt{3} - 1)^{1/3}$ satisfies $(y^3 + 1)^2 = 3$, whence $y^6 + 2y^3 - 2 = 0$. If $y = \frac{p}{q}$ were rational in lowest terms, then $p \mid 2$ and $q \mid 1$, whence $y = \pm 1, \pm 2$; none of which satisfy the polynomial.

3. $z = \left(\frac{4+\sqrt{3}}{5}\right)^{1/2}$ satisfies $5z^2 - 4 = \sqrt{3}$, from which $25z^4 - 40z^2 + 13 = 0$. If $z = \frac{p}{q}$ were rational in lowest terms, then $p \mid 13$ and $q \mid 25$. There are twelve possibilities: it is tedious to check, but none satisfy the required polynomial,

$$z = \pm 1,\ \pm 13,\ \pm\tfrac{1}{5},\ \pm\tfrac{13}{5},\ \pm\tfrac{1}{25},\ \pm\tfrac{13}{25}$$

In this case it is easier to bypass the theorem: if $z \in \mathbb{Q}$ then $\sqrt{3} = 5z^2 - 4$ would also be rational!

4. We use the theorem to factorize the polynomial $3x^3 + x^2 + x - 2 = 0$. If $x = \frac{p}{q}$ is a rational root, then $p \mid 2$ and $q \mid 3$ give several possibilities:

$$x \in \left\{\pm 1, \pm 2, \pm\tfrac{1}{3}, \pm\tfrac{2}{3}\right\}$$

It doesn't take long to check that $x = \frac{2}{3}$ is the only rational root. A factor of $3x - 2$ may be extracted by long division to obtain

$$3x^3 + x^2 + x - 2 = (3x - 2)(x^2 + x + 1)$$

The quadratic has no real roots: absent complex numbers, the factorization is complete.

It is far from clear that non-algebraic (*transcendental*) numbers exist: e and π are the most famous. These satisfy no polynomial equation with integer coefficients, though demonstrating such is tricky.

Exercises 1.2. *Key concepts: Algebraic Numbers, Rational Roots Theorem/Testing for Irrationality*

1. Describe all linear equations corresponding to the rational number $\frac{101}{29}$.
2. Show that $\sqrt{3}$, $\sqrt{5}$ and $\sqrt{24}$ are not rational numbers: what are the relevant polynomials?
3. Show that $2^{1/3}$ and $13^{1/4}$ are not rational numbers.
4. Show that $(2 + \sqrt{2})^{1/2}$ and $(5 - \sqrt{3})^{1/3}$ are irrational.
5. Explain why $4 - 7b^2$ must be rational if b is rational.
6. Given rational numbers (p, q), (r, s) as ordered pairs, what are $(p, q) + (r, s)$ and $(p, q) \cdot (r, s)$?
7. Let $n \in \mathbb{N}$. Use the rational roots theorem to prove that $\sqrt{n} \in \mathbb{Q} \iff \sqrt{n} \in \mathbb{N}$.
8. In the proof of the rational roots theorem, explain why the condition $\gcd(p, q) = 1$ allows us to conclude that $p \mid a_0 q^n \implies p \mid a_0$

1.3 Ordered Fields

We have thus far formally constructed the natural numbers and used them to build the integers and rational numbers. It is a significantly greater challenge to *construct* the real numbers. We start by thinking about ordered fields, of which both $\mathbb{Q}$ and $\mathbb{R}$ are examples.

Axioms 1.7. A *field* $\mathbb{F}$ is a set with two binary operations $+, \cdot$ which satisfy (for all $a, b, c \in \mathbb{F}$),[4]

	Addition	Multiplication
Closure	$a + b \in \mathbb{F}$	$ab \in \mathbb{F}$
Associativity	$a + (b + c) = (a + b) + c$	$a(bc) = (ab)c$
Commutativity	$a + b = b + a$	$ab = ba$
Identity	$\exists 0 \in \mathbb{F}$ such that $a + 0 = a$	$\exists 1 \in \mathbb{F}$ such that $a \cdot 1 = a$
Inverse	$\exists -a \in \mathbb{F}$ such that $a + (-a) = 0$	If $a \neq 0$, $\exists a^{-1} \in \mathbb{F}$ such that $aa^{-1} = 1$
Distributivity	$a(b + c) = ab + ac$	

A field $\mathbb{F}$ is *ordered* if we also have a binary relation $\leq$ which satisfies (again for all $a, b, c \in \mathbb{F}$):

O1 $a \leq b$ or $b \leq a$

O2 $a \leq b$ and $b \leq a \implies a = b$

O3 $a \leq b$ and $b \leq c \implies a \leq c$

O4 $a \leq b \implies a + c \leq b + c$

O5 $a \leq b$ and $0 \leq c \implies ac \leq bc$

For an ordered field, the symbol $<$ is used in the usual manner: $x < y \iff x \leq y$ and $x \neq y$.

As with Peano's axioms for the natural numbers, these are not worth memorizing. Instead you should quickly check that you believe all of them for your current understanding of the real numbers; you can't *prove* anything since the real numbers are yet to be defined!

Example 1.8. It is worth considering the rational numbers in a little more detail. These inherit a natural ordering from $\mathbb{Z}$ and $\mathbb{N}$:

$$\frac{p}{q} \leq \frac{r}{s} \iff ps \leq qr \qquad \text{(remember that } q, s > 0\text{)}$$

It is now possible, though tedious, to *prove* that each of the axioms of an ordered field holds for $\mathbb{Q}$, using only basic facts about multiplication, addition and ordering *within the integers.* For instance,

[4] Write multiplication $\cdot$ as juxtaposition unless necessary, and use the common shorthand $a^2 = a \cdot a$. The field axioms are very easy to remember if you know some abstract algebra:

- The addition axioms say that $(\mathbb{F}, +)$ is an abelian group.
- The multiplication axioms say that $(\mathbb{F} \setminus \{0\}, \cdot)$ is an abelian group.
- The distributive axiom describes how addition and multiplication interact.

Commutativity of Multiplication Given $a = \frac{p}{q}$ and $b = \frac{s}{t}$ rational, we have

$$ab = \frac{ps}{qt} = \frac{sp}{tq} = ba$$

since multiplication of integers (numerator and denominator) is commutative.

O3 Suppose $a \leq b$ and $b \leq c$. Write $a = \frac{p}{q}$, $b = \frac{r}{s}$ and $c = \frac{t}{u}$ where all three denominators are positive. By assumption,

$$\begin{aligned} ps \leq qr \text{ and } ru \leq st &\implies psu \leq qru \leq qst \\ &\implies pu \leq qt && \text{(divide by } s \neq 0) \\ &\implies a = \frac{p}{q} \leq \frac{t}{u} = c \end{aligned}$$

Basic Results about ordered fields

As with the axioms of an ordered field, these are not worth memorizing.

Theorem 1.9. *Let $\mathbb{F}$ be a ordered field with at least two elements $0 \neq 1$. Then:*

1. $a + c = b + c \implies a = b$
2. $a \cdot 0 = 0$
3. $(-a)b = -(ab)$
4. $(-a)(-b) = ab$
5. $ac = bc$ and $c \neq 0 \implies a = b$
6. $ab = 0 \implies a = 0$ or $b = 0$
7. $a \leq b \implies -b \leq -a$
8. $a \leq b$ and $c \leq 0 \implies bc \leq ac$
9. $0 \leq a$ and $0 \leq b \implies 0 \leq ab$
10. $0 \leq a^2$
11. $0 < 1$
12. $0 < a \implies 0 < a^{-1}$
13. $0 < a < b \implies 0 < b^{-1} < a^{-1}$

All these statements should be intuitive for the fields $\mathbb{Q}$ and $\mathbb{R}$. Try proving a few using only the axioms; they are most easily done in the order presented. For instance, part 2 might be proved as follows:

$$\begin{aligned} & a \cdot 0 + 0 = a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0 && \text{(additive identity/distibutive axioms)} \\ \implies & 0 = a \cdot 0 && \text{(part 1)} \end{aligned}$$

We finish with a final useful ingredient.

Definition 1.10. In an ordered field $\mathbb{F}$, the *absolute value* of an element a is

$$|a| := \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$$

Theorem 1.11. *In any ordered field:*

1. $|a| \geq 0$
2. $|ab| = |a| \cdot |b|$
3. $|a + b| \leq |a| + |b|$ *($\triangle$-inequality)*
4. $|a - b| \geq ||a| - |b||$ *(reverse/extended $\triangle$-inequality)*

All parts are straightforward if you consider the $\pm$-cases separately for a, b.

Exercises 1.3. *Key concepts: Ordered Field ($\mathbb{Q}$ an example arising naturally from $\mathbb{Z}$), $\triangle$-inequality*

1. Which of the axioms of an ordered field fail for $\mathbb{N}$? For $\mathbb{Z}$?

2. Prove parts 11 and 13 of Theorem 1.9.
   (*Hint: You can use any of the parts that come before...*)

3. (a) Prove that $|a + b + c| \leq |a| + |b| + |c|$ for all $a, b, c \in \mathbb{R}$.
   (*Hint: Apply the triangle inequality twice. Don't consider eight separate cases!*)
   (b) For any $a_1, \ldots, a_n \in \mathbb{R}$, use induction to prove
   $$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|$$

4. (a) Show that $|b| < a \iff -a < b < a$.
   (b) Show that $|a - b| < c \iff b - c < a < b + c$.
   (c) Show that $|a - b| \leq c \iff b - c \leq a \leq b + c$.

5. Let $a, b \in \mathbb{R}$. Show that if $a \leq b_1$ for every $b_1 > b$, then $a \leq b$.
   (*Hint: draw a picture if you're stuck. This is a very important example!*)

6. In an ordered field, suppose that $0 \leq a$ and $0 \leq b$. Explain carefully why $0 \leq a + b$.

7. Following Example 1.8, prove that $\mathbb{Q}$ satisfies axiom O5.
   (*Hint: if $a = \frac{p}{q}$, etc., what is meant by $ac \leq bc$?*)

8. (Hard!) The complex numbers $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ form a field. The lexicographic ordering of $\mathbb{C}$ is defined by
   $$x + iy \leq p + iq \iff \begin{cases} x < p \text{ or} \\ x = p \text{ and } y \leq q \end{cases}$$
   Which of the order axioms O1–O5 are satisfied by the lexicographic ordering?
   (*Provide a counter-example if an axiom is not satisfied; don't prove your claims if an axiom is satisfied.*)

1.4 The Completeness Axiom, or Least Upper Bound Principle

Though we haven't provided an explicit *definition* of the real numbers, you should be comfortable that both $\mathbb{Q}$ and $\mathbb{R}$ are ordered fields. We now ask how these might be distinguished *axiomatically*. Perhaps surprisingly, only one additional axiom is required! We first need some terminology.

Definition 1.12 (Maxima, Minima & Boundedness). Let $S \subseteq \mathbb{R}$ be non-empty.

1. S is *bounded above* if it has an *upper bound* M:

$$\exists M \in \mathbb{R} \text{ such that } \forall s \in S,\ s \leq M$$

2. We write $M = \max S$, the *maximum* of S, if M is an upper bound for S **and** $M \in S$.
3. S *bounded below*, a *lower bound* m, and the *minimum* $\min S$ are defined similarly.
4. S is *bounded* if it is bounded both above and below. It is *bounded by* M if

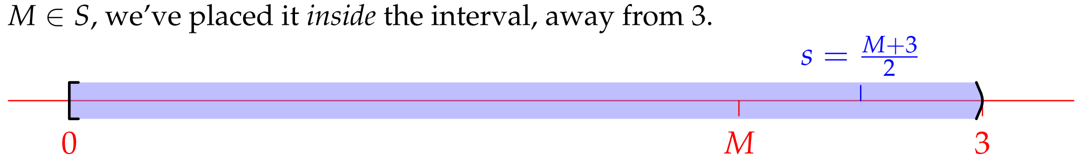
$$\forall s \in S,\ |s| \leq M \qquad (M \text{ is an upper bound, } -M \text{ a lower bound})$$

Examples 1.13. 1. If S is a finite set, then it is bounded and has both a maximum and a minimum. For instance, $S = \{-3, \pi, 12\}$ has $\min S = -3$ and $\max S = 12$.

2. $\mathbb{N}$ has minimum 1, but no maximum. $\mathbb{Z}$ and $\mathbb{Q}$ have neither: both are *unbounded.*

3. The half-open interval $S = [0, 3)$ is bounded, e.g. by $M = 5$; it has minimum 0 but no maximum. While this last is intuitive, it is worth giving an explicit argument, in this case by contradiction.[5]

   Suppose $M = \max S$ exists; necessarily $0 \leq M < 3$. We draw a picture to get the lay of the land: since $M \in S$, we've placed it *inside* the interval, away from 3.



   The crux of the argument is to observe that there must be some $s \in S$ which is *larger* than M, the natural choice being the average $s := \frac{1}{2}(M+3)$. Now observe that

$$3 - s = s - M = \frac{1}{2}(3 - M) > 0$$

   In particular, $s \in S$ and $s > M$. Since S contains an element larger than M, it follows that M cannot be the maximum of S. In conclusion, S has no maximum.

Lemma 1.14. 1. *If M is an upper bound for S, so is $M + \varepsilon$ for any $\varepsilon \geq 0$.*

2. *If $M = \max S$ exists, then it is unique.*

Try proving these basic facts yourself.

[5] S has a maximum means: $\exists M \in S$ such that $\forall s \in S,\ s \leq M$. We prove the negation $\forall M \in S, \exists s \in S$ such that $s > M$.

Example 1.15. In a variation on the previous example, we show that the set

$$S = \mathbb{Q} \cap [0, \sqrt{2}) = \{x \in \mathbb{Q} : 0 \le x < \sqrt{2}\}$$

has no maximum. The approach is similar to before: given a hypothetical maximum M, we find some $s \in S$ between M and $\sqrt{2}$. The challenge is that we can't use the *average* $\frac{1}{2}(M + \sqrt{2})$: this isn't rational (*why?*) and so doesn't lie in S!

To fix this, we informally construct a sequence. Define s_n to be $\sqrt{2}$ to n decimal places:

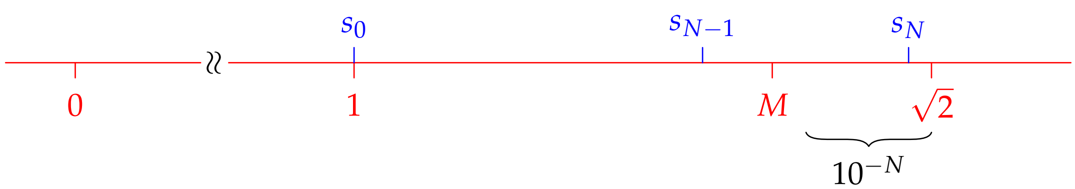
$$s_0 = 1, \quad s_1 = 1.4 = \frac{14}{10}, \quad s_2 = 1.41 = \frac{141}{100}, \quad s_3 = 1.414 = \frac{1414}{1000}, \quad \ldots$$

Since any finite decimal is rational and $0 \le s_n < \sqrt{2}$, we see that $s_n \in S$. Moreover, $\sqrt{2} - s_n \le 10^{-n}$ can be made arbitrarily small by choosing N sufficiently large.

Now suppose $M = \max S$ exists. Since $M \in S$, we have $M < \sqrt{2}$. Choose any $N \in \mathbb{N}$ large enough so that $10^{-N} < \sqrt{2} - M$ (any integer $N > -\log_{10}(\sqrt{2} - M)$ will do!). Certainly $s_N \in S$ and moreover,

$$\sqrt{2} - s_N \le 10^{-N} < \sqrt{2} - M \implies M < s_N$$

The purported maximum M is plainly not an upper bound for S: contradiction.

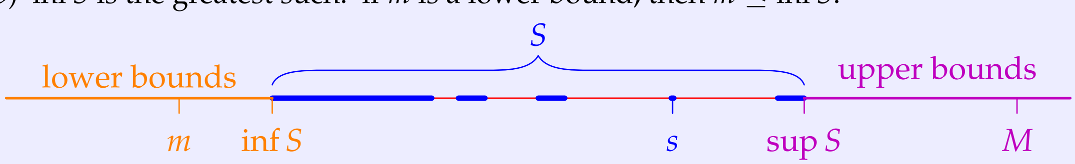


Suprema and Infima

We generalize the idea of maximum and minimum values to any bounded sets.

Definition 1.16. Let $S \subseteq \mathbb{R}$ be non-empty.

1. If S is bounded above, its *supremum* $\sup S$ is its *least upper bound.* Otherwise said,
   (a) $\sup S$ is an upper bound: $\forall s \in S,\ s \le \sup S$,
   (b) $\sup S$ is the least such: if M is an upper bound, then $\sup S \le M$.
2. Similarly, if S is bounded below, its *infimum* $\inf S$ is its *greatest lower bound*:
   (a) $\inf S$ is a lower bound: $\forall s \in S,\ \inf S \le s$,
   (b) $\inf S$ is the greatest such: if m is a lower bound, then $m \le \inf S$.

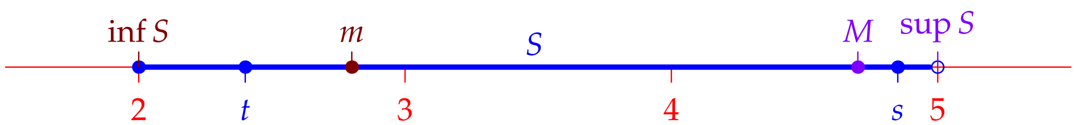


Example 1.17. The interval $S = [2,5)$ has $\sup S = 5$ and $\inf S = 2$ $(= \min S)$. We verify these claims: (a), (b) are the properties in the definition.

(a) Since $s \in S \iff 2 \le s < 5$, we see that 5 is indeed an upper bound and 2 a lower bound.

(b) We demonstrate the contrapositive. Suppose $M < 5$ and define[6] $s = \max\{\frac{1}{2}(M+5), 4\}$. Then $M < s < 5$ and $s \in S$. It follows that M is *not* an upper bound for S. The least upper bound is therefore $\sup S = 5$.

For the infimum: if $m > 2$, define $t = \min\{\frac{1}{2}(m+2), 4\}$ to see that $2 < t < m$ and $t \in S$, whence m is not a lower bound.



Axiom 1.18 (Completeness of $\mathbb{R}$). If $S \subseteq \mathbb{R}$ is non-empty and bounded above, then $\sup S$ exists (and is a real number!).

It is precisely this property that distinguishes the real numbers from the rationals.[7] Certainly every bounded set S of *rational* numbers has a supremum; the issue is that $\sup S$ *need not be rational*!

By reflecting across zero (Exercise 9), we obtain the same thing for the infimum.

Theorem 1.19 (Existence of Infima). *If $S \subseteq \mathbb{R}$ non-empty and bounded below, then $\inf S \in \mathbb{R}$ exists.*

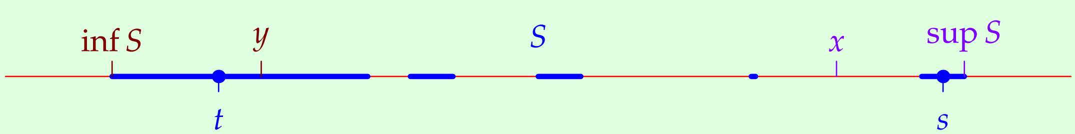
A Useful Contrapositive Part (b) of the Definition is plainly a biconditional: if $\sup S \le M$, then M is at least as large as an upper bound and is therefore also an upper bound for S (Lemma 1.14)! As in Example 1.17, one often uses the contrapositive of part (b):

$M < \sup S$ if and only if M is *not* an upper bound for S.

Unpacking this further using the meaning of upper bound (and substituting x for M) we recover a useful result that will be used repeatedly.

Lemma 1.20. 1. *Let S be bounded above. Then $x < \sup S \iff \exists s \in S$ such that $x < s$.*

2. *Let S be bounded below. Then $y > \inf S \iff \exists t \in S$ such that $t < y$.*



[6]The number 4 is merely an arbitrary element to make sure $s \in S$ in case M were huge and negative!

[7]More formally (the details are too much for us): if $\mathbb{F}$ is an ordered field with $0 \neq 1$ and which satisfies the completeness axiom, then $\mathbb{F}$ is isomorphic to the real numbers.

Examples 1.21. We state the following without proof or calculation. You should be able to justify everything using the definition, or by mirroring Example 1.17.

1. A bounded set has many possible bounds, but only one supremum or infimum.
2. If S has a maximum, then $\max S = \sup S$. Similarly, if a minimum exists, then $\min S = \inf S$.
3. (Example 1.15) $S = \mathbb{Q} \cap [0, \sqrt{2})$ has $\sup S = \sqrt{2}$: this is a set of rational numbers whose supremum is not rational.
4. $S = \mathbb{Q} \cap (\pi, 4)$ has $\sup S = 4$, $\inf S = \pi$, and no maximum nor minimum.
5. $S = \{\frac{1}{n} : n \in \mathbb{N}\} = \{\ldots, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1\}$ has $\sup S = \max S = 1$, $\inf S = 0$, and no minimum.
6. $S = \bigcup_{n=1}^{\infty} [n, n + \frac{1}{2}) = [1, 1.5) \cup [2, 2.5) \cup [3, 3.5) \cup \cdots$ has $\inf S = 1$. It is not bounded above.
7. $S = \bigcap_{n=1}^{\infty} [\frac{1}{n}, 1 + \frac{1}{n})$ has $\inf S = 1 = \sup S$ since $S = \{1\}$.

The Archimedean Property and the Density of the Rationals

We finish this section by discussing a crucial property related to completeness, and of the distribution of the rational numbers among the reals.

Theorem 1.22 (Archimedean Property). *If $b > 0$ is a real number, then $\exists n \in \mathbb{N}$ such that $n > b$. More generally: $a, b > 0 \implies \exists n \in \mathbb{N}$ such that $an > b$.*

We assume nothing about $\mathbb{R}$ except that is an ordered field satisfying the completeness axiom and where $0 \neq 1$ (footnote 7). The natural numbers in this context are *defined* as the subset

$$\mathbb{N} = \{1, 1+1, 1+1+1, \ldots\} \subseteq \mathbb{R}$$

and Peano's axioms are a *theorem*.

Proof. Suppose the result were false. Then $\exists b > 0$ such that $n \leq b$ for all $n \in \mathbb{N}$; that is, $\mathbb{N}$ is bounded above! By completeness, $\sup \mathbb{N}$ exists, and we trivially see that

$$0 < 1 \implies \sup \mathbb{N} < \sup \mathbb{N} + 1 \implies \sup \mathbb{N} - 1 < \sup \mathbb{N}$$

By Lemma 1.20, $\exists n \in \mathbb{N}$ such that $n > \sup \mathbb{N} - 1$. But then $\sup \mathbb{N} < n + 1$ which is clearly a natural number! Thus $\sup \mathbb{N}$ is not an upper bound for $\mathbb{N}$: contradiction.
For the more general statement, simply replace b with $\frac{b}{a}$. ∎

The use of completeness is *necessary*: there exist non-Archimedean ordered fields!

Example (1.15, cont.). The Archimedean property is precisely what is needed to justify the existence of an integer $N > -\log_{10}(\sqrt{2} - M)$.

Corollary 1.23 (Density of $\mathbb{Q}$ in $\mathbb{R}$). *Between any two real numbers, there exists a rational number.*

The idea is hopefully straightforward: given $a < b$, stretch the interval by an integer factor n until it contains an integer m, before dividing by n to obtain $a < \frac{m}{n} < b$. We use the Archimedean property to establish the existence of the scale factors m, n.

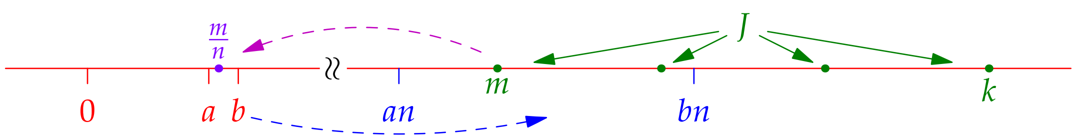
Proof. Suppose WLOG that $0 \le a < b$, and apply the Archimedean property to $\frac{1}{b-a} > 0$:

$$\exists n \in \mathbb{N} \text{ such that } n > \frac{1}{b-a}$$

A second application (or trivially if $a = 0$) says $\exists k \in \mathbb{N}$ such that $k > an$. Now consider the set

$$J := \{j \in \mathbb{N} : an < j \le k\}$$

and define $m = \min J$: this exists since J is a finite non-empty set of natural numbers.[8]



Clearly $m > an > m - 1$, since $m = \min J$. But then $m \le an + 1 < bn$. We conclude that

$$an < m < bn \implies a < \frac{m}{n} < b$$

■

By iterating this result we see that any interval (a, b) contains *infinitely many* rational numbers. It can moreover be established that the irrational numbers are also dense in $\mathbb{R}$ (Exercise 6).

Exercises 1.4. *Key concepts: Suprema, Completeness (distinguishes $\mathbb{R}$), Contrapositive criterion, Archimedean property/Density of $\mathbb{Q} \subset \mathbb{R}$*

1. Decide whether each set is bounded above and/or below. If so, state its supremum and/or infimum (no working is required).

   (a) $(0, 1)$ (b) $\{2, 7\}$ (c) $\{0\}$

   (d) $\bigcup_{n=1}^{\infty} [2n, 2n+1]$ (e) $\{1 - \frac{1}{3^n} : n \in \mathbb{N}\}$ (f) $\{r \in \mathbb{Q} : r^2 < 2\}$

   (g) $\bigcup_{n=1}^{\infty} \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ (h) $\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\}$ (i) $\{\cos(\frac{n\pi}{3}) : n \in \mathbb{N}\}$

2. Modelling Example 1.15, *sketch* an argument that $S = \mathbb{Q} \cap (\pi, 4]$ has no minimum.

   (*Hint: let s_n be π rounded up to n decimal places*)

3. Let S be a non-empty, bounded subset of $\mathbb{R}$.

   (a) Prove that $\inf S \le \sup S$.

   (b) What can you say about S if $\inf S = \sup S$?

[8] This part of the argument is necessary since, in this context, we haven't established the well-ordering property of $\mathbb{N}$ (essentially Peano's fifth axiom).

4. Let S and T be non-empty subsets of $\mathbb{R}$ with the property that $s \leq t$ for all $s \in S$ and $t \in T$.

   (a) Prove that S is bounded above and T bounded below.

   (b) Prove that $\sup S \leq \inf T$.

   (c) Give an example of such sets S, T where $S \cap T$ is non-empty.

   (d) Give an example of such sets S, T where $S \cap T$ is empty, and $\sup S = \inf T$.

5. Prove that if $a > 0$ then there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < a < n$.

6. Let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the set of *irrational* numbers. Given real numbers $a < b$, prove that there exists $x \in \mathbb{I}$ such that $a < x < b$.

   (*Hint: First show* $\{r + \sqrt{2} : r \in \mathbb{Q}\} \subseteq \mathbb{I}$)

7. Let A, B be non-empty bounded subsets of $\mathbb{R}$, and let S be the set of all sums

   $$S := \{a + b : a \in A, b \in B\}$$

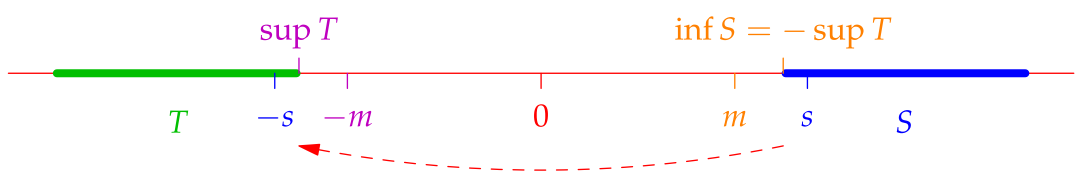
   (a) Prove that $\sup S = \sup A + \sup B$.

   (b) Prove that $\inf S = \inf A + \inf B$.

8. Show that $\sup\{r \in \mathbb{Q} : r < a\} = a$ for each $a \in \mathbb{R}$.

9. We prove Theorem 1.19 on the existence of the infimum.

   Let $S \subseteq \mathbb{R}$ be non-empty and let m be a lower bound for S. Define $T = \{t \in \mathbb{R} : -t \in S\}$ by reflecting S across zero.



   (a) Prove that $-m$ is an upper bound for T.

   (b) By completeness (Axiom 1.18), $\sup T$ exists. Prove that $\inf S = -\sup T$ by verifying Definition 1.16 parts 2(a) and (b).

1.5 The Symbols $\pm\infty$

Thus far the only subsets of the real numbers that have a supremum are those which are *non-empty* and *bounded above.* In this very short section, we introduce the ∞-symbol to provide all subsets of the real numbers with both a supremum and an infimum.

Definition 1.24. Let $S \subseteq \mathbb{R}$ be any subset. If S is bounded above/below, then $\sup S/\inf S$ are as in Definition 1.16. Otherwise:

1. We write $\sup S = \infty$ if S is *unbounded above*, that is
$$\forall x \in \mathbb{R}, \exists s \in S \text{ such that } s > x$$
2. We write $\inf S = -\infty$ if S is *unbounded below*,
$$\forall y \in \mathbb{R}, \exists t \in S \text{ such that } t < y$$
3. By convention, $\sup \emptyset := -\infty$ and $\inf \emptyset := \infty$, though these will rarely be of use to us.

The symbols $\pm\infty$ have *no other meaning* (as yet); in particular, they are *not numbers*. If one is willing to abuse notation and write $x < \infty$ and $y > -\infty$ for any real numbers x, y, then the conclusions of Lemma 1.20 are precisely statements 1 & 2 above!

Examples 1.25. 1. $\sup \mathbb{R} = \sup \mathbb{Q} = \sup \mathbb{Z} = \sup \mathbb{N} = \infty$, since all are unbounded above. We also have $\inf \mathbb{R} = \inf \mathbb{Q} = \inf \mathbb{Z} = -\infty$ (recall that $\inf \mathbb{N} = \min \mathbb{N} = 1$).

2. If $a < b$, then *any* interval $[a, b]$, (a, b), $[a, b)$ or $(a, b]$ has supremum b and infimum a, even if one end is infinite. For example,
$$S = (7, \infty) = \{x \in \mathbb{R} : x > 7\}$$
has $\sup S = \infty$ and $\inf S = 7$.

3. Let $S = \{x \in \mathbb{R} : x^3 - 4x < 0\}$. With a little factorization, we see that
$$x^3 - 4x = x(x-2)(x+2) < 0 \iff x < -2 \text{ or } 0 < x < 2$$
It follows that $S = (-\infty, -2) \cup (0, 2)$, from which $\sup S = 2$ and $\inf S = -\infty$.

Exercises 1.5. *Key concepts:* $\pm\infty$ *are shorthands for* ***unboundedness****: they are* ***not numbers****!*

1. Give the infimum and supremum of each of the following sets:
   (a) $\{x \in \mathbb{R} : x < 0\}$
   (b) $\{x \in \mathbb{R} : x^3 \leq 8\}$
   (c) $\{x^2 : x \in \mathbb{R}\}$
   (d) $\{x \in \mathbb{R} : x^2 < 8\}$
2. Let $S \subseteq \mathbb{R}$ be non-empty, and let $-S = \{-s : s \in S\}$. Prove that $\inf S = -\sup(-S)$.
3. Let $S, T \subseteq \mathbb{R}$ be non-empty such that $S \subseteq T$. Prove that $\inf T \leq \inf S \leq \sup S \leq \sup T$.
4. If $\sup S < \inf S$, what can you say about S?

1.6 A Development of $\mathbb{R}$ (non-examinable)

The comment in footnote 7 constitutes a *synthetic* definition of the real numbers: there is essentially just one set with the required properties. While this might satisfy an algebra-addict, it is nice to be able to provide an explicit construction. The following approach uses so-called *Dedekind cuts.*

First one defines $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$. Use Peano's axioms and proceed as in sections 1.1 and 1.2. The operations $+, \cdot$ and $\leq$ are defined, first on $\mathbb{N}$ and then for $\mathbb{Z}$ and $\mathbb{Q}$ building on these concepts for the integers.

Definition 1.26. A *Dedekind cut* α^* is a non-empty proper subset of $\mathbb{Q}$ with the properties:

1. (Closed downwards) If $r \in \alpha^*$ and $s \in \mathbb{Q}$ with $s < r$, then $s \in \alpha^*$.
2. (No maximum) If M is an upper bound for α^*, then $M \notin \alpha^*$.

Define $\mathbb{R}$ to be the set of all Dedekind cuts!

The rough idea is that a real number α corresponds to the Dedekind cut α^* of all *rational numbers less than α.*

Examples 1.27. 1. For any *rational number* r, the corresponding *real number* is the Dedekind cut

$$r^* = \{x \in \mathbb{Q} : x < r\}$$

For instance $4^* = \{x \in \mathbb{Q} : x < 4\}$ is the Dedekind cut definition of the *real number* 4.

2. It is a little trickier to explicitly define cuts corresponding to irrational numbers, though some are relatively straightforward. For instance the real number $\sqrt{2}$ would be the set

$$\sqrt{2}^* = \{x \in \mathbb{Q} : x < 0 \text{ or } x^2 < 2\}$$

It remains to *prove* that the set of Dedekind cuts satisfies the axioms of a complete ordered field. The full details are too much, so here is a rough overview.

- Define the ordering of Dedekind cuts via

$$\alpha^* \leq \beta^* \iff \alpha^* \subseteq \beta^*$$

  One can now prove axioms O1–O3 and that the ordering corresponds to that of $\mathbb{Q}$.

- Define addition of cuts via

$$\alpha^* + \beta^* := \{a + b : a \in \alpha^*, b \in \beta^*\}$$

  This suffices to prove the addition axioms and O4: a careful definition of $-\alpha^*$ is required.

- Multiplication is horrible: if $\alpha^*, \beta^* \geq 0^*$ then

$$\alpha^*\beta^* := \{ab : a \geq 0, a \in \alpha^*, b \geq 0, b \in \beta^*\} \cup \{q \in \mathbb{Q} : q < 0\}$$

  which may be carefully extended to cover situations when α^* or $\beta^* < 0^*$. Once this has been done, one can then prove the multiplication axioms, the final order axiom O5, and the distributive axiom.

- The completeness axiom must also be verified, though it comes almost for free! If $A \subseteq \mathbb{R}$ (a set of Dedekind cuts), then the supremum of A is simply

  $$\sup A = \bigcup_{\alpha^* \in A} \alpha^*$$

  Think about it...

An alternative approach to $\mathbb{R}$ using sequences of rational numbers will be given later.

Exercises 1.6. *Key concepts:* $\mathbb{R}$ *is unnatural and difficult to construct in a logical manner*

1. Show that if α^*, β^* are Dedekind cuts, then so is

   $$\alpha^* + \beta^* = \{r_1 + r_2 : r_1 \in \alpha^*, r_2 \in \beta^*\}$$

2. Let α^*, β^* be Dedekind cuts and define the 'product':

   $$\alpha^* \cdot \beta^* = \{r_1 r_2 : r_1 \in \alpha^*, r_2 \in \beta^*\}$$

   (a) Calculate some 'products' using the cuts $0^*, 1^*$ and $(-1)^*$.

   (b) Discuss why this 'product' is unsatisfactory for defining multiplication in $\mathbb{R}$.

3. We verify the Archimedean property (Theorem 1.22) using the Dedekind cut definition of $\mathbb{R}$ (it is somewhat easier since the unboundedness of $\mathbb{N}$ and $\mathbb{Q}$ are baked in).

   (a) Explain why every cut β^* is bounded above by some rational number.
   (*Hint: if β^* satisfies Definition 1.26 parts 1 & 2 but is unbounded above, then what is it?*)

   (b) If $\beta^* > 0^*$ is a positive cut bounded above by $\frac{p}{q}$ with $p, q \in \mathbb{N}$, show that $n := p + 1$ corresponds to a cut for which $n^* > \beta^*$.

2 Sequences

Sequences are the fundamental tool in our approach to analysis.

2.7 Limits of Sequences

Definition 2.1. A *sequence* of real numbers is a list indexed by the natural numbers

$$(s_n) = (s_1, s_2, s_3, \ldots)$$

We call s_1 the *initial term/element*.

More formally, we could view a sequence as a function $s_n : \mathbb{N} \to \mathbb{R}$. Other letters may be used (a_n, b_n, etc.), though s_n is typical in the abstract. It is also common to have sequences which start with a different initial term (e.g., $n = 0$). If you need to be explicit, write, e.g., $(s_n)_{n=0}^{\infty}$.

Examples 2.2. 1. Explicit sequences are often defined by providing a formula for the n^{th} term. For instance, $s_n = \left(1 + \frac{1}{n}\right)^n$ defines a sequence whose first three terms are

$$s_1 = 2, \quad s_2 = \tfrac{9}{4}, \quad s_3 = \tfrac{64}{27}, \quad \ldots$$

Since each term is a rational number, (s_n) could be described as a *rational sequence.*

2. Sequences can be defined inductively. For instance, if $t_1 = 1$ and $t_{n+1} = 3t_n - 1$, then

$$(t_n) = (1, 2, 5, 14, 41, \ldots)$$

3. $u_n = \frac{1}{n^2-4}$ defines a sequence with initial term $u_3 = \frac{1}{5}$:

$$(u_n)_{n=3}^{\infty} = \left(\tfrac{1}{5}, \tfrac{1}{12}, \tfrac{1}{21}, \ldots\right)$$

Limits We are typically most interested in what happens to the terms of a sequence when n gets *large* (one reason it is common to be non-explicit as to the initial term). In elementary calculus you should have become used to examples such as[9]

$$\lim \frac{2n^2 + 3n - 1}{3n^2 - 2} = \frac{2}{3}$$

which encapsulates the idea that the expression $s_n = \frac{2n^2+3n-1}{3n^2-2}$ gets 'close to' $\frac{2}{3}$ when n is large. We can easily convince ourselves of this with a calculator/computer: to 4 decimal places,

$$(s_n) = \big(4,\ 1.3,\ 1.04,\ 0.9348,\ 0.8767,\ 0.8396,\ 0.8138,\ 0.7947,\ \ldots\big), \qquad s_{1000} = 0.6677$$

Our primary business is to make this idea logically watertight, the major issue being what is meant by 'close to.' In Section 2.8 we will do so by developing the formal definition of limit. Before seeing this, we quickly refresh a few simple examples. All these examples can be made formal later, but for the present just rely on your intuition and experience: it is essential to have a good idea of the correct answer *before* you try to prove it!

[9] If there are multiple letters around, writing $\lim\limits_{n\to\infty}$ with a subscript can aid the reader.

Examples 2.3. 1. $\lim \frac{1}{n} = 0$. Our instinct is that $s_n = \frac{1}{n}$ becomes arbitrarily small as n becomes large.

2. $\lim \frac{7n+9}{2n-4} = \frac{7}{2}$. To convince yourself of this, you might write $\frac{7n+9}{2n-4} = \frac{7+\frac{9}{n}}{2-\frac{4}{n}}$ and observe that the $\frac{1}{n}$ terms become tiny as n increases.

3. The sequence with n^{th} term $s_n = (-1)^n$ does not converge to anything: it *diverges*.

$$(s_n)_{n=0}^{\infty} = (1, -1, 1, -1, 1, -1, \ldots)$$

4. If $c_n = \frac{1}{n}\cos\left(\frac{\pi n}{6}\right)$, then $\lim c_n = 0$. To see this, observe that the cosine term lies between ± 1, while $\frac{1}{n}$ has limit 0.

5. The sequence defined inductively by $s_0 = 2$, $s_{n+1} := \frac{1}{2}s_n + 3$ begins

$$(s_n) = \left(2, 4, 5, \tfrac{11}{2}, \tfrac{23}{4}, \tfrac{47}{8}, \ldots\right)$$

This appears to have limit $\lim s_n = 6$. It is not hard to spot the pattern $s_n = 6 - \frac{4}{2^n}$, which may easily be verified by induction: for the induction step, observe that

$$\frac{1}{2}s_n + 3 = \frac{1}{2}\left(6 - \frac{4}{2^n}\right) + 3 = 6 - \frac{4}{2^{n+1}}$$

Exercises 2.7. *Key concepts: Sequences, Use your intuition!*

1. Decide whether each sequence converges; if it does, state the limit. No proofs are required; if you're unsure what's going on, try writing out the first few terms.

   (a) $a_n = \dfrac{1}{3n+1}$ (b) $b_n = \dfrac{3n+1}{4n-1}$ (c) $c_n = \dfrac{n}{3^n}$ (d) $d_n = \sin\left(\dfrac{n\pi}{4}\right)$

2. Repeat the previous question for sequences whose n^{th} term is as follows:

   (a) $\dfrac{n^2+3}{n^2-3}$ (b) $1 + \dfrac{2}{n}$ (c) $2^{1/n}$ (d) $(-1)^n n$ (e) $\dfrac{7n^3+8n}{2n^3-31}$

   (f) $\sin\left(\dfrac{n\pi}{2}\right)$ (g) $\sin\left(\dfrac{2n\pi}{3}\right)$ (h) $\dfrac{2^{n+1}+5}{2^n-7}$ (i) $\left(1+\dfrac{1}{n}\right)^2$ (j) $\dfrac{6n+4}{9n^2+7}$

3. Give an example of:

   (a) A sequence (x_n) of irrational numbers having a limit $\lim x_n$ that is a rational number.
   (b) A sequence (r_n) of rational numbers having a limit $\lim r_n$ that is an irrational number.

4. Prove by induction that the sequence defined in Example 2.2.2 has n^{th} term $t_n = \frac{1}{2}(3^{n-1}+1)$.

5. In future courses, you'll meet sequences of *functions*. For instance, we could define a sequence (f_n) of functions $f_n : \mathbb{R} \to \mathbb{R}$ inductively via

$$f_0(x) \equiv 1, \qquad f_{n+1}(x) := 1 + \int_0^x f_n(t)\,\mathrm{d}t$$

   Compute the functions f_1, f_2 and f_3. The sequence (f_n) should seem familiar if you think back to elementary calculus; why?

2.8 The Formal Definition of Limit

While intuition regarding limits is essential, mathematics is about *proving* things rigorously; for this one needs a formal *definition*. The essential difficulty is that 'close to' is poorly defined without context; we get round this by considering all possible measures of closeness simultaneously.

Definition 2.4. To say that a sequence (s_n) *converges to a limit* $s \in \mathbb{R}$ means,[10]

$$\forall \epsilon > 0, \ \exists N \text{ such that } n > N \implies |s_n - s| < \epsilon$$

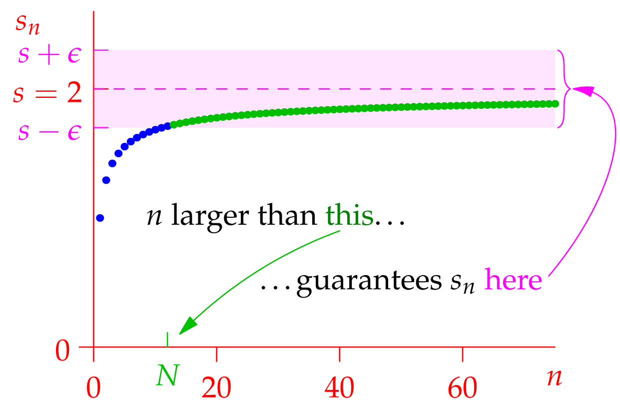
We write $\lim s_n = s$ or simply $s_n \to s$; both are read "s_n *approaches* (or *tends to*) s."
A sequence *converges* if it has a limit, and *diverges* otherwise.

Try reading it this way: given any measure of closeness, we can make n large enough so that (by our given measure) s_n is close to s. This isn't as hard as it looks, though a *lot* of examples will likely be necessary for it to sink in...

Example 2.5. We prove that the sequence with

$$s_n = 2 - \frac{1}{\sqrt{n}}$$

converges to $s = 2$.



By plotting the sequence, we see how ϵ controls the distance (closeness) from s_n to the limit $s = 2$; no matter the size of ϵ, we must show that (s_n) has some tail whose terms are closer to s.

Proving such 'for all, there exists' statements requires a specific argument structure:

- Suppose $\epsilon > 0$ has been given and describe N as a function of ϵ (ϵ smaller means N larger).
- Verify algebraically that $n > N \implies |s_n - s| < \epsilon$.

Scratch work. To find a suitable N, start with what you want to be true and let it inspire you.

Whenever $n > N$, we want $\epsilon > |s_n - s| = \left|\left(2 - \frac{1}{\sqrt{n}}\right) - 2\right| = \left|\frac{1}{\sqrt{n}}\right|$.

Choosing $N = \frac{1}{\epsilon^2}$ should be enough to complete the proof!

Warning! "$N = \frac{1}{\epsilon^2}$" is not the correct conclusion! To make it clear that we've satisfied the definition, we rearrange our scratch work: these last three lines are all you *need* to write!

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \frac{1}{\epsilon^2}$. Then

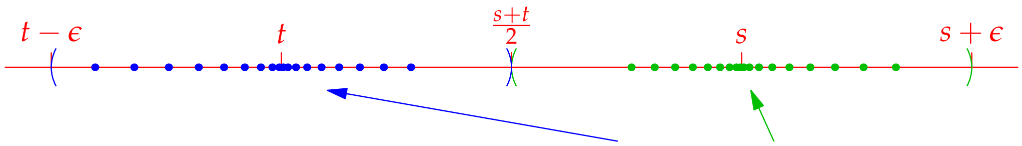
$$n > N \implies |s_n - s| = \left|2 - \frac{1}{\sqrt{n}} - 2\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \epsilon$$

Thus $\lim s_n = 2$, as required.

[10] N may be a real number or a natural number (equivalent by the Archimedean property, Theorem 1.22). It tends to be easier to use $N \in \mathbb{R}$ for convergence and $N \in \mathbb{N}$ when directly proving *divergence* (see Definition 2.9 and Examples 2.10).

Lemma 2.6 (Uniqueness of Limit). *If (s_n) converges, then its limit is unique.*

The proof structure should be familiar from other uniqueness arguments: assume there are two limits $s \neq t$ and obtain a contradiction. The picture explains the strategy: by choosing $\epsilon = \frac{|s-t|}{2}$ in the definition we obtain a *tail* of the sequence which must be simultaneously close to *both limits*.



For all $n > N$, s_n must lie both here and here!

Proof. Suppose $s \neq t$ are two limits. Take $\epsilon = \frac{|s-t|}{2}$ and apply Definition 2.4 twice: $\exists N_1, N_2$ such that

$$n > N_1 \implies |s_n - s| < \frac{|s-t|}{2} \qquad \text{and} \qquad n > N_2 \implies |s_n - t| < \frac{|s-t|}{2}$$

Define $N := \max(N_1, N_2)$. Taking any $n > N$ quickly yields a contradiction:

$$\begin{aligned} n > N \implies |s-t| &= |s - s_n + s_n - t| \le |s_n - s| + |s_n - t| && (\triangle\text{-inequality}) \\ &< \frac{|s-t|}{2} + \frac{|s-t|}{2} = |s-t| \end{aligned}$$

■

Theorem 2.7. *If $k > 0$ is constant, then* $\lim \frac{1}{n^k} = 0$.

Proof. In this, and the examples that follow, only the formal arguments are required; scratch work is included to show the thought process.

Scratch work. We want $n > N \implies \frac{1}{n^k} < \epsilon$. That is, $n > \frac{1}{\epsilon^{1/k}}$. It should be enough to choose $N = \frac{1}{\epsilon^{1/k}}$.

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \frac{1}{\epsilon^{1/k}}$. Then

$$n > N \implies \left| \frac{1}{n^k} - 0 \right| = \frac{1}{n^k} < \frac{1}{N^k} = \epsilon$$

■

We conclude that $\lim \frac{1}{n^k} = 0$.

Examples 2.8. 1. We prove that $\lim \left(\sqrt{n+4} - \sqrt{n} \right) = 0$.

Scratch work. We use a (hopefully) familiar trick for manipulating surd expressions:

$$\sqrt{n+4} - \sqrt{n} = \frac{4}{\sqrt{n+4} + \sqrt{n}} < \frac{4}{2\sqrt{n}} = \frac{2}{\sqrt{n}}$$

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \frac{4}{\epsilon^2}$. Then

$$n > N \implies \left| \sqrt{n+4} - \sqrt{n} \right| = \frac{4}{\sqrt{n+4} + \sqrt{n}} < \frac{4}{2\sqrt{n}} = \frac{2}{\sqrt{n}} < \frac{2}{\sqrt{N}} = \epsilon$$

Thus $\lim \left(\sqrt{n+4} - \sqrt{n} \right) = 0$.

2. We prove that $\lim \frac{3n+1}{n-7} = 3$.

Scratch work. Given $\epsilon > 0$, we want to choose N such that

$$n > N \implies \left| \frac{3n+1}{n-7} - 3 \right| = \left| \frac{(3n+1) - 3(n-7)}{n-7} \right| = \left| \frac{22}{n-7} \right| < \epsilon \qquad (*)$$

For large n ($n > 7$) everything is positive, so it is sufficient to have $n - 7 > \frac{22}{\epsilon} \ldots$

Formal argument 1. Suppose $\epsilon > 0$ is given and let $N = 7 + \frac{22}{\epsilon}$. Then

$$n > N \implies \left| \frac{3n+1}{n-7} - 3 \right| = \frac{22}{n-7} < \frac{22}{N-7} = \epsilon$$

The absolute values are dropped since $n > 7$. We conclude that $\lim \frac{3n+1}{n-7} = 3$.

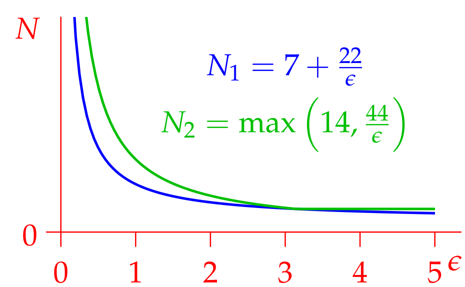
Scratch work 2. An alternative approach is available if we play with $(*)$ a little. By insisting that $n \geq 14$, we can simplify the denominator $n - 7 \geq \frac{1}{2}n \implies \frac{22}{n-7} \leq \frac{44}{n}$.

Formal argument 2. Suppose $\epsilon > 0$ is given and let $N = \max\left(14, \frac{44}{\epsilon}\right)$. Then

$$n > N \implies \left| \frac{3n+1}{n-7} - 3 \right| = \left| \frac{22}{n-7} \right| \leq \frac{22}{\frac{1}{2}n} = \frac{44}{n} \qquad \text{(since } n \geq 14\text{)}$$

$$< \frac{44}{N} \leq \epsilon \qquad \text{(since } N \geq \tfrac{44}{\epsilon}\text{)}$$

We again conclude that $\lim \frac{3n+1}{n-7} = 3$.



The plot illustrates the two choices of N as functions of ϵ. The second is always larger than the first: if $N = N_1(\epsilon)$ works in a proof, then any larger choice $N_2(\epsilon)$ will work also; use this to your advantage to produce simpler arguments!

3. Given $s_n = \frac{2n^4-3n+1}{3n^4+n^2+4}$, we prove that $\lim s_n = \frac{2}{3}$.

Scratch work. We want to conclude that $\left| \frac{2n^4-3n+1}{3n^4+n^2+4} - \frac{2}{3} \right| = \left| \frac{-2n^2-9n-5}{3(3n^4+n^2+4)} \right| < \epsilon$. Attempting to solve for n is crazy! Instead we simplify the fraction using $n \geq 1$ and the $\triangle$-inequality:

$$\left| \frac{-2n^2 - 9n - 5}{3(3n^4 + n^2 + 4)} \right| \overset{\triangle}{\leq} \frac{16n^2}{3(3n^4 + n^2 + 4)} < \frac{16n^2}{9n^4} < \frac{2}{n^2} \qquad (1 \leq n \leq n^2 \text{ and } n^2 + 4 > 0)$$

The final simplification is merely for additional cleanliness.

Formal argument. Suppose $\epsilon > 0$ is given and let $N = \sqrt{\frac{2}{\epsilon}}$. Then,

$$n > N \implies \left| \frac{2n^4 - 3n + 1}{3n^4 + n^2 + 4} - \frac{2}{3} \right| = \left| \frac{-2n^2 - 9n - 5}{3(3n^4 + n^2 + 4)} \right| \overset{\triangle}{\leq} \frac{16n^2}{3(3n^4 + n^2 + 4)} \qquad (n \geq 1)$$

$$< \frac{16n^2}{9n^4} < \frac{2}{n^2} < \frac{2}{N^2} = \epsilon$$

Other choices of N are feasible (e.g., Exercise 3); everything depends on how you want to simplify things in your scratch work.

Divergent sequences

By negating Definition 2.4, we obtain explicit criteria for divergence.

Definition 2.9. A sequence (s_n) *does not converge to* $s \in \mathbb{R}$ if,

$$\exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

A sequence is *divergent* if it does not converge to *any* limit $s \in \mathbb{R}$. Otherwise said,

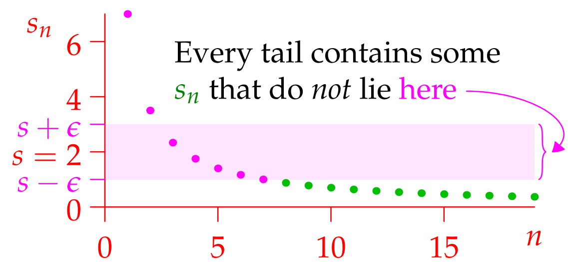
$$\forall s \in \mathbb{R},\ \exists \epsilon > 0 \text{ such that } \forall N,\ \exists n > N \text{ with } |s_n - s| \geq \epsilon$$

Examples 2.10. 1. We prove that the sequence with $s_n = \frac{7}{n}$ does not converge to $s = 2$.

Scratch work. Of course the limit is really 0.

If ϵ is anything smaller than 2, then s_n will eventually be further than ϵ from $s = 2$.

Choosing $\epsilon = 1$ should be enough. Indeed, since we only care about large n,



$$|s_n - 2| = \left|\frac{7}{n} - 2\right| = 2 - \frac{7}{n} \geq 1 \iff n \geq 7$$

Direct proof. Let $\epsilon = 1$. Given $N \in \mathbb{N}$, let $n = \max(7, N+1)$. Then $n > N$ and

$$|s_n - 2| = \left|\frac{7}{n} - 2\right| = 2 - \frac{7}{n} \geq 1 = \epsilon$$

Contradiction proof. For an alternative approach, suppose $\lim s_n = 2$ and let $\epsilon = 1$ in Definition 2.4. Then $\exists N$ such that

$$n > N \implies \left|\frac{7}{n} - 2\right| < 1 \implies 1 < \frac{7}{n} < 3 \implies \frac{7}{3} < n < 7$$

This last is false for large n, in particular for $n := \max(7, N+1)$. Contradiction.

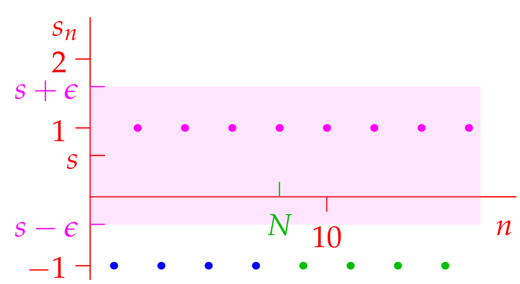
While the two arguments are similar, the contradiction approach has the advantage in that you need only remember *one definition*!

2. We prove (just by contradiction this time) that $s_n = (-1)^n$ defines a divergent sequence.

Suppose that $\lim s_n = s$ and let $\epsilon = 1$ in the definition of limit. Then $\exists N \in \mathbb{N}$ such that

$$\begin{aligned} n > N &\implies |(-1)^n - s| < 1 \\ &\implies (-1)^n - 1 < s < 1 + (-1)^n \end{aligned}$$

Choosing any even $n > N$ forces $0 < s$; any odd $n > N$ forces $s < 0$: contradiction.



The choice of $\epsilon = 1$ was suggested by the picture: if $\epsilon = 1$, then, regardless of N, there exist $n > N$ with $|s_n - s| > 1$.

3. We prove directly that the sequence defined by $s_n = \ln n$ is divergent.

   Scratch work. Since logarithms increase unboundedly,[11] for large n we should have $\ln n \geq s + 1$, for any purported limit $s \in \mathbb{R}$.

   Proof. Suppose $s \in \mathbb{R}$ is given and let $\epsilon = 1$. Given $N \in \mathbb{N}$, define $n = \max(N+1, e^{s+1})$. Then

$$n > N \quad \text{and} \quad \ln n \geq \ln(e^{s+1}) = s + 1 \qquad (\ln \text{ is } \textit{increasing})$$
$$\implies |s_n - s| = \ln n - s \geq 1 = \epsilon$$

   We conclude that (s_n) is divergent.

Bounded Sequences

As a first taste using the limit definition abstractly, we consider several related results regarding the boundedness of sequences.

Theorem 2.11. *Suppose (s_n) is convergent: $\lim s_n = s$.*

1. *If $s_n \geq m$ for all large[12] n, then $\lim s_n \geq m$.*
2. *If $s_n \leq M$ for all large n, then $\lim s_n \leq M$.*
3. *(s_n) is bounded ($\exists M$ such that $\forall n, |s_n| \leq M$).*

Proof. 1. We prove the contrapositive. Suppose $s < m$ and let $\epsilon = \frac{m-s}{2} > 0$. Then $\exists N$ such that

$$n > N \implies |s_n - s| < \frac{m-s}{2} \implies s_n - s < \frac{m-s}{2}$$
$$\implies s_n - m < \frac{s-m}{2} < 0 \implies s_n < m$$

2. This is almost identical to part 1.

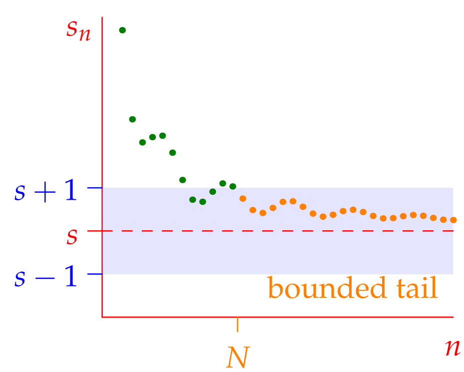
3. The picture shows the strategy: taking $\epsilon = 1$ in the limit definition bounds an infinite tail of the sequence; the finitely many terms that come before are a non-issue.

   Let $\epsilon = 1$ in the definition of limit. Then $\exists N$ such that

$$n > N \implies |s_n - s| < 1 \implies s - 1 < s_n < s + 1$$
$$\implies |s_n| < \max\{|s-1|, |s+1|\}$$

   It follows that every term of the sequence is bounded by

$$M := \max\big(|s-1|, |s+1|, |s_n| : n \leq N\big)$$



∎

[11] Definition 2.19 will state what it means for a sequence to *diverge to* ∞: this isn't (yet) what we're trying to demonstrate.

[12] Equivalently $s_n \geq m$ for all but finitely many n. In the language of the proof, for all n expect perhaps when $n \leq N$. Since we typically care only about large n, this caveat is sometimes left unstated: e.g., $s_n \geq m \implies \lim s_n \geq m$.

Theorem 2.12 (Squeeze Theorem). *Suppose sequences satisfy $a_n \le s_n \le b_n$ (for all large n), where the two outer sequences converge to s. Then $\lim s_n = s$.*

Proof. Subtract s from the assumed inequality to obtain

$$a_n - s \le s_n - s \le b_n - s \implies |s_n - s| \le \max(|a_n - s|, |b_n - s|)$$

Suppose $\epsilon > 0$ is given. Since $\lim a_n = s = \lim b_n$, there exist N_a, N_b such that

$$n > N_a \implies |a_n - s| < \epsilon \qquad \text{and} \qquad n > N_b \implies |b_n - s| < \epsilon$$

Now let $N = \max(N_a, N_b)$ to see that

$$n > N \implies |s_n - s| \le \max(|a_n - s|, |b_n - s|) < \epsilon$$

■

Example 2.13. Since $0 \le \frac{1+\sin n}{n} \le \frac{2}{n}$ for all n, the squeeze theorem forces $\lim \frac{1+\sin n}{n} = 0$.

Exercises 2.8. *Key concepts: ϵ–N definition, divergence, boundedness, squeeze theorem*

1. For each sequence, determine the limit and prove your claim.

   (a) $a_n = \frac{n}{n^2+1}$ (b) $b_n = \frac{7n-19}{3n+7}$ (c) $c_n = \frac{4n+3}{7n-5}$

   (d) $d_n = \frac{2n+4}{5n+2}$ (e) $e_n = \frac{1}{n}\sin n$ (f) $f_n = \frac{n^2+n-1}{3n^2-10}$

2. Prove:

   (a) $\lim[\sqrt{n^2+1} - n] = 0$ (b) $\lim[\sqrt{n^2+n} - n] = \frac{1}{2}$ (c) $\lim[\sqrt{4n^2+n} - 2n] = \frac{1}{4}$

3. (a) Show that $n \ge 2 \implies 2n^2 + 9n + 5 \le 9n^2$.

   (b) (Example 2.8.3) Give another proof that $\lim \frac{2n^4-3n+1}{3n^4+n^2+4} = \frac{2}{3}$ by choosing $N = \max(2, \frac{1}{\sqrt{\epsilon}})$.

4. (a) Prove that the sequence with n^{th} term $s_n = \frac{2}{n^2}$ does not converge to -1.

   (b) Prove that (s_n) does not converge to 1.

5. Prove that the sequence defined by $t_n = n^2$ diverges.

6. (Example 2.10.3) Prove by contradiction that $(\ln n)$ diverges.

7. True or false: if (s_n) is bounded, then it is convergent. Explain.

8. Let (t_n) be bounded, and let (s_n) satisfy $\lim s_n = 0$. Prove that $\lim(s_n t_n) = 0$.

9. We extend Theorem 2.11.

   (a) Suppose $\lim s_n = s$ where every $s_n > m$. Can we conclude that $s > m$? Explain.

   (b) Let (s_n) be convergent and suppose $\lim s_n > a$. Prove that $\exists N$ such that $n > N \implies s_n > a$.

10. Suppose $s \in \mathbb{R}$. Prove:

    (a) $\lim s_n = s \iff \lim(s_n - s) = 0$

    (b) $\lim s_n = s \implies \lim |s_n| = |s|$

2.9 Limit Theorems for Sequences

We'd like to develop rules for limits so that we don't have to resort to ϵ–N proofs every time. The rough idea of these results is that limits respect the basic rules of algebra. Rather than dive straight into abstract proofs, we start with two special cases that illustrate commonly encountered tricks.

Examples 2.14. Suppose that (s_n) converges to s. We prove that $\lim 5s_n = 5s$ and that $\lim s_n^2 = s^2$.

1. The sequence $(5s_n)$ is obtained by multiplying the original terms by 5. To prove $\lim 5s_n = 5s$ we must show:

   $$\forall \epsilon > 0, \exists N \text{ such that } n > N \implies |5s_n - 5s| < \epsilon \qquad (*)$$

   This last amounts to observing that $|s_n - s| < \frac{\epsilon}{5}$. Since $\frac{\epsilon}{5}$ is simply another small number, $(*)$ is just the statement $\lim s_n = s$ in disguise! Here is a formal argument.

   Let $\epsilon > 0$ be given. Since $\lim s_n = s$, we know that[13]

   $$\exists N \text{ such that } n > N \implies |s_n - s| < \frac{\epsilon}{5} \implies |5s_n - 5s| < \epsilon$$

2. The challenge is to make $|s_n^2 - s^2| = |s_n - s|\,|s_n + s|$ small. The first term can be made arbitrarily small by the hypothesis $\lim s_n = s$. To control the second, we use the triangle-inequality:

   $$|s_n + s| = |s_n - s + 2s| \le |s_n - s| + 2\,|s|$$

   Assuming $|s_n - s| \le 1$ gives a bound $|s + s_n| \le 1 + 2\,|s|$. We now have enough for a proof.

   Suppose $\lim s_n = s$, that $\epsilon > 0$ is given and let $\delta = \min(1, \frac{\epsilon}{1+2|s|})$. Then $\exists N$ such that,

   $$\begin{aligned} n > N &\implies |s_n - s| < \delta \le 1 \\ &\implies |s_n^2 - s^2| = |s_n - s|\,|s_n + s| \overset{\triangle}{\le} |s_n - s|\,(|s_n - s| + 2\,|s|) \\ &\qquad\qquad < \delta(1 + 2\,|s|) \le \epsilon \end{aligned}$$

Theorem 2.15 (Limit laws). *Limits respect algebraic operations: $\pm, \times, \div$ and rational roots. More specifically, if (s_n) converges to s and (t_n) to t, then,*

1. $\lim(s_n \pm t_n) = s \pm t$
2. $\lim(s_n t_n) = st$; *in particular, if k is constant, then* $\lim ks_n = ks$
3. *If $t \neq 0$, then* $\lim \frac{s_n}{t_n} = \frac{s}{t}$
4. *If $k \in \mathbb{N}$, then* $\lim \sqrt[k]{s_n} = \sqrt[k]{s}$*, provided the roots exist ($s_n, s \ge 0$ if k even)*

By part 4 and induction on part 2, $\lim s_n^q = s^q$ *for any $q \in \mathbb{Q}$.*

Examples 2.14 are special cases of part 2 ($k = 5$ and then $t_n = s_n$).

[13] It is non-standard, but if this approach makes you squeamish, you can introduce a second $\tilde{\epsilon}$:

Given $\epsilon > 0$, let $\tilde{\epsilon} = \frac{\epsilon}{5}$, then $\exists N$ such that $n > N \implies |s_n - s| < \tilde{\epsilon} \implies |5s_n - 5s| < \epsilon$.

Rigorously proving the limit laws takes some work. Before engaging in some of this, we advertise their benefit by performing some calculations as you might have seen in elementary calculus.

Examples 2.16. 1. We evaluate $\lim \frac{3n^2+2\sqrt{n}-1}{5n^2-2}$ using the limit laws.

$$\lim \frac{3n^2+2\sqrt{n}-1}{5n^2-2} = \lim \frac{3+\frac{2}{n^{3/2}}-\frac{1}{n^2}}{5-\frac{2}{n^2}} \qquad (n>0)$$
$$= \frac{\lim\left(3+\frac{2}{n^{3/2}}-\frac{1}{n^2}\right)}{\lim\left(5-\frac{2}{n^2}\right)} \qquad \text{(part 3)}$$
$$= \frac{\lim 3+\lim\frac{2}{n^{3/2}}-\lim\frac{1}{n^2}}{\lim 5-\lim\frac{2}{n^2}} \qquad \text{(part 1)}$$
$$= \frac{3+0-0}{5-0} = \frac{3}{5} \qquad \text{(parts 2, 4 and Theorem 2.7))}$$

The calculation involves some generally accepted sleight of hand: none of the limits should really be written until one knows they exist. To be completely logical would require us to rewrite the argument upside down, though to sacrifice readability in this manner would be ill-advised.

2. Suppose (s_n) is defined inductively via $s_1 = 2$ and $s_{n+1} = \frac{1}{2}(s_n + \frac{2}{s_n})$:

$$(s_n) = \left(2, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \ldots\right)$$

This sequence in fact converges, though a proof requires ideas from Section 2.10. Given this fact, the limit laws allow us to compute the limit s:

$$s = \lim s_{n+1} = \frac{1}{2}\left(\lim s_n + \frac{2}{\lim s_n}\right) = \frac{1}{2}\left(s+\frac{2}{s}\right) \implies \frac{1}{2}s = \frac{1}{s} \implies s^2 = 2$$

Since s_n is plainly always positive, we conclude that $\lim s_n = \sqrt{2}$.

We now commence our assault on the limit laws.

Proof. 1. We use a trick similar to that in Example 2.14.1: control both sequences so that both $|s_n - s|, |t_n - t| < \frac{\epsilon}{2}$, then sum.

Suppose $\epsilon > 0$ is given. Since $\lim s_n = s$ and $\lim t_n = t$, we see that $\exists N_1, N_2$ such that

$$n > N_1 \implies |s_n - s| < \frac{\epsilon}{2} \qquad \text{and} \qquad n > N_2 \implies |t_n - t| < \frac{\epsilon}{2}$$

Let $N = \max(N_1, N_2)$, then

$$n > N \implies |s_n + t_n - (s+t)| \overset{\triangle}{\le} |s_n - s| + |t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

The argument for $s_n - t_n$ is almost identical.

2. Exercise 2.8.8 deals with (and extends) the case when $s = 0$. We therefore suppose $s \neq 0$. The approach is similar to part 1, we just need to be a bit cleverer to break up $|s_n t_n - st|$.

   Suppose $\epsilon > 0$ is given. By Theorem 2.11, (t_n) is bounded:

   $$\exists M \text{ such that } \forall n,\ |t_n| \leq M$$

   We make assume, WLOG, that $M > 0$. Since $\lim s_n = s$ and $\lim t_n = t$, $\exists N_1, N_2$ such that

   $$n > N_1 \implies |s_n - s| < \frac{\epsilon}{2M} \qquad \text{and} \qquad n > N_2 \implies |t_n - t| < \frac{\epsilon}{2\,|s|}$$

   Let $N = \max(N_1, N_2)$, then

   $$|s_n t_n - st| = |s_n t_n - st_n + st_n - st| \overset{\triangle}{\leq} |s_n - s|\,|t_n| + |s|\,|t_n - t| < \frac{\epsilon}{2M} M + |s| \frac{\epsilon}{2\,|s|} = \epsilon$$

3 & 4.: See Exercise 7.

■

With a few general examples, the limit laws allow us to rapidly compute a great variety of limits.

Theorem 2.17. 1. *If* $|a| < 1$ *then* $\lim a^n = 0$

2. *If* $a > 0$ *then* $\lim a^{1/n} = 1$

3. $\lim n^{1/n} = 1$

Examples 2.18. 1. $\lim (3n)^{2/n} = (\lim 3^{1/n})^2 (\lim n^{1/n})^2 = 1$.

2. Observe from the squeeze theorem (2.12) that $\left|\frac{\sin n}{n}\right| \leq \frac{1}{n} \to 0$. We conclude:

$$\lim \frac{n^{2/n} + \left(3 - n^{-1}\sin n\right)^{1/5}}{4n^{-3/2} + 7} = \frac{(\lim n^{1/n})^2 - \left(3 - \lim \frac{\sin n}{n}\right)^{1/5}}{4 \lim \frac{1}{n^{3/2}} + 7} = \frac{1 - \sqrt[5]{3}}{7}$$

Proof. 1. The $a = 0$ case is trivial. Otherwise, given $\epsilon > 0$, let $N = \log_{|a|} \epsilon$, then

$$n > N \implies |a^n| < \left|a^N\right| = |a|^N = \epsilon$$

2. Suppose $a \geq 1$ and let $s_n = a^{1/n} - 1$. Since $s_n > 0$, the binomial theorem[14] shows that

   $$a = (1 + s_n)^n \geq 1 + ns_n \implies 0 < s_n \leq \frac{a - 1}{n}$$

   The squeeze theorem shows that $\lim s_n = 0$, whence $\lim a^{1/n} = 1$.

   The $a < 1$ case and part 3 are in Exercise 8.

■

[14] $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{2\cdot 3}x^3 + \cdots + x^n$.

Divergence to $\pm\infty$ and the 'divergence laws'

We consider unbounded sequences and provide a positive definition of a type of divergence.

Definition 2.19. We say that (s_n) *diverges to* ∞ if,

$$\forall M > 0,\ \exists N \text{ such that } n > N \implies s_n > M$$

We write $\lim s_n = \infty$, or $s_n \to \infty$, and say that s_n 'tends to' ∞. The definition for $s_n \to -\infty$ is similar. If (s_n) neither converges nor diverges to $\pm\infty$, we say that it *diverges by oscillation*: you likely wrote $\lim s_n = \text{DNE}$ ("does not exist") in elementary calculus.

Consider how M describes "closeness" to infinity analogously to how ϵ measures closeness to s in the original definition of limit (2.4).

Examples 2.20. As with convergence arguments, some scratch work might be helpful.

1. We show that $\lim(n^2 + 4n) = \infty$.

   Let $M > 0$ be given, and let $N = \sqrt{M}$. Then

   $$n > N \implies n^2 + 4n > n^2 > N^2 = M$$

2. We prove that $s_n = n^5 - n^4 - 2n + 1$ diverges to ∞.

   The negative terms cause some trouble, though our solution should be familiar from previous calculations:

   $$s_n > \frac{1}{2}n^5 \iff n^5 > 2(n^4 + 2n - 1) \iff n > 2 + \frac{4}{n^3} - \frac{1}{n^4}$$

   Certainly this holds if $n > 6$. We can now complete the proof.

   Let $M > 0$ be given, and let $N = \max(6, \sqrt[5]{2M})$. Then

   $$n > N \implies s_n > \frac{1}{2}n^5 > \frac{1}{2}(2M) = M$$

3. We prove that $s_n = n^2 - n^3$ diverges to $-\infty$.

   First observe that

   $$s_n = n^2(1-n) < -\frac{1}{2}n^3 \iff 1 - n < -\frac{1}{2}n \iff n \geq 2$$

   Now let $M > 0$ be given,[15] and define $N = \max(2, \sqrt[3]{2M})$. Then

   $$n > N \implies n > 2 \implies s_n < -\frac{1}{2}n^3 < -\frac{1}{2}N^3 \leq -M$$

[15] You may prefer to phrase $\lim s_n = -\infty$ instead as

$$\forall m < 0,\ \exists N \text{ such that } n > N \implies s_n < m \qquad \text{(in our argument } M = -m\text{)}$$

Several of the limit laws can be adapted to sequences which diverge to $\pm\infty$.

Theorem 2.21. *Suppose* $\lim s_n = \infty$.

1. *If* $t_n \geq s_n$ *for all (large)* n*, then* $\lim t_n = \infty$
2. *If* $\lim t_n$ *exists and is* finite*, then* $\lim s_n + t_n = \infty$.
3. *If* $\lim t_n > 0$ *then* $\lim s_n t_n = \infty$.
4. $\lim \frac{1}{s_n} = 0$
5. *If* $\lim t_n = 0$ *and* $t_n > 0$ *for all (large)* n*, then* $\lim \frac{1}{t_n} = \infty$

Similar statements when $\lim s_n = -\infty$ *should be clear.*

Proof. We prove two parts; try the rest yourself.

2. Since (t_n) converges, it is bounded (below): $\exists m$ such that $\forall n,\ t_n \geq m$. Let M be given. Since $\lim s_n = \infty$, $\exists N$ such that

$$n > N \implies s_n > M - m \implies s_n + t_n > M - m + m = M$$

4. Let $\epsilon > 0$ be given, and let $M = \frac{1}{\epsilon}$. Then $\exists N$ such that

$$n > N \implies s_n > M = \frac{1}{\epsilon} \implies \frac{1}{s_n} < \epsilon$$

■

Rational Sequences

We can now describe the limit of any sequence $\frac{p_n}{q_n}$ where (p_n), (q_n) are polynomials in n.

Example 2.22. By applying Theorem 2.21 (part 3) to

$$s_n := 3n + 4n^{-2} \to \infty \quad \text{and} \quad t_n = \frac{1}{2 - n^{-2}} \to \frac{1}{2}$$

we see that

$$\lim \frac{3n^3 + 4}{2n^2 - 1} = \lim \frac{3n + 4n^{-2}}{2 - n^{-2}} = \lim(3n + 4n^{-2}) \cdot \lim \frac{1}{2 - n^{-2}} = \infty$$

More generally, you should be able to confirm a familiar result from elementary calculus:

Corollary 2.23. *If* p_n, q_n *are polynomials in* n *with leading coefficients* p, q *respectively then*

$$\lim \frac{p_n}{q_n} = \begin{cases} 0 & \text{if } \deg(p_n) < \deg(q_n) \\ \frac{p}{q} & \text{if } \deg(p_n) = \deg(q_n) \\ \operatorname{sgn}(\frac{p}{q})\infty & \text{if } \deg(p_n) > \deg(q_n) \end{cases}$$

Exercises 2.9. *Key concepts: Divergence to $\pm\infty$, Limit/divergence laws,* $\lim a^{1/n} = 1 = \lim n^{1/n}$

1. Suppose $\lim x_n = 3$, $\lim y_n = 7$ and that all y_n are non-zero. Determine the following:

   (a) $\lim(x_n + y_n)$ (b) $\lim \frac{3y_n - x_n}{y_n^2}$ (c) $\lim \sqrt{x_n y_n + 4}$

2. Suppose $s \in \mathbb{R}$. Prove that $\lim s_n = s \implies \lim s_n^3 = s^3$ by mimicking Example 2.14.2.

3. Let $s_n = (100n)^{\frac{100}{n}}$. Describe s_1 and s_{10} (1 followed by how many zeros?). Now compute $\lim s_n$.

4. Define (s_n) inductively via $s_1 = 1$ and $s_{n+1} = \sqrt{s_n + 1}$ for $n \geq 1$.

   (a) List the first four terms of (s_n).
   (b) It turns out that (s_n) converges. Assume this and prove that $\lim s_n = \frac{1}{2}(1 + \sqrt{5})$.

5. Prove:

   (a) $\lim(n^3 - 98n) = \infty$ (b) $\lim\left(\sqrt{n} - n + \frac{4}{n}\right) = -\infty$

6. Let $x_1 = 1$ and $x_{n+1} = 3x_n^2$ for $n \geq 1$.

   (a) Show that if (x_n) converges with limit a, then $a = \frac{1}{3}$ or $a = 0$.
   (b) What is $\lim x_n$? Prove your assertion and explain what is going on.

7. We prove parts 3 and 4 of the limit laws (Theorem 2.15). Assume $\lim s_n = s$ and $\lim t_n = t$.

   (a) Suppose $t \neq 0$. Explain why $\exists N_1$ such that $n > N_1 \implies |t_n| > \frac{1}{2}|t|$.
   (b) Let $\epsilon > 0$ be given. Since $\lim t_n = t$, $\exists N_2$ such that $n > N_2 \implies |t_n - t| < \frac{1}{2}|t|^2 \epsilon$. Combine N_1 and N_2 to prove that $\lim \frac{1}{t_n} = \frac{1}{t}$.
   (c) Explain how to conclude part 3: $\lim \frac{s_n}{t_n} = \frac{s}{t}$.
   (d) Use the following inequality (valid when $s_n, s > 0$) to construct a proof for part 4

   $$\left|s_n^{1/k} - s^{1/k}\right| = \frac{|s_n - s|}{s_n^{\frac{k-1}{k}} + s_n^{\frac{k-2}{k}} s^{\frac{1}{k}} + \cdots + s^{\frac{k-1}{k}}} \leq \frac{|s_n - s|}{s^{\frac{k-1}{k}}}$$

8. We finish the proof of Theorem 2.17.

   (a) Suppose $0 < a < 1$. By considering $b = \frac{1}{a}$, prove that $\lim a^{1/n} = 1$.
   (b) Let $s_n = n^{1/n} - 1$. Apply the binomial theorem to $n = (1 + s_n)^n$ to prove that $s_n < \sqrt{\frac{2}{n-1}}$. Hence conclude that $\lim n^{1/n} = 1$.

9. Prove the remaining parts of Theorem 2.21.

10. Assume $s_n \neq 0$ for all n and that the limit $L = \lim \left|\frac{s_{n+1}}{s_n}\right|$ exists.

    (a) Show that if $L < 1$, then $\lim s_n = 0$.
    (*Hint: if $L < a < 1$, obtain N so that $n > N \implies |s_n| < a^{n-N}|s_N|$*)
    (b) Show that if $L > 1$, then $\lim |s_n| = +\infty$.
    (*Hint: apply (a) to the sequence $t_n = \frac{1}{|s_n|}$*)

11. Let $p > 0$ and $a \in \mathbb{R}$ be given. How does $\lim\limits_{n\to\infty} \frac{a^n}{n^p}$ depend on the value of a?

2.10 Monotone and Cauchy Sequences

The definition of limit (Definition 2.4) has a major weakness: to demonstrate the convergence of a sequence we must already know its limit! What we'd like is a method for determining whether a sequence converges *without* first guessing a suitable limit.[16] In this section we consider two important classes of sequence for which this can be done.

Definition 2.24. A sequence (s_n) is said to be:

- *Monotone-up*[17] if $s_{n+1} \geq s_n$ for all n.
- *Monotone-down* if $s_{n+1} \leq s_n$ for all n.
- *Monotone* if either of the above is true.

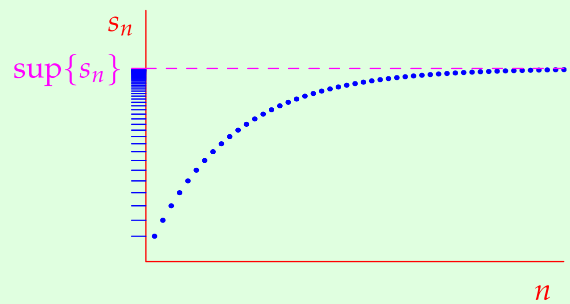
Examples 2.25. 1. The sequence with n^{th} term $s_n = \frac{7}{n} + 4$ is (strictly) monotone-down:

$$s_{n+1} = \frac{7}{n+1} < \frac{7}{n} = s_n$$

2. A constant sequence $(s_n) = (s, s, s, s, \ldots)$ is *both* monotone-up and monotone-down.

Theorem 2.26 (Monotone Convergence).
Every bounded monotone sequence is convergent. Specifically:

- *If (s_n) is bounded above and monotone-up, then $\lim s_n$ exists and equals $\sup\{s_n\}$.*
- *If (s_n) is bounded below and monotone-down, then $\lim s_n$ exists and equals $\inf\{s_n\}$.*



In fact the conclusion $\lim s_n = \sup\{s_n\}$ holds for all monotone-up sequences: if unbounded above, then the result is ∞ (Exercise 5).

Proof. If (s_n) is bounded above, then $s := \sup\{s_n\}$ exists by the completeness axiom (s is finite!). Let $\epsilon > 0$ be given. By Lemma 1.20, there exists some $s_N > s - \epsilon$. Since (s_n) is monotone-up,

$$n > N \implies s_n \geq s_N > s - \epsilon \implies 0 \leq s - s_n < \epsilon \implies |s - s_n| < \epsilon$$

The monotone-down case is similar. ■

[16] This gets at a typical application of sequences: given a sequence whose elements have a useful property, one demonstrates the existence of a new object (the limit) to which (hopefully!) the useful property transfers. For instance, if (f_n) is a sequence of differentiable functions, we'd like to know if $\lim f_n(x)$ exists and is itself differentiable with derivative $\lim f_n'(x)$. Discussions of this ilk dominate Math 140B.

[17] Some authors describe a monotone-up sequence as either *non-decreasing* or *increasing*. We prefer *monotone-up* since it directly describes the direction of any possible movement in the sequence and prevents confusion over whether the inequality is strict. If necessary, a sequence with $s_{n+1} > s_n$ may be described as *strictly increasing* or *strictly monotone-up*.

Examples 2.27. 1. Define (s_n) via $s_n = 1$ and $s_{n+1} = \frac{1}{5}(s_n + 8)$:

$$(s_n) = \big(1, 1.8, 1.96, 1.992, 1.9984, 1.99968, \ldots\big)$$

This sequence certainly appears to be monotone-up and converging to 2. Here is a proof:

Bounded above: $s_n < 2 \implies s_{n+1} < \frac{1}{5}\left[2+8\right] = 2$. By induction, (s_n) is bounded above by 2.

Monotone-up: $s_{n+1} - s_n = \frac{4}{5}\left[2 - s_n\right] > 0$ since $s_n < 2$.

Convergence: By monotone convergence, $s = \lim s_n$ exists. Now use the limit laws to find s:

$$s = \lim s_{n+1} = \frac{1}{5}\left(\lim s_n + 8\right) = \frac{1}{5}(s+8) \implies s = 2$$

2. (Example 2.16.2, cont.) Let $s_1 = 2$ and $s_{n+1} = \frac{1}{2}\big(s_n + \frac{2}{s_n}\big)$.

   Bounded below: The sequence is plainly always positive and thus bounded below by zero.

   Monotone-down: We first obtain an improved lower bound:

$$s_{n+1}^2 = \frac{1}{4}\left(s_n + \frac{2}{s_n}\right)^2 = 2 + \frac{1}{4}\left(s_n - \frac{2}{s_n}\right)^2 \geq 2$$

   shows[18] that $s_n^2 \geq 2$ for all n. It follows that

$$\frac{s_{n+1}}{s_n} = \frac{1}{2}\left(1 + \frac{2}{s_n^2}\right) \leq 1 \implies s_{n+1} \leq s_n$$

   Convergence: By monotone convergence, $s = \lim s_n$ exists. Example 2.16.2 provides the limit:

$$s = \frac{1}{2}\left(s + \frac{2}{s}\right) \implies s = \sqrt{2}$$

   This shows the necessity of completeness: (s_n) is a monotone, bounded sequence of *rational* numbers, but its limit is *irrational*.

3. A *decimal number* $d_0.d_1d_2d_3\ldots$ is the limit of a monotone-up sequence of *rational numbers*:

$$d_0.d_1d_2d_3\ldots = d_0 + \lim_{n\to\infty} \sum_{k=1}^{n} \frac{d_k}{10^k}$$

   This is bounded above (by $d_0 + 1 \in \mathbb{Z}$) and so converges.

4. The sequence with $s_n = \left(1 + \frac{1}{n}\right)^n$ is particularly famous. In Exercise 10 we show that (s_n) is monotone-up and bounded above. The limit provides, arguably, the oldest definition of e:

$$e := \lim \left(1 + \frac{1}{n}\right)^n$$

[18] This is the famous AM–GM inequality $\frac{x+y}{2} \geq \sqrt{xy}$ with $x = s_n$ and $y = \frac{2}{s_n}$.

Limits Superior and Inferior

One interpretation of $\lim s_n$ is that it approximately describes s_n for large n. Even when a sequence does not have a limit, it is useful to be able to describe its long-term behavior.

Definition 2.28. Let (s_n) be a sequence and define two related sequences (v_N) and (u_N):

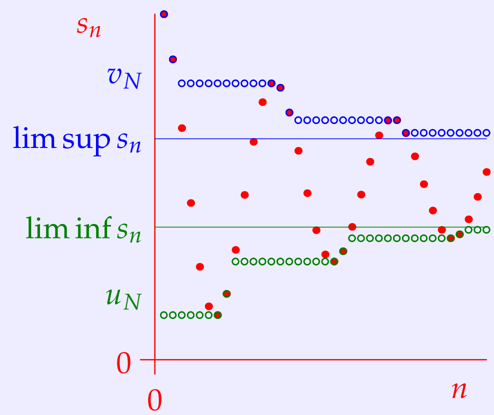
$$v_N := \sup\{s_n : n \geq N\}, \qquad u_N := \inf\{s_n : n \geq N\}$$

1. The *limit superior* of (s_n) is

$$\limsup s_n = \begin{cases} \lim\limits_{N\to\infty} v_N & \text{if } (s_n) \text{ bounded above} \\ \infty & \text{if } (s_n) \text{ unbounded above} \end{cases}$$

2. The *limit inferior* of (s_n) is

$$\liminf s_n = \begin{cases} \lim\limits_{N\to\infty} u_N & \text{if } (s_n) \text{ bounded below} \\ -\infty & \text{if } (s_n) \text{ unbounded below} \end{cases}$$



The original sequence (s_n) is wedged between (v_n) and (u_n) in a manner reminiscent of the squeeze theorem (though lim sup and lim inf need not be equal). The next result summarizes the situation more formally; we omit the proof since these claims should be clear from the definition and previous results, particularly the monotone convergence theorem.

Lemma 2.29. 1. *(v_N) is monotone-down and (u_N) monotone-up.*

2. *$\limsup s_n$ and $\liminf s_n$ exist for any sequence (they might be infinite).*

3. *If $n \geq N$, then $u_N \leq s_n \leq v_N$.*

4. *$\liminf s_n \leq \limsup s_n$.*

Examples 2.30. 1. If $s_n = \frac{1}{n}$, then $v_N = s_N$ and $u_N = 0$, whence $\limsup s_n = \liminf s_n = 0$.

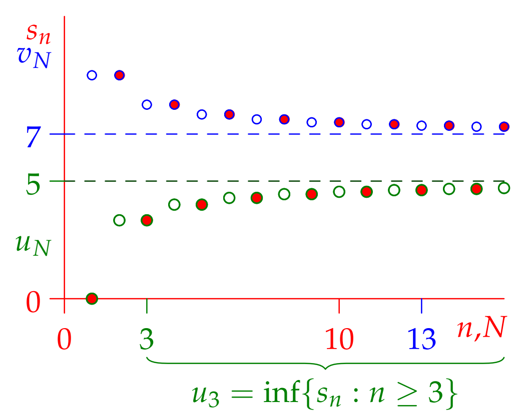
2. The picture shows the sequences (s_n), (u_N) and (v_N) when $s_n = 6 + (-1)^n \left(1 + \frac{5}{n}\right)$

We won't compute everything precisely, but the picture suggests (s_n) has two "sub"-sequences: the odd terms increase while the even terms decrease towards, respectively

$$\liminf s_n = 5, \qquad \limsup s_n = 7$$

Here is one value from each derived sequence:

$$u_3 = \inf\{s_n : n \geq 3\} = s_3 \approx 3.333$$
$$v_{13} = \sup\{s_n : n \geq 13\} = s_{14} \approx 7.357$$



3. Let $s_n = (-1)^n$. This time the calculation is easy: for any N,

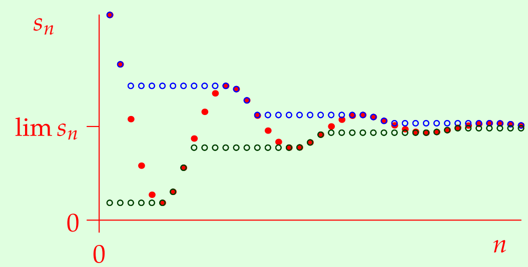
$$u_N = \inf\{s_n : n \geq N\} = -1 \quad \text{and} \quad v_N = \sup\{s_n : n \geq N\} = 1$$

Therefore $\limsup s_n = 1$ and $\liminf s_n = -1$.

Theorem 2.31. *For any sequence* (s_n)*,*

$$\limsup s_n = \liminf s_n \iff \lim s_n \text{ exists}$$

In such a case all three values are equal.
Note that the limits can be $\pm\infty$*.*



Proof. ($\Rightarrow$) Suppose $s := \limsup s_n = \liminf s_n$.

- If s is finite, apply the squeeze theorem to $u_n \leq s_n \leq v_n$ (both extremes converge to s).
- If $s = \infty$, then $u_n \leq s_n$ for all n. Theorem 2.21.1 shows that $\lim s_n = \infty = s$.
- If $s = -\infty$, instead use $s_n \leq v_n$.

($\Leftarrow$) We could prove this now, but it will come almost for free a little later... ■

Cauchy Sequences

We now come to a class of sequences whose analogues will dominate your future studies.

Definition 2.32. A sequence (s_n) is *Cauchy*[19] if

$$\forall \epsilon > 0,\ \exists N \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon$$

A sequence is Cauchy when terms in the tails of the sequence are constrained to stay close to one another. As we'll see shortly, this will provide an alternative way to detect and describe *convergence*.

Examples 2.33. 1. Let $s_n = \frac{1}{n}$. Let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then

$$m > n > N \implies |s_m - s_n| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon \qquad \text{(WLOG } m > n\text{)}$$

Thus (s_n) is Cauchy. A similar argument works for any $s_n = \frac{1}{n^k}$ for positive k.

2. Suppose $s_1 = 5$ and $s_{n+1} = s_n + \frac{1}{n(n+1)}$. As before, let $\epsilon > 0$ be given and let $N = \frac{1}{\epsilon}$. Then,

$$|s_{n+1} - s_n| = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\implies |s_m - s_n| \overset{\triangle}{\leq} |s_{n+1} - s_n| + \cdots + |s_m - s_{m-1}| = \frac{1}{n} - \frac{1}{m} < \frac{1}{n} < \frac{1}{N} = \epsilon$$

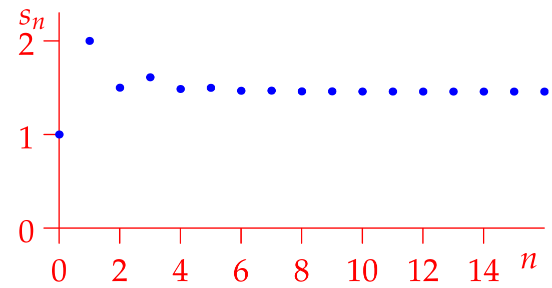
Again we have a Cauchy sequence.

[19] Augustin-Louis Cauchy (1789–1857) was a French mathematician, responsible (in part) for the ϵ-N definition of limit.

3. Define $(s_n)_{n=0}^{\infty}$ inductively:

$$s_0 = 1, \quad s_{n+1} = \begin{cases} s_n + 3^{-n} & \text{if } n \text{ even} \\ s_n - 2^{-n} & \text{if } n \text{ odd} \end{cases}$$

$$(s_n) = \left(1, 2, \frac{3}{2}, \frac{29}{18}, \frac{107}{72}, \ldots\right)$$



Since $|s_{n+1} - s_n| \le 2^{-n}$, we see that

$$m > n \implies |s_m - s_n| \overset{\triangle}{\le} |s_{n+1} - s_n| + \cdots + |s_m - s_{m-1}| = \sum_{k=n}^{m-1} |s_{k+1} - s_k|$$
$$\le \sum_{k=n}^{m-1} 2^{-k} = \frac{2^{-n} - 2^{-m}}{1 - 2^{-1}} < 2^{1-n}$$

where we used the familiar geometric sum formula from calculus: $\sum\limits_{k=a}^{b-1} r^k = \frac{r^a - r^b}{1-r}$.
Suppose $\epsilon > 0$ is given, and let $N = 1 - \log_2 \epsilon = \log_2 \frac{2}{\epsilon}$. Then

$$m > n > N \implies |s_m - s_n| < 2^{1-n} < 2^{1-N} = \epsilon$$

We conclude that (s_n) is Cauchy.

The last picture illustrates the essential point of Cauchy sequences: (s_n) appears to converge...

Theorem 2.34 (Cauchy Completeness). *A sequence of real numbers is convergent if and only if it is Cauchy.*

Proof. $(\Rightarrow)$ Suppose $\lim s_n = s$ (is finite). Given $\epsilon > 0$, we may choose N such that

$$m, n > N \implies |s_n - s| < \frac{\epsilon}{2} \quad \text{and} \quad |s_m - s| < \frac{\epsilon}{2}$$
$$\implies |s_n - s_m| = |s_n - s + s - s_m| \overset{\triangle}{\le} |s_n - s| + |s - s_m| < \epsilon$$

Otherwise said, (s_n) is Cauchy.

$(\Leftarrow)$ To discuss the convergence of (s_n) we need a potential limit. In view of Theorem 2.31, the obvious candidates are $\limsup s_n$ and $\liminf s_n$. We have two goals: show that (s_n) is bounded whence the limits superior and inferior are *finite*; then show that these are *equal*.

(Boundedness of (s_n)) Take $\epsilon = 1$ in Definition 2.32:

$$\exists N \text{ such that } m, n > N \implies |s_n - s_m| < 1$$

It follows that

$$n > N \implies |s_n - s_{N+1}| < 1 \implies s_{N+1} - 1 < s_n < s_{N+1} + 1$$

whence (s_n) is bounded. It follows that $\limsup s_n$ and $\liminf s_n$ are both *finite*.

(lim sup $s_n = \liminf s_n$) Suppose $\epsilon > 0$ is given. Since (s_n) is Cauchy,

$$\exists N \in \mathbb{N} \text{ such that } m, n > N \implies |s_n - s_m| < \epsilon \implies s_n < s_m + \epsilon$$

Take the supremum over all $n > N$: since $v_{N+1} = \sup\{s_n : n \geq N+1\}$, we see that

$$m > N \implies v_{N+1} \leq s_m + \epsilon$$

Now take the infimum of the right hand side over all $m > N$ to obtain

$$v_{N+1} \leq u_{N+1} + \epsilon \qquad \text{(since } u_{N+1} = \inf\{s_m : m \geq N+1\}\text{)}$$

Since (v_{N+1}) is monotone-down and (u_{N+1}) monotone-up, we see that

$$\limsup s_n \leq v_{N+1} \leq u_{N+1} + \epsilon \leq \liminf s_n + \epsilon \implies \limsup s_n \leq \liminf s_n + \epsilon$$

Since $\epsilon > 0$ was arbitrary, we conclude that $\limsup s_n \leq \liminf s_n$. By Lemma 2.29 we have equality.

By Theorem 2.31, we conclude that (s_n) converges to $\limsup s_n = \liminf s_n$. ∎

By the Theorem, Examples 2.33 all converge. All three limits can be found precisely (for instance, see Exercise 7). With a small modification to the second example, however, we obtain something genuinely new:

Example (2.33.2 cont). Let $s_1 = 5$ and, for each n, define $s_{n+1} := s_n + \frac{\sin n}{n(n+1)}$. Since $|\sin n| \leq 1$, the computation proceeds almost the same as before:

$$|s_{n+1} - s_n| = \frac{|\sin n|}{n(n+1)} \leq \frac{1}{n(n+1)} = \cdots$$

The new sequence is Cauchy and thus convergent, though good luck explicitly finding its limit!

The main point is easy to miss: the Cauchy condition is a powerful tool for determining whether a sequence converges *without first guessing a limit.* While the proof depends on monotone convergence (via limit superior/inferior), Cauchy completeness is more powerful in that it applies even to non-monotone sequences.

An Alternative Definition of $\mathbb{R}$ Cauchy sequences suggest a *definition* of the real numbers which does not rely on Dedekind cuts (Section 1.6).

Define an equivalence relation $\sim$ on the collection $\mathcal{C}$ of all Cauchy sequences of rational numbers:[20]

$$(s_n) \sim (t_n) \iff \lim(s_n - t_n) = 0$$

Now define $\mathbb{R} := {}^{\mathcal{C}}/_{\sim}$ to be the set of equivalence classes. All this is done without reference to Cauchy completeness, though it certainly informs our intuition that (s_n) and (t_n) have the same limit (as real numbers). Significant work is still required to properly define $+, \cdot, \leq$, etc., and to verify the axioms of a complete ordered field—we won't pursue this.

[20] We don't need real numbers to define the limit of the *rational* sequence $(s_n - t_n)$: $\forall \epsilon \in \mathbb{Q}^+$ is enough...

Exercises 2.10. *Key concepts: Monotone sequences & Convergence, Cauchy sequences & completeness, Limits superior/inferior*

1. Use Definition 2.32 to show that the sequence with $s_n = \frac{1}{n^2}$ is Cauchy. Repeat for $t_n = \frac{1}{n(n-2)}$.
2. Let $s_1 = 1$ and $s_{n+1} = \frac{n}{n+1}s_n^2$ for $n \geq 1$.
   (a) Find s_2, s_3 and s_4.
   (b) Show that $\lim s_n$ exists and hence prove that $\lim s_n = 0$.
3. Let $s_1 = 1$ and $s_{n+1} = \frac{1}{3}(s_n + 1)$ for $n \geq 1$.
   (a) Find s_2, s_3 and s_4.
   (b) Use induction to show that $s_n > \frac{1}{2}$ for all n, and conclude that (s_n) is monotone-down.
   (c) Show that $\lim s_n$ exists and find $\lim s_n$.
4. (a) Let (s_n) be a sequence such that $\forall n,\ |s_{n+1} - s_n| \leq 3^{-n}$. Prove that (s_n) is Cauchy.
   (b) Let $s_1 = 10$ and, for each n, let $s_{n+1} = s_n + \frac{\cos n}{3^n}$. Explain why (s_n) is convergent.
   (c) Is the result in (a) true if we only assume that $|s_{n+1} - s_n| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$?
5. Suppose (s_n) is *unbounded* and monotone-up. Prove that $\lim s_n = \infty$.
   (*Thus* $\lim s_n = \sup\{s_n\}$ *for any monotone-up sequence*)
6. Let $s_n = \frac{(-1)^n}{n}$. Find the sequences $(u_N), (v_N)$ and explicitly compute $\limsup s_n$ and $\liminf s_n$.
7. Consider the sequence in Example 2.33.3. Explain why $s_{2n} = s_{2n-2} - \frac{2}{4^n} + \frac{9}{9^n}$.
   Now use the geometric sum formula to evaluate $\lim s_{2n}$.
   (*Since* (s_n) *converges, this means the original sequence has the same limit*)
8. Let S be a bounded nonempty set for which $\sup S \notin S$. Prove that there exists a monotone-up sequence (s_n) of points in S such that $\lim s_n = \sup S$.
   (*Hint: for each* n*, use* $\sup S - \frac{1}{n}$ *to build* s_n)
9. Let (s_n) be a monotone-up sequence of positive numbers and define $\sigma_n = \frac{1}{n}(s_1 + s_2 + \cdots + s_n)$. Prove that (σ_n) is monotone-up.
10. (Hard!) We prove that the sequence defined by $s_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.
    (a) Show that
    $$\frac{1 + \frac{1}{n+1}}{1 + \frac{1}{n}} = 1 - \frac{1}{(n+1)^2} \qquad \text{and} \qquad \frac{1 + \frac{1}{n}}{1 + \frac{1}{n+1}} = 1 + \frac{1}{n(n+2)}$$
    (b) Prove *Bernoulli's inequality* by induction:
    For all real $x > -1$ and $n \in \mathbb{N}_0$ we have $(1 + x)^n \geq 1 + nx$.
    (c) By considering $\frac{s_{n+1}}{s_n}$, use parts (a) and (b) to prove that (s_n) is monotone-up.
    (d) Similarly, show that $t_n := \left(1 + \frac{1}{n}\right)^{n+1} = \left(1 + \frac{1}{n}\right) s_n$ defines a monotone-down sequence.
    (e) Prove that (s_n) and (t_n) converge, and to the *same* limit (this is Bernoulli's definition of e).
    (f) Prove that $\lim(1 - \frac{1}{n})^n = e^{-1}$.

2.11 Subsequences

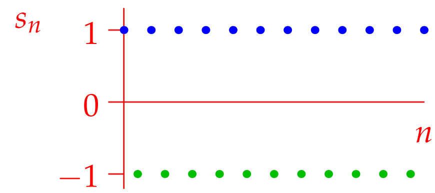
The overall behavior of a sequence is often hard to describe, but if we delete some of its terms we might obtain a *subsequence* with much simpler behavior.

Definition 2.35. Let (s_n) be a sequence. A *subsequence* (s_{n_k}) is a subset $(s_{n_k}) \subseteq (s_n)$, where

$$n_1 < n_2 < n_3 < \cdots$$

A subsequence is simply an infinite subset whose order is inherited from the original sequence.

Example 2.36. Take $s_n = (-1)^n$ (recall Example 2.10.2) and let $n_k = 2k$. Then $s_{n_k} = 1$ for all k. Note two important facts:



- The subsequence $(s_{n_k})_{k=0}^{\infty}$ is indexed by k, not n.
- The subsequence is constant and thus *convergent*.

Our main goal in this section is to prove the famous Bolzano–Weierstraß theorem (illustrated in the example): that every bounded sequence has a convergent subsequence.

Lemma 2.37. *If* $\lim_{n\to\infty} s_n = s$, *then every subsequence* (s_{n_k}) *satisfies* $\lim_{k\to\infty} s_{n_k} = s$.

Proof. Suppose s is finite and suppose $\epsilon > 0$ is given. Then $\exists N$ such that $n > N \implies |s_n - s| < \epsilon$. Since $n_k \geq k$ for all k, we see that

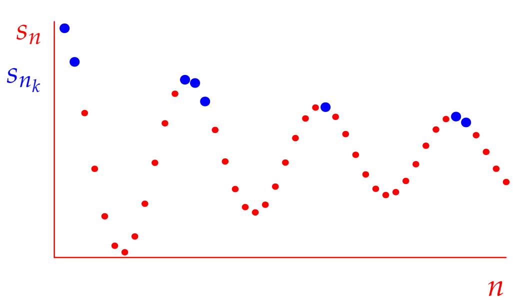
$$k > N \implies n_k > N \implies |s_{n_k} - s| < \epsilon$$

The case where $s = \pm\infty$ is an exercise. ∎

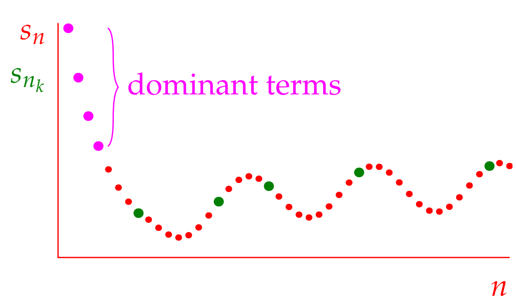
Lemma 2.38. *Every sequence has a monotonic subsequence.*

Proof. Given (s_n), we call the term s_n 'dominant' if $m > n \implies s_m < s_n$. There are two cases:

1. If there are infinitely many dominant terms, then the subsequence of such is monotone-down.
2. If there are finitely many dominant terms, choose s_{n_1} after all such. Since s_{n_1} is not dominant, $\exists n_2 > n_1$ such that $s_{n_2} \geq s_{n_1}$. Induct to obtain a monotone-up subsequence.



Case 1: monotone-down subsequence



Case 2: monotone-up subsequence

∎

Theorem 2.39. *Given a sequence (s_n), there exist subsequences (s_{n_k}) and (s_{n_l}) such that*

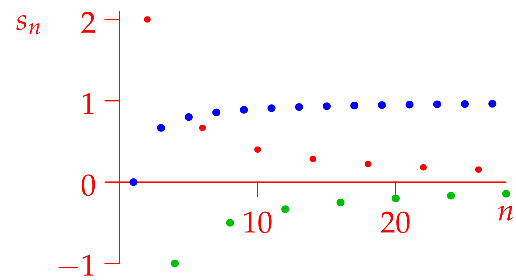
$$\lim s_{n_k} = \limsup s_n \quad \textit{and} \quad \lim s_{n_l} = \liminf s_n$$

By the lemmas, we may moreover assume that these subsequences are monotonic.

Example 2.40. The picture shows the sequence with n^{th} term

$$s_n = \begin{cases} \frac{4}{n}(-1)^{\frac{n}{2}+1} & \text{when } n \text{ is even} \\ 1 - \frac{1}{n} & \text{when } n \text{ is odd} \end{cases}$$

Monotonic subsequences with limits $\limsup s_n = 1$ and $\liminf s_n = 0$ are indicated.



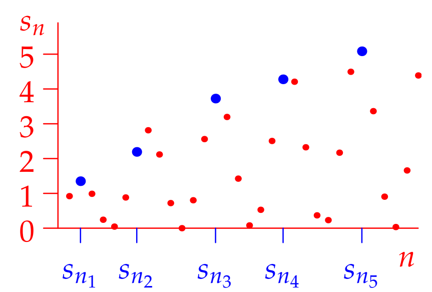
Proof. We prove only the lim sup claim, since the other is similar. There are three cases to consider; visualizing the third is particularly difficult and may take several readings.

($\limsup s_n = \infty$) Since (s_n) is unbounded above, for any $k > 0$ there exist *infinitely many* terms $s_n > k$. We may therefore inductively choose a subsequence (s_{n_k}) via

$$n_1 = \min\{n \in \mathbb{N} : s_{n_1} > 1\}$$
$$n_k = \min\{n \in \mathbb{N} : n_k > n_{k-1}, \text{ and } s_{n_k} > k\}$$

Choosing the minimum isn't necessary, though it keeps the subsequence explicit. Clearly

$$s_{n_k} > k \implies \lim_{k\to\infty} s_{n_k} = \infty = \limsup s_n$$



Example: $\limsup \frac{\sqrt{n}}{2}(1 + \sin n) = \infty$

($\limsup s_n = -\infty$) Since $\liminf s_n \le \limsup s_n = -\infty$, Lemma 2.31 says that $\lim s_n = -\infty$, whence (s_n) itself is a suitable subsequence.

($\limsup s_n = v$ finite) Let $n_1 = 1$ and define s_{n_k} for $k \ge 2$ inductively:

- Since (v_N) is monotone-down and converges to v, take $\epsilon = \frac{1}{2k}$ to see that[21]

$$\exists N_k > n_{k-1} \text{ such that } v \le v_{N_k} < v + \frac{1}{2k}$$

- Since $v_{N_k} = \sup\{s_n : n \ge N_k\}$, Lemma 1.20 says

$$\exists n_k \ge N_k \text{ such that } s_{n_k} > v_{N_k} - \frac{1}{2k}$$

But then $|v - s_{n_k}| \overset{\triangle}{\le} |v - v_{N_k}| + |v_{N_k} - s_{n_k}| < \frac{1}{k}$. The squeeze theorem says that $\lim_{k\to\infty} s_{n_k} = v$. ∎

[21] (v_N) being monotone-down is crucial: if N satisfies $v_N - v < \frac{1}{2k}$, so does $N_k := \max(N, 1 + n_{k-1})$.

Example (2.40 cont.). The example shows why the two-step construction is necessary. It may seem that we should simply be able to choose subsequences of (u_N) and (v_N). Indeed,

$$(u_N) = (\underbrace{-1,-1,-1,-1}_{s_4}, \underbrace{-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}}_{s_8}, \underbrace{-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3},-\tfrac{1}{3}}_{s_{12}},\ldots)$$

contains a subsequence of (s_n) converging to $\liminf s_n = 0$. Unfortunately, $(v_N) = (2,2,1,1,1,\ldots)$ does not contain a subsequence of (s_n). Taking $n_k = 2k+1$ ($k \geq 2$) results in the displayed sequence:

$$s_{n_k} = 1 - \frac{1}{2k+1} > 1 - \frac{1}{2k} = v_{N_k} - \frac{1}{2k}$$

There are two immediate corollaries. The first (see Exercise 3) fully establishes Theorem 2.31

$$\limsup s_n = \liminf s_n \iff \lim s_n \text{ exists} \qquad \text{(could be } \pm\infty)$$

The second is the main goal of this section.

Theorem 2.41 (Bolzano–Weierstraß). *Every bounded sequence has a convergent subsequence.*

Proof 1. Lemma 2.38 says there exists a monotone subsequence. This is bounded and thus converges by the monotone convergence theorem. ■

Proof 2. By Theorem 2.39, there exists a subsequence converging to the *finite* value $\limsup s_n$. ■

For a third proof(!) we present the classic 'shrinking-interval' argument which has the benefit of easily generalizing to higher dimensions.

Proof 3. Suppose (s_n) is bounded by M. One of the intervals $[-M,0]$ or $[0,M]$ must contain infinitely many terms of the sequence (perhaps both do!). Call this interval E_0 and choose any $n_0 \in E_0$.

Split E_0 into left- and right half-intervals, one of which must contain infinitely many terms of the sequence for which $n > n_0$;[22] call this half-interval E_1 and choose any $s_{n_1} \in E_1$ with $n_1 > n_0$.

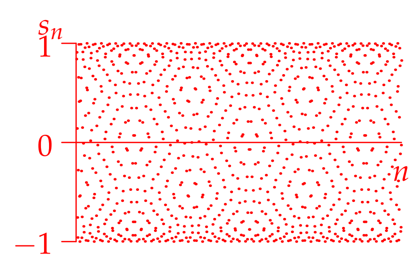
Repeat this *ad infinitum* to obtain a subsequence (s_{n_k}) and a family of nested intervals

$$[-M,M] \supset E_0 \supset E_1 \supset E_2 \supset \cdots \quad \text{of width} \quad |E_k| = \frac{M}{2^k} \quad \text{with} \quad s_{n_k} \in E_k$$

It remains only to see that (s_{n_k}) converges; we leave this to Exercise 5. ■

Example 2.42. $(s_n) = (\sin n)$ is bounded and therefore has a convergent subsequence! Its limit s must lie in the interval $[-1,1]$.

The picture shows the first 1000 terms—remember that n is measured in *radians*. It is not at all clear from the picture what s or our mystery subsequence should be! There is a reason for this, as we'll see momentarily...



[22]Only *finitely many* terms in (s_n) could come before s_{n_0} ...

Subsequential Limits, Divergence by Oscillation & Closed Sets

Recall Definition 2.19: a sequence (s_n) *diverges by oscillation* if it neither converges nor diverges to $\pm\infty$. We can now give a positive statement of this idea.

$$(s_n) \text{ diverges by oscillation} \overset{\text{Thm 2.31}}{\iff} \liminf s_n \neq \limsup s_n$$
$$\overset{\text{Thm 2.39}}{\iff} (s_n) \text{ has subsequences tending to different limits}$$

The word *oscillation* comes from the third interpretation: if $s_1 \neq s_2$ are limits of two subsequences, then any tail of the sequence $\{s_n : n > N\}$ contains infinitely many terms arbitrarily close to s_1 and infinitely many (other) terms arbitrarily close to s_2. The original sequence (s_n) therefore *oscillates* between neighborhoods of s_1 and s_2. Of course there could be many other *subsequential limits*...

Definition 2.43. We call $s \in \mathbb{R} \cup \{\pm\infty\}$ a *subsequential limit* of a sequence (s_n) if there exists a subsequence (s_{n_k}) such that $\lim\limits_{k\to\infty} s_{n_k} = s$.

Examples 2.44. 1. The sequence defined by $s_n = \frac{1}{n}$ has only one subsequential limit, namely zero. Recall Lemma 2.37: $\lim s_n = 0$ implies that every subsequence also converges to 0.

2. If $s_n = (-1)^n$, then the subsequential limits are ± 1.

3. The sequence $s_n = n^2(1 + (-1)^n)$ has subsequential limits 0 and ∞.

4. All positive even integers are subsequential limits of $(s_n) = (2, 4, 2, 4, 6, 2, 4, 6, 8, 2, 4, 6, 8, 10, \ldots)$.

5. (Hard!) Recall the countability of $\mathbb{Q}$ from a previous class: the standard argument enumerates the rationals by constructing a sequence

$$(r_n) = \Big(\frac{0}{1}, \underbrace{\frac{1}{1}, -\frac{1}{1}}_{|p|+q=2}, \underbrace{\frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}}_{|p|+q=3}, \underbrace{\frac{1}{3}, -\frac{1}{3}, \frac{3}{1}, -\frac{3}{1}}_{|p|+q=4}, \underbrace{\frac{1}{4}, -\frac{1}{4}, \frac{2}{3}, -\frac{2}{3}, \frac{3}{2}, -\frac{3}{2}, \frac{4}{1}, -\frac{4}{1}}_{|p|+q=5}, \ldots\Big)$$

We claim that the set of subsequential limits of (r_n) is in fact the set $\mathbb{R} \cup \{\pm\infty\}$!

To see this, let $a \in \mathbb{R}$ be given and choose a subsequence (r_{n_k}) inductively:

- By the density of $\mathbb{Q}$ in $\mathbb{R}$ (Corollary 1.23), the set $S_n = \mathbb{Q} \cap (a - \frac{1}{n}, a + \frac{1}{n})$ contains infinitely many rational numbers and thus infinitely many terms of the sequence (r_n).
- Choose any $r_{n_1} \in S_1$ and, for each $k \geq 2$, choose any

$$r_{n_k} \in S_k \text{ such that } n_k > n_{k-1}$$

- Since $|r_{n_k} - a| < \frac{1}{n_k} \leq \frac{1}{k}$, we conclude that $\lim\limits_{k\to\infty} r_{n_k} = a$.

An argument for the subsequential limits $\pm\infty$ is in the Exercises. Somewhat amazingly, the *specific* sequence (r_n) is irrelevant: the conclusion is the same for *any* sequence enumerating $\mathbb{Q}$!

6. (Even harder—Example 2.42, cont.) We won't prove it, but the set of subsequential limits of $(s_n) = (\sin n)$ is the *entire interval* $[-1, 1]$! Otherwise said, for any $s \in [-1, 1]$ there exists a subsequence $(\sin n_k)$ such that $\lim\limits_{k\to\infty} \sin n_k = s$.

Theorem 2.45. *Let (s_n) be a sequence in $\mathbb{R}$ and let S be its set of subsequential limits. Then*

1. *S is non-empty (as a subset of $\mathbb{R} \cup \{\pm\infty\}$).*
2. *$\sup S = \limsup s_n$ and $\inf S = \liminf s_n$.*
3. *$\lim s_n$ exists iff S has only one element: namely $\lim s_n$.*

Proof. 1. By Theorem 2.39, $\limsup s_n \in S$.

2. By part 1, $\limsup s_n \leq \sup S$. For any convergent subsequence (s_{n_k}) we have $n_k \geq k$, whence

$$\forall N,\ \{s_{n_k} : k \geq N\} \subseteq \{s_n : n \geq N\} \implies \lim s_{n_k} = \limsup s_{n_k} \leq \limsup s_n$$

Since this holds for *every* convergent subsequence, we have $\sup S \leq \limsup s_n$ and therefore equality. The result for $\inf S$ is similar.

3. Applying Theorem 2.31, we see that $\lim s_n$ exists if and only if

$$\limsup s_n = \liminf s_n \iff \sup S = \inf S \iff S \text{ has only one element}$$

∎

Closed Sets You should be comfortable with the notion of a *closed interval* (e.g. $[0, 1]$) from elementary calculus. Sequences allow us to make a formal definition.

Definition 2.46. Let $A \subseteq \mathbb{R}$.

- We say that $s \in \mathbb{R}$ is a *limit point* of A if there exists a sequence $(s_n) \subseteq A$ converging to s.
- The *closure* $\overline{A}$ is the set of limit points of A. Plainly $A \subseteq \overline{A}$ for any set.
- A is *closed* if it equals its closure: $A = \overline{A}$.

Examples 2.47. 1. The interval $[0, 1]$ is closed. If $(s_n) \subseteq [0, 1]$ has $\lim s_n = s$, then

$$0 \leq s_n \leq 1 \overset{\text{Thm 2.11}}{\implies} s \in [0, 1]$$

More generally, every 'closed interval' $[a, b]$ is closed, as are *finite* unions of closed intervals, for instance $[1, 5] \cup [7, 11]$.

2. The 'half-open' interval $(0, 1]$ is not closed: its closure is $\overline{(0, 1]} = [0, 1]$. In particular, the sequence $s_n = \frac{1}{n}$ lies in $(0, 1]$, but $\lim s_n = 0 \notin (0, 1]$.

3. Example 2.44.5 shows that the closure of the rational numbers is the reals: $\overline{\mathbb{Q}} = \mathbb{R}$.

Theorem 2.48. *If (s_n) is a sequence, then its set of (finite) subsequential limits is closed.*

We omit the proof since it involve unpleasantly many subscripts (subsequences of subsequences...).

Exercises 2.11. *Key concepts: Subsequences of a convergent sequence all have the same limit, Existence of (monotone) subsequences tending to* $\limsup s_n$ / $\liminf s_n$, *Subsequential Limits, Bolzano–Weierstraß: boundedness* $\Longrightarrow \exists$ *convergent subsequence*

1. Consider the sequences with the following n^{th} terms:

$$a_n = (-1)^n \qquad b_n = \frac{1}{n} \qquad c_n = n^2 \qquad d_n = \frac{6n+4}{7n-3}$$

   For each sequence: state whether it converges, diverges to $\pm\infty$, or diverges by oscillation; give an example of a monotone subsequence; state the set of subsequential limits; state the limits superior and inferior.

2. Prove the case of Lemma 2.37 when $\lim s_n = \infty$

3. Suppose that $\lim s_n = s$ (could be $\pm\infty$). Use Theorem 2.39 and Lemma 2.37 to prove that $\limsup s_n = s = \liminf s_n$.

   (*This completes the proof of Theorem 2.31*)

4. Suppose that $L = \lim s_n^2$ exists and is finite.

   (a) Given an example of such a sequence where (s_n) is *divergent.*

   (b) Prove that (s_n) contains a convergent *subsequence.* What are the possible limits of this subsequence? Why?
   (*Hint: use Bolzano–Weierstraß*)

5. Complete the third proof of Bolzano–Weierstraß (Theorem 2.41) by proving that the constructed subsequence (s_{n_k}) is Cauchy.

6. (a) Show that the closed interval $[a, b]$ is a closed set in the sense of Definition 2.46.

   (b) Is there a sequence (s_n) such that $(0, 1)$ is its set of subsequential limits?

7. By considering Example 2.47.2, or otherwise, show that an *infinite* union of closed intervals need not be closed.

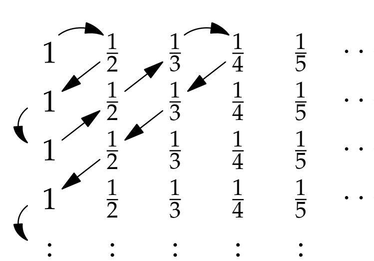
8. Let (r_n) be any sequence enumerating of the set $\mathbb{Q}$ of rational numbers. Show that there exists a subsequence (r_{n_k}) such that $\lim\limits_{k\to\infty} r_{n_k} = +\infty$.

   (*Hint: modify the argument in Example 2.44.5*)

9. (Hard) Let (s_n) be the sequence of numbers defined in the figure, listed in the indicated order.

   (a) Find the set S of subsequential limits of (s_n).

   (b) Determine $\limsup s_n$ and $\liminf s_n$.



2.12 Lim sup and Lim inf

In this short section we collect a couple of useful results, mostly for later use. First we observe that the limit laws do not work as tightly for limits superior and inferior.

Theorem 2.49. *Let (s_n), (t_n) be bounded sequences. Then:*

1. $\limsup(s_n + t_n) \le \limsup s_n + \limsup t_n$
2. *If, in addition, (s_n) is convergent to s, then we have equality*

$$\limsup(s_n + t_n) = s + \limsup t_n$$

Natural modifications can be made for infima and products of sequences (see Exercise 3).

Example 2.50. To convince yourself that equality is unlikely, consider $s_n = (-1)^n = -t_n$. Plainly

$$\limsup(s_n + t_n) = 0 < 2 = \limsup s_n + \limsup t_n$$

Proof.

1. For each N, the set $\{s_n + t_n : n \ge N\}$ is bounded above by

$$\sup\{s_n : n \ge N\} + \sup\{t_n : n \ge N\}$$

from which

$$\sup\{s_n + t_n : n \ge N\} \le \sup\{s_n : n \ge N\} + \sup\{t_n : n \ge N\}$$

Take limits as $N \to \infty$ for the first result.

2. By part 1, we know that

$$\limsup(s_n + t_n) \le s + \limsup t_n$$

For the other direction, rearrange and apply part 1 again:

$$\begin{aligned}\limsup t_n &= \limsup\big((s_n + t_n) - s_n\big) \le \limsup(s_n + t_n) + \limsup(-s_n)\\ &= \limsup(s_n + t_n) - s\end{aligned}$$

∎

The next result will be critical when we study infinite series, particularly the ratio and root tests.

Theorem 2.51. *Let (s_n) be a non-zero sequence. Then*

$$\liminf\left|\frac{s_{n+1}}{s_n}\right| \le \liminf|s_n|^{1/n} \le \limsup|s_n|^{1/n} \le \limsup\left|\frac{s_{n+1}}{s_n}\right|$$

In particular, $\lim\left|\frac{s_{n+1}}{s_n}\right| = L \implies \lim|s_n|^{1/n} = L.$ (†)

Examples 2.52. 1. Here is a quick proof that $\lim n^{1/n} = 1$ (recall Theorem 2.17). Let $s_n = n$, then

$$\lim \left| \frac{s_{n+1}}{s_n} \right| = \lim \frac{n+1}{n} = 1 \implies \lim n^{1/n} = \lim |s_n|^{1/n} = 1$$

2. Apply the corollary to $s_n = n!$ to see that

$$\lim (n!)^{1/n} = \lim \left| \frac{s_{n+1}}{s_n} \right| = \lim (n+1) = \infty$$

Proof. We prove the third inequality. Assume $\limsup \left| \frac{s_{n+1}}{s_n} \right| = L \neq \infty$ (otherwise the inequality is trivial). Suppose $\epsilon > 0$ is given, and denote $a = L + \epsilon$. Then

$$L = \lim_{N \to \infty} \sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n \geq N \right\} < a \implies \exists N \text{ such that } \sup \left\{ \left| \frac{s_{n+1}}{s_n} \right| : n \geq N \right\} < a$$

Now let $b = a^{-N} |s_N|$. For any $n \geq N$, we therefore have $\left| \frac{s_{n+1}}{s_n} \right| < a$, whence

$$\begin{aligned} n > N &\implies |s_n| < a^{n-N} |s_N| \implies |s_n|^{1/n} < a \left(a^{-N} |s_N| \right)^{1/n} = ab^{1/n} \\ &\implies \limsup |s_n|^{1/n} \leq a \lim b^{1/n} = a = L + \epsilon \end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude the third inequality: $\limsup |s_n|^{1/n} \leq L$.
The second inequality is trivial and the first is similar to the third. ∎

Exercises 2.12. *Key concepts: "Limit laws" for* $\limsup$ *and* $\liminf$, $\lim \left| \frac{s_{n+1}}{s_n} \right| = \lim |s_n|^{1/n}$

1. Compute $\lim \frac{1}{n} (n!)^{1/n}$
   (*Hint: let* $s_n = \frac{n!}{n^n}$ *in Theorem 2.51 and recall that* $\lim \left(1 + \frac{1}{n}\right)^n = e$)

2. Evaluate $\lim \left(\frac{(2n)!}{(n!)^2} \right)^{1/n}$

3. Let (s_n) and (t_n) be non-negative, bounded sequences.
   (a) Prove that $\limsup(s_n t_n) \leq (\limsup s_n)(\limsup t_n)$
   (b) Give an example which shows that we do not expect equality in part (a).
   (c) If, in addition, $\lim s_n = s$, prove that $\limsup(s_n t_n) = s \limsup t_n$.

4. Consider the sequence with $s_{2m} = s_{2m+1} = 2^{-m}$:

$$(s_n)_{n=0}^{\infty} = \left(1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \ldots\right)$$

   Compute $|s_n|^{1/n}$ and $\left| \frac{s_{n+1}}{s_n} \right|$ when n is even and then when it is odd. Thus find all expressions in Theorem 2.51 and conclude that the converse of (†) is *false*.

3 Series

3.14 Infinite Series and the Series Tests

For millennia, certainly since Zeno's paradoxes of c. 430 BC, mathematicians have been interested in the meaning and evaluation of infinite sums such as

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$$

The standard approach in modern mathematics is to outsource the definition to that of *limits.*

Definition 3.1. The n^{th} *partial sum* s_n of a sequence $(a_n)_{n=m}^{\infty}$ is the **finite sum**

$$s_n := \sum_{k=m}^{n} a_k = a_m + a_{m+1} + \cdots + a_n$$

- The *(infinite) series*[23] $\sum\limits_{n=m}^{\infty} a_n$ is the limit $\lim s_n$ of the sequence (s_n) of partial sums.
- A series *converges*, *s to* $\pm\infty$ or *diverges by oscillation* as does the sequence (s_n).
- $\sum a_n$ *converges absolutely* if $\sum |a_n|$ converges.
- $\sum a_n$ *converges conditionally* if it converges but not absolutely ($\sum |a_n|$ diverges to ∞).

We don't (yet) know whether our motivating example converges, but at least we have a meaning:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim s_n \quad \text{where} \quad s_n = \sum_{k=1}^{n} \frac{1}{k^2} = 1 + \frac{1}{4} + \cdots + \frac{1}{n^2}$$

Theorem 3.2 (Basic Series Laws). *Infinite series behave nicely with respect to addition and scalar multiplication. For instance:*

1. *If $\sum a_n$ is convergent and k is constant, then $\sum ka_n = k \sum a_n$ is convergent.*
2. *If $\sum a_n$ and $\sum b_n$ are convergent, then $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$ are also convergent.*
3. *If $\sum a_n = \infty$ and $k > 0$, then $\sum ka_n = \infty$.*
4. *If $\sum a_n = \infty$ and $\sum b_n$ converges, then $\sum(a_n + b_n) = \infty$.*

Proof. Simply apply the limit/divergence laws to the sequence of partial sums. E.g. for 1,

$$\sum ka_n = \lim_{n\to\infty} \sum_{j=m}^{n} ka_j \overset{\text{finite sum}}{=} \lim_{n\to\infty} k \sum_{j=m}^{n} a_j \overset{\text{limit laws}}{=} k \lim_{n\to\infty} \sum_{j=m}^{n} a_j = k \sum a_n$$

The others may be proved similarly. ∎

Series **do not** behave nicely with respect to multiplication (see also Exercise 3):

$$a_1b_1 + a_2b_2 + \cdots = \sum a_nb_n \neq \left(\sum a_n\right)\left(\sum b_n\right) = (a_1 + a_2 + \cdots)(b_1 + b_2 + \cdots)$$

[23] If the initial term is understood or is irrelevant to the situation, it is common to write $\sum a_n$.

Series which may be evaluated exactly

Given a series $\sum a_n$, our primary goal is to answer a simple question: "Does it converge?" Even when the answer is *yes*, a precise computation of the limit will usually be beyond us. We instead develop techniques (the upcoming *series tests*) which typically rely on comparing $\sum a_n$ to some 'standard' series whose properties are completely understood: in particular…

Definition 3.3 (Geometric series). A sequence (a_n) is *geometric* if the ratio of successive terms is constant: $a_n = ba^n$ for some constants a, b. A *geometric series* is the sum of a geometric sequence.

The computation of the sequence of partial sums should be familiar (for simplicity assume $b = 1$)

$$(1-a)s_n = \left(a^m + a^{m+1} + \cdots + a^n\right) - \left(a^{m+1} + a^{m+2} + \cdots + a^n + a^{n+1}\right) = a^m - a^{n+1}$$

from which we quickly conclude:

Theorem 3.4. *Suppose a is constant. Then*

$$s_n = \sum_{k=m}^{n} a^k = \begin{cases} \dfrac{a^m - a^{n+1}}{1-a} & \text{if } a \neq 1 \\ n+1-m & \text{if } a = 1 \end{cases} \implies \sum_{n=m}^{\infty} a^n \begin{cases} \text{converges to } \dfrac{a^m}{1-a} & \text{if } |a| < 1 \\ \text{diverges to } \infty & \text{if } a \geq 1 \\ \text{diverges by oscillation} & \text{if } a \leq -1 \end{cases}$$

In particular, $\sum a^n$ converges absolutely if $|a| < 1$ and diverges otherwise.

Examples 3.5. 1. $\displaystyle\sum_{n=-1}^{\infty} 2\left(-\frac{4}{5}\right)^n = 2\frac{\left(-\frac{4}{5}\right)^{-1}}{1+\frac{4}{5}} = -\frac{5}{2}\cdot\frac{5}{9} = -\frac{25}{18}$

2. Consider the series $\sum a_n = \sum_{n=3}^{\infty} \left(\frac{2}{5}\right)^n + 2^n$. If this were convergent, then

$$\sum 2^n = \sum a_n - \sum \left(\frac{2}{5}\right)^n$$

would converge (Theorem 3.2); a contradiction.

Telescoping series A rarer type of series can be evaluated using the algebra of partial fractions.

Example 3.6. To compute $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$, first observe that

$$s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \frac{1}{k} - \frac{1}{k+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim\left(1 - \frac{1}{n+1}\right) = 1$$

Similar arguments can be made for other series such as $\sum \frac{1}{n(n+2)}$.

The Cauchy Criterion

The starting point for our general series tests uses Cauchy completeness.

Example 3.7. Consider again the series $\sum \frac{1}{n^2}$. We show that the sequence of partial sums (s_n) is Cauchy. Suppose $\epsilon > 0$ is given and let $N = \frac{1}{\epsilon}$. Then,

$$\begin{aligned} m > n > N \implies |s_m - s_n| &= \sum_{k=n+1}^{m} \frac{1}{k^2} < \sum_{k=n+1}^{m} \frac{1}{k(k-1)} = \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} \\ &= \frac{1}{n} - \frac{1}{m} < \frac{1}{N} = \epsilon \end{aligned}$$

where most terms cancel analogous to the telescoping series approach. By Cauchy completeness (Theorem 2.34), (s_n) converges and we conclude

$$\boxed{\sum \frac{1}{n^2} \text{ converges}}$$

Computing the value of this series is significantly harder: a sketch argument for why $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ is in Exercise 10.

Theorem 3.8 (Cauchy criterion for series). *A series $\sum a_n$ converges precisely when*

$$\forall \epsilon > 0,\ \exists N \text{ such that } m > n > N \implies |s_m - s_n| = \left| \sum_{k=n+1}^{m} a_k \right| < \epsilon$$

The previous example essentially verified the Cauchy criterion for $\sum \frac{1}{n^2}$.

Proof. Let (s_n) be the sequence of partial sums. Then

$$\begin{aligned} \sum a_n \text{ converges} &\iff (s_n) \text{ converges} \\ &\iff (s_n) \text{ is a Cauchy sequence} \qquad \text{(Theorem 2.34)} \\ &\iff \Big(\forall \epsilon > 0,\ \exists N \text{ such that } m > n > N \implies |s_m - s_n| < \epsilon\Big) \end{aligned}$$

■

Example 3.9. For contradiction, suppose that the *harmonic series* $\sum \frac{1}{n}$ converges. Take $\epsilon = \frac{1}{2}$ in the Cauchy criterion to observe that

$$\exists N \text{ such that } m > n > N \implies \left| \sum_{k=n+1}^{m} \frac{1}{k} \right| < \frac{1}{2}$$

However, taking $m = 2n$ (plainly $m > n$ since $n > N \geq 1$) results in a contradiction:

$$\frac{1}{2} > \left| \sum_{k=n+1}^{m} \frac{1}{k} \right| = \left| \frac{1}{n+1} + \cdots + \frac{1}{m} \right| \geq \frac{m-n}{m} = 1 - \frac{n}{m} = \frac{1}{2}$$

We conclude that the harmonic series diverges to ∞.

The Series Tests

For the remainder of this section we develop tests for the convergence/divergence of an infinite series: the n^{th}-term, comparison, root and ratio tests. The first follows quickly from the Cauchy criterion.

Theorem 3.10 (Divergence/n^{th}-term test). *If $\lim a_n \neq 0$ then $\sum a_n$ is divergent.*

Proof. We prove the contrapositive. Suppose $\sum a_n$ is convergent, and that $\epsilon > 0$ is given. Take $m = n+1$ in the Cauchy criterion. Then

$$\exists \tilde{N} \text{ such that } m > \tilde{N} \implies |a_m| < \epsilon \qquad (\text{let } \tilde{N} = N+1)$$

Otherwise said, $\lim a_n = 0$. ■

Examples 3.11. 1. The series $\sum \sin(\frac{n\pi}{9})$ diverges.

2. The n^{th}-term test tells us that the geometric series $\sum a^n$ diverges whenever $|a| \geq 1$. We still need our earlier analysis (Theorem 3.4) for when $|a| < 1$.

3. The **converse** of the n^{th}-term test is **false!** Example 3.9 provides the canonical example: the **divergent** harmonic series $\sum \frac{1}{n}$ also satisfies $\lim \frac{1}{n} = 0$.

Theorem 3.12 (Comparison test). 1. *Let $\sum b_n$ be a convergent series of non-negative terms and assume $|a_n| \leq b_n$ for all (large) n. Then both $\sum a_n$ and $\sum |a_n|$ are convergent.*

2. *If $\sum a_n = \infty$ and $a_n \leq b_n$ for all (large) n, then $\sum b_n = \infty$.*

Proof. Suppose "large n" means $n > M$ for some fixed M.

1. Let $\epsilon > 0$ be given. Since $\sum b_n$ converges, $\exists N \geq M$ such that

$$m > n > N \implies \left| \sum_{k=n+1}^{m} a_k \right| \stackrel{\triangle}{\leq} \sum_{k=n+1}^{m} |a_k| \leq \sum_{k=n+1}^{m} b_k < \epsilon$$

2. The n^{th} partial sum (post M) of $\sum b_n$ is

$$\sum_{k=M+1}^{n} b_k \geq \sum_{k=M+1}^{n} a_k \to \infty$$

■

Corollary 3.13. 1. *(Absolute convergence implies convergence) Take $|a_n| = b_n$ in part 1 to see that $\sum |a_n|$ convergent $\implies \sum a_n$ convergent.*

2. *(Estimation of series) Suppose $\sum b_n$ is a convergent series of non-negative terms and that $|a_n| \leq b_n$ for **all** n. Then*

$$\sum a_n \leq \sum |a_n| \leq \sum b_n$$

Examples 3.14. 1. Since the geometric series $\sum \frac{2}{3^n}$ converges, $\frac{2n+1}{(n+2)3^n} \leq \frac{2}{3^n}$, we see that

$$\sum_{n=0}^{\infty} \frac{2n+1}{(n+2)3^n} \leq 2\sum_{n=0}^{\infty} 3^{-n} = \frac{2}{1-\frac{1}{3}} = 3$$

That is, the first series converges (absolutely) to some value ≤ 3.

2. One can sometimes find a sensible comparison series by considering how a_n behaves for large n. For instance, when n is large, $a_n = \frac{(n^2+1)^{1/2}}{(1+\sqrt{n})^4}$ behaves like $\frac{n}{n^2} = \frac{1}{n}$. Indeed, when $n \geq 2$,

$$a_n > \frac{n}{(1+\sqrt{n})^4} > \frac{n}{(2\sqrt{n})^4} = \frac{1}{16n}$$

Comparison with the divergent series $\frac{1}{16}\sum \frac{1}{n}$ shows that $\sum a_n$ also diverges to ∞.

3. Since $\ln n < n \implies \frac{1}{\ln n} > \frac{1}{n}$, we see that $\sum \frac{1}{\ln n}$ diverges to ∞ by comparison with $\sum \frac{1}{n}$.

4. $\sum \frac{\sin n}{n^2}$ converges absolutely by comparison to $\sum \frac{1}{n^2}$ (Example 3.7). Corollary 3.13 estimates its value ($\leq \frac{\pi^2}{6}$):

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^2} \leq \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(approximately } 1.014 \leq 1.280 \leq 1.645\text{)}$$

5. The *alternating harmonic series* $s = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges via a sneaky comparison. The series $t = \sum_{n=1}^{\infty} \frac{1}{2n(2n-1)}$ converges by comparison with $\sum \frac{1}{4(n-1)^2}$. Its n^{th} partial sum

$$t_n = \sum_{k=1}^{n} \frac{1}{2k(2k-1)} = \sum_{k=1}^{n} \left[\frac{1}{2k-1} - \frac{1}{2k}\right]$$

is precisely the *even* partial sum of the alternating harmonic series $s_{2n} = \sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k}$. Plainly $\lim s_{2n} = t$. Moreover $s_{2n+1} = s_{2n} + \frac{1}{2n+1} \implies \lim s_{2n+1} = t \implies \lim s_n = t$. Since the harmonic series $\sum \frac{1}{n}$ diverges (Example 3.9), we conclude that the alternating harmonic series **converges conditionally**. We'll revisit this discussion in the next section.

6. $\sum \left(\frac{n}{n+1}\right)^{n^2}$ converges by comparison with $\sum 2^{-n}$. To see this, recall Exercise 2.10.10:

$$\left(\frac{n}{n+1}\right)^n = \frac{n+1}{n}\left(1 - \frac{1}{n+1}\right)^{n+1} \xrightarrow[n\to\infty]{} e^{-1}$$

Plainly $e^{-1} < \frac{1}{2}$, whence for large n,

$$\left(\frac{n}{n+1}\right)^n \leq \frac{1}{2} \implies \left(\frac{n}{n+1}\right)^{n^2} \leq 2^{-n}$$

In fact $\left(\frac{n}{n+1}\right)^n$ is monotone-down, whence $e^{-1} \leq \left(\frac{n}{n+1}\right)^n \leq \frac{1}{2}$ **for all** n, and so

$$0.58198 \approx \frac{e^{-1}}{1-e^{-1}} = \sum_{n=1}^{\infty} e^{-n} \leq \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \leq \sum_{n=1}^{\infty} 2^{-n} = \frac{1/2}{1-1/2} = 1$$

A computer estimate yields $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \approx 0.8174$.

Our last two tests in this section are less powerful but often easier to use.

Theorem 3.15 (Root test). *Suppose* $\limsup |a_n|^{1/n} = L$.

1. *If* $L < 1$*, then* $\sum a_n$ *converges absolutely.*
2. *If* $L > 1$*, then* $\sum a_n$ *diverges.*

If $L = 1$*, then no conclusion can be drawn.*

We defer the proof until after some examples. By combining with the inequalities of Theorem 2.51

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

we obtain a second familiar test.

Corollary 3.16 (Ratio test). *Suppose* (a_n) *is a sequence of non-zero terms.*

1. *If* $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$*, then* $\sum a_n$ *converges absolutely.*
2. *If* $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$*, then* $\sum a_n$ *diverges.*

In elementary calculus you likely saw the special cases when

$$L = \lim |a_n|^{1/n} = \lim \left|\frac{a_{n+1}}{a_n}\right|$$

Our versions are more general since these limits *might not exist.*

Examples 3.17. 1. The ratio test is particularly useful for series involving *factorials* and *exponentials.*

   (a) $\sum \frac{n^4}{2^n}$ converges, since $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)^4}{2n^4} = \frac{1}{2} < 1$.

   (b) $\sum \frac{n!}{2^n}$ diverges, since $\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+1)!}{2n!} = \lim \frac{n+1}{2} = \infty$.

2. Both tests are inconclusive for rational sequences: if $a_n = \frac{b_n}{c_n}$ where b_n, c_n are polynomials, then

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = 1 = \lim |a_n|^{1/n}$$

   For example, attempting to apply the ratio test to $\sum \frac{n+5}{n^2}$ results in

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \frac{(n+6)n^2}{(n+5)(n+1)^2} = 1$$

   This series is divergent by comparison with the harmonic series $\sum \frac{1}{n}$.

3. In Example 3.14.6, our use of the comparison test was really the root test in disguise:

$$a_n = \left(\frac{n}{n+1}\right)^{n^2} \implies \lim |a_n|^{1/n} = \lim \left(\frac{n}{n+1}\right)^n = e^{-1} < 1 \implies \sum a_n \text{ converges}$$

   In this case the root test was much easier to apply.

4. The ratio test is the weakest test thus far; certainly it does not apply if any of the terms a_n are zero! For a more subtle example of its failure, consider

$$a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even} \\ 3^{-n} & \text{if } n \text{ is odd} \end{cases} \implies \frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{3}\left(\frac{2}{3}\right)^n & \text{if } n \text{ is even} \\ \frac{1}{2}\left(\frac{3}{2}\right)^n & \text{if } n \text{ is odd} \end{cases}$$
$$\implies \liminf \left|\frac{a_{n+1}}{a_n}\right| = 0, \quad \limsup \left|\frac{a_{n+1}}{a_n}\right| = \infty$$

The ratio test is therefore inconclusive. However, applying the root test it almost trivial!

$$|a_n|^{1/n} = \begin{cases} \frac{1}{2} & \text{if } n \text{ is even} \\ \frac{1}{3} & \text{if } n \text{ is odd} \end{cases} \implies \limsup |a_n|^{1/n} = \frac{1}{2} < 1 \implies \sum a_n \text{ converges}$$

We need not even have used the root test: $\sum a_n$ plainly converges by comparison with $\sum 2^{-n}$!

For a precise value, note that the sequence of n^{th} partial sums converges monotone up to the sum of two geometric series,

$$\sum_{n=0}^{\infty} a_n = \sum_{k=0}^{\infty} 2^{-2k} + 3^{-2k-1} = \frac{1}{1-1/4} + \frac{1/3}{1-1/9} = \frac{11}{8}$$

Proof of the Root Test. 1. Suppose $L < 1$. Choose any $\epsilon > 0$ such that $L + \epsilon < 1$ (say $\epsilon = \frac{1-L}{2}$). Since $v_N = \sup\{|a_n|^{1/n} : n \geq N\}$ defines a *monotone-down* sequence converging to L, we see that

$$\exists N \text{ such that } v_N - L < \epsilon$$

But then

$$n \geq N \implies |a_n|^{1/n} - L < \epsilon \implies |a_n| < (L+\epsilon)^n$$

$\sum |a_n|$ therefore converges by comparison with the geometric series $\sum (L+\epsilon)^n$.

2. If $L > 1$ then there exists some subsequence (a_{n_k}) such that $|a_{n_k}|^{1/n_k} \to L > 1$. In particular, infinitely many terms of this subsequence must be greater than 1 and so (a_n) does not converge to zero. $\sum a_n$ thus diverges by the n^{th}-term test.

∎

Summary The logical flow of the tests in this section is as follows:

$$\begin{array}{llcl}
\text{(divergence testing)} & \text{Comparison} & n^{\text{th}}\text{-term} \Longrightarrow \text{Root} \Longrightarrow \text{Ratio} \\
 & \Uparrow & \Uparrow \\
 & \text{Definition of } \sum a_n \iff & \text{Cauchy criterion} \\
 & & \Downarrow \\
\text{(convergence testing)} & & \text{Comparison} \Longrightarrow \text{Root} \Longrightarrow \text{Ratio}
\end{array}$$

The ratio test is typically the easiest to use, but the least powerful. Every series which converges by the ratio test can be seen to converge by the root and comparison tests, and the Cauchy criterion. If you find that a series diverges by the ratio test, you could have just used the n^{th}-term test!

Exercises 3.14. *Key concepts: Infinite series, Cauchy criterion, Comparison/root/ratio tests*

1. Determine which of the following series converge. Justify your answers.

   (a) $\sum \frac{n-1}{n^2}$ (b) $\sum (-1)^n$ (c) $\sum \frac{3^n}{n^3}$ (d) $\sum \frac{n^3}{3^n}$ (e) $\sum \frac{n^2}{n!}$

   (f) $\sum \frac{1}{n^n}$ (g) $\sum \frac{n}{2^n}$ (h) $\sum \frac{n!}{n^n}$ (i) $\sum\limits_{n=2}^{\infty} \big[n+(-1)^n\big]^{-2}$ (j) $\sum \big[\sqrt{n+1}-\sqrt{n}\big]$

2. Let $\sum a_n$ and $\sum b_n$ be convergent series of non-negative terms. Prove that $\sum \sqrt{a_n b_n}$ converges.
   (*Hint: start by showing that* $\sqrt{a_n b_n} \le a_n + b_n$)

3. (a) If $\sum a_n$ converges absolutely, prove that $\sum a_n^2$ converges.
   (b) More generally, if $\sum |a_n|$ converges and (b_n) is a bounded sequence, prove that $\sum a_n b_n$ converges absolutely.

4. Find a series $\sum a_n$ which diverges by the root test but for which the ratio test is inconclusive.

5. Suppose $\liminf |a_n| = 0$. Prove that there is a subsequence (a_{n_k}) such that $\sum a_{n_k}$ converges.
   (*Hint: Try to construct a subsequence which converges to zero faster than* $\frac{1}{k^2}$.

6. Prove that the harmonic series $\sum \frac{1}{n}$ diverges by comparing with the series $\sum a_n$, where

$$(a_n) = \left(1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{32}, \ldots\right)$$

7. Suppose $b_n \le a_n$ for all n and that $\sum b_n$ and $\sum a_n$ converge. Prove that $\sum b_n \le \sum a_n$.
   (*This also proves part 2 of Corollary 3.13*)

8. Given $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, find the values of $\sum \frac{1}{(2n)^2}$, $\sum \frac{1}{(2n+1)^2}$ and $\sum \frac{(-1)^{n+1}}{n^2}$.

9. The *limit comparison test* states:

   Suppose $\sum a_n, \sum b_n$ are series of positive terms and that $L = \lim \frac{a_n}{b_n} \in (0, \infty)$. Then the series have the same convergence status (both converge or both diverge to ∞).

   (a) Use the limit comparison test with $b_n = \frac{1}{n^2}$ to show that the series $\sum \frac{1}{n} \ln\left(1+\frac{1}{n}\right)$ converges.
   (*Hint: Recall that* $e = \lim\left(1+\frac{1}{n}\right)^n$)
   (b) Prove the limit comparison test.
   (*Hint: first show that* $\frac{L}{2} < \frac{a_n}{b_n} < \frac{3L}{2}$ *for large* n)
   (c) What can you say about the series $\sum a_n$ and $\sum b_n$ if $L = 0$ or $L = \infty$? Explain.

10. Euler asserted that the sine function, written as an infinite polynomial in the form of a Maclaurin series, could also be expressed as an infinite product,

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

   By considering the solutions to $\sin x = 0$, give some weight to Euler's claim. By comparing coefficients in these expressions, deduce the fact $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.
   (*As presented, this argument is non-rigorous!*)

3.15 The Integral and Alternating Series Tests

In this section we develop two further standalone series tests, both with narrower applications than our previous tests.

The first is a little out of place given that it requires (improper) integration.[24]

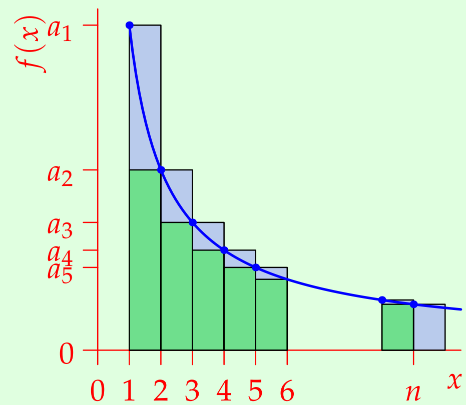
Theorem 3.18 (Integral test). *Let $a_n = f(n)$, where f is non-negative, non-increasing, and integrable on $[1, \infty)$. Then*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x)\,dx \text{ converges}$$

In such a situation,

$$\int_1^{\infty} f(x)\,dx \le \sum_{n=1}^{\infty} a_n \le a_1 + \int_1^{\infty} f(x)\,dx$$

The statement is easily modified if the initial term is not a_1.



Proof. We need only interpret the picture as describing upper and lower Riemann sums:

$$\int_1^{n+1} f(x)\,dx \le \sum_{k=1}^{n} a_k = s_n = a_1 + \sum_{k=2}^{n} a_k \le a_1 + \int_1^{n} f(x)\,dx \qquad (*)$$

Take limits as $n \to \infty$ for the result. ∎

Even for divergent sums, $(*)$ allows us to estimate the growth of (s_n). For greater accuracy, we may evaluate the first few terms explicitly and modify the integral test to estimate the remainder.

An important application of the integral test is to provide a complete description of the convergence status of p-**series**: a useful family of series to which others may be compared.

Corollary 3.19 (p-series). *Let $p > 0$. The series $\sum \frac{1}{n^p}$ converges if and only if $p > 1$.*

Examples 3.20. 1. $\sum \frac{1}{n^3}$ converges (it is a p-series with $p > 1$). Moreover,

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \left[-\frac{1}{2}x^{-2}\right]_1^b = \frac{1}{2} \implies \frac{1}{2} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le \frac{3}{2}$$

This is a poor estimate, particularly the lower bound. For a quick improvement, evaluate the first term and re-run the test starting at $n = 2$:

$$1 + \int_2^{\infty} \frac{1}{x^3}\,dx \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1 + \frac{1}{8} + \int_2^{\infty} \frac{1}{x^3}\,dx \implies 1 + \frac{1}{8} \le \sum_{n=1}^{\infty} \frac{1}{n^3} \le 1 + \frac{1}{4}$$

If greater accuracy is required, more terms can be explicitly evaluated.

[24] Which in turn requires limits of functions: $\int_1^{\infty} f(x)\,dx := \lim_{b\to\infty} \int_1^b f(x)\,dx$. While we haven't rigorously developed these concepts, the relevant computations should be familiar from elementary calculus.

2. In Example 3.9, we used the Cauchy criterion to show that the harmonic series diverges to ∞. The integral test makes this much easier and allows us to estimate how many terms are required for the partial sum to s_n to reach a certain threshold: 10 say. Since

$$\ln(n+1) = \int_1^{n+1} \frac{1}{x}\,\mathrm{d}x \le s_n = \sum_{k=1}^{n} \frac{1}{k} \le 1 + \int_1^n \frac{1}{x}\,\mathrm{d}x = 1 + \ln n$$

we see that $s_n \approx 10$ requires

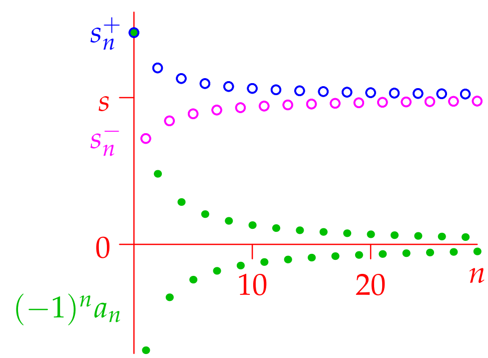
$$\ln(n+1) \le 10 \le 1 + \ln n \implies e^9 \le n \le e^{10} - 1 \implies 8104 \le n \le 22025$$

The harmonic series diverges to infinity, but it does so *very slowly.*

3. The integral test shows that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$. To exceed 10, somewhere between 10^{3223} and 10^{6631} terms are required. To exceed 100 requires 'roughly' $10^{6\times 10^{42}}$ terms: 1 followed by 1000 zeros for each water molecule in Lake Tahoe puts you at least in the right ballpark...

4. The series $\sum \frac{2n+1}{\sqrt{4n^3-1}}$ diverges to ∞ by comparison with the p-series $\sum \frac{1}{\sqrt{n}}$.

Alternating Series and Conditional Convergence



Our final test is unique in that it can detect *conditional convergence.* The canonical example is the *alternating harmonic series* (Example 3.14.5). With an eye on generalization, we re-index so that the first term is $a_0 = 1$:

$$s = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n a_n = a_0 - a_1 + a_2 - a_3 + \cdots$$

The *alternating* $\pm$-signs give the series its name. Consider the behavior of the sequence of partial sums (s_n), in particular two subsequences $(s_n^+) = (s_{2n})$ and $(s_n^-) = (s_{2n-1})$:

$$s_n^+ = \sum_{k=0}^{2n} (-1)^k a_k = 1 - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_1 - a_2} - \underbrace{\left(\frac{1}{2} - \frac{1}{3}\right)}_{a_3 - a_4} - \cdots - \underbrace{\left(\frac{1}{2n} - \frac{1}{2n+1}\right)}_{a_{2n-1} - a_{2n}} \qquad (n \ge 0)$$

$$s_n^- = \sum_{k=0}^{2n-1} (-1)^k a_k = \underbrace{\left(1 - \frac{1}{2}\right)}_{a_0 - a_1} + \underbrace{\left(\frac{1}{3} - \frac{1}{4}\right)}_{a_2 - a_3} + \cdots + \underbrace{\left(\frac{1}{2n-1} - \frac{1}{2n}\right)}_{a_{2n-2} - a_{2n-1}} \qquad (n \ge 1)$$

Each bracketed term is non-negative, so (s_n^+) is monotone-down and (s_n^-) monotone-up. Moreover,

$$\frac{1}{2} = s_1^- \le s_n^- \le s_n^- + a_{2n} = s_n^+ \le s_0^+ = 1 \qquad (\dagger)$$

from which both subsequences are *bounded* and thus *convergent.* Not only this, but

$$\lim(s_n^+ - s_n^-) = \lim a_{2n} = 0$$

shows that the limits of both subsequences are *identical* (of course both are s).

The above discussion depends only on two simple properties of the sequence (a_n). We've therefore proved a general statement.

Theorem 3.21 (Alternating series test). *Suppose (a_n) is monotone-down and that $\lim a_n = 0$. Then:*

1. *The alternating series $s = \sum (-1)^n a_n$ converges.*
2. *If (s_n) is the sequence of partial sums, then $|s - s_n| \le a_{n+1}$.*

Think about where the hypotheses regarding (a_n) are used in the proof.

It can be shown that the alternating harmonic series converges to $\ln 2$, though the estimates provided by the alternating series test are very poor: to guarantee accuracy to two decimal place requires us to sum 100 terms of the series!

Examples 3.22. 1. Since $a_n = \frac{1}{n!}$ converges monotone-down to zero, the alternating series $\sum \frac{(-1)^n}{n!}$ converges. By evaluating s_8 and s_9 explicitly, we see that

$$0.3678791887\ldots \le \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \le 0.3678819444\ldots$$

yielding an estimate of 0.36788 to 5 decimal places (the exact value is in fact e^{-1}). The alternating series test is only needed for the estimate, since the series converges absolutely.

2. The series $\sum_{n=2}^{\infty} \frac{\sin \frac{\pi}{2} n}{\ln n}$ can be viewed as an alternating series since every *even* term is zero. Writing $m = 2n+1$, we obtain

$$\sum_{n=2}^{\infty} \frac{\sin \frac{\pi}{2} n}{\ln n} = \sum_{m=1}^{\infty} \frac{\sin(\pi m + \frac{\pi}{2})}{\ln(2m+1)} = \sum_{m=1}^{\infty} \frac{(-1)^m}{\ln(2m+1)}$$

Since $\frac{1}{\ln(2m+1)}$ decreases to zero, the alternating series test demonstrates convergence.

Rearranging Infinite Series

A *rearrangement* of an infinite series $\sum a_n$ is a series that results from changing the *order* of the terms of the sequence (a_n) *before* computing the partial sums. The new series must still use every term of the original. Since the new sequence of partials sums is likely different, we shouldn't assume that the rearranged series has the same convergence properties as the old.

Example 3.23. We rearrange the alternating harmonic series by summing two positive terms before each negative term:

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} + \cdots$$

Every term of the original sequence is used here, so this is a genuine rearrangement. It is perhaps surprising to discover that the new series converges, though its limit is *not the same* as the original alternating harmonic series! We leave the details to Exercise 11. This behavior is quite different to that of finite sums, where the order of summation makes no difference at all.

The general situation is summarized in a famous result of Riemann. The first part says that absolutely convergent series behave just like finite sums. Conditionally convergent series are much stranger.[25]

Theorem 3.24 (Riemann rearrangement).

1. *If a series converges absolutely, then all rearrangements converge to the same limit.*
2. *If a series converges conditionally and $s \in \mathbb{R} \cup \{\pm\infty\}$ is given, then there exists a rearrangement which tends to s.*

We omit the proofs since they are prohibitively lengthy. Instead we illustrate the rough idea of part 2 via an example.

Example 3.25. We show how to construct a rearrangement of the alternating harmonic series which converges to $s = \sqrt{2} = 1.41421\ldots$

First we convince ourselves that the sum of the positive terms $\sum a_n^+$ diverges to infinity. The comparison test makes this easy:

$$\frac{1}{2n-1} > \frac{1}{2n} \implies \sum a_n^+ = \sum \frac{1}{2n-1} > \frac{1}{2}\sum \frac{1}{n} = \infty$$

The negative terms also diverge: $\sum a_n^- = -\infty$. Construction of the rearrangement is inductive.

1. Sum just enough positive terms $S_1 = a_1^+ + a_2^+ + \cdots + a_{n_1}^+$ *in order* until the partial sum exceeds s: plainly $S_1 = 1 + \frac{1}{3} + \frac{1}{5} \approx 1.53333$ will do here.
2. Add negative terms starting at the beginning of the sequence until the sum is less than s:
$$S_2 = S_1 + (a_1^- + a_2^- + \cdots + a_{m_1}^-) = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} = 1.0333\cdots < s$$
3. Repeat: add positive terms until the sum just exceeds s, then add negative terms, etc.,
$$S_3 = S_2 + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} = 1.4551\ldots > s, \qquad S_4 = S_3 - \frac{1}{4} = 1.2051\ldots < s$$

Continuing the process ad infinitum, we claim that

$$s = \sqrt{2} = 1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} - \frac{1}{4} + \frac{1}{15} + \frac{1}{17} + \frac{1}{19} + \frac{1}{21} - \frac{1}{6} + \frac{1}{23} + \frac{1}{25} + \cdots$$

To see why, observe:

- Since $\sum a_n^+ = \infty$ and $\sum a_n^- = -\infty$, at each stage we need only add/subtract *finitely many terms.*
- All terms of the original sequence (a_n) are eventually used since we add the positive (negative) terms *in order.* E.g., $a_{495} = \frac{1}{495}$ appears, *at the latest*, during the 495^{th} positive-addition phase.
- $|S_n - s| \leq |a_{m_n}|$, where a_{m_n} is the final term used at the n^{th} stage. The right hand side converges to zero (n^{th}-term test!), whence $\lim S_n = s$.

[25]Riemann's second result is in fact even stronger. Conditionally convergent series also have rearrangements whose sequence of partial sums diverges by oscillation to any given $\liminf s_n < \limsup s_n$!

Exercises 3.15. *Key concepts: Integral test and approximation, Alternating series and approximation*

1. Use the integral test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converges or diverges.
2. Prove Corollary 3.19 regarding the convergence/divergence of p-series.
3. Let $s_n = \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$. Estimate how many terms are required before $s_n \geq 100$.
4. (Example 3.20.3) Verify the claim that $\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \infty$ and the claim regarding the estimate.
5. (a) Use calculus to show that $a_n = \frac{\ln n}{n^2}$ is monotone-down whenever $n \geq 2$.
   (b) Show that $\lim a_n = 0$, and that the hypotheses of the integral test are therefore satisfied.
   (c) Determine whether the series $\sum_{n=2}^{\infty} \frac{\ln n}{n^2}$ converges or diverges.
6. (a) Give an example of a series $\sum a_n$ which converges, but for which $\sum a_n^2$ diverges.
   (*Exercise 3.14.3 really requires that $\sum a_n$ be absolutely convergent!*)
   (b) Give an example of a divergent series $\sum b_n$ for which $\sum b_n^2$ converges.
7. Suppose (a_n) satisfies the hypotheses of the alternating series test except that $\lim a_n = a$ is **strictly positive**. What can you say about the sequences (s_n^+) and (s_n^-) and the series $\sum (-1)^n a_n$?
8. Let $a_n = \frac{1}{n}$ have partial sum $s_n = \sum_{k=1}^{n} a_n$, and define a new sequence (t_n) by

   $$t_n = s_n - \ln n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$$

   Prove that (t_n) is a positive, monotone-down sequence, which therefore converges.[26]

   (*Hint: You'll need the mean value theorem from elementary calculus*)
9. Suppose $\sum a_n$ is conditionally convergent and let $\sum a_n^+$ be the series obtained by summing, in order, the *positive* terms of the sequence (a_n). Prove that $\sum a_n^+ = \infty$.
10. (a) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$ is conditionally convergent to some real number s.
    (b) How many terms are required for the partial sum s_n to approximate s to within 0.01.
    (c) Following Example 3.25, use a calculator to state the first twelve terms in a rearrangement of the series in part (a) which converges to 0.
11. Recall the rearrangement of the alternating harmonic series in Example 3.23.
    (a) Verify that the *subsequence* of partial sums (s_{3n}) is monotone-up, by checking that

    $$b_n := \frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n} > 0, \qquad \text{for all } n \in \mathbb{N}$$

    (b) Use the comparison test to show that $\sum b_n$ converges.
    (c) Prove that the rearranged series converges to some value $s > \frac{5}{6}$.
    (*Thus $s > \ln 2 \approx 0.69$, the limit of the original alternating harmonic series*)

[26] The limit $\gamma := \lim t_n \approx 0.5772$ is the *Euler–Mascheroni constant.* It appears in several mathematical identities, and yet very little about it is understood; it is not even known whether γ is irrational!

4 Continuity

In this chapter we discuss continuous functions. Functions themselves should be familiar. For reference, we begin with a review of some basic concepts and conventions.

We are concerned with functions $f : U \to V$ where both U, V are subsets of the real numbers $\mathbb{R}$ and f is some *rule* assigning to each real number $x \in U$ a real number $f(x) \in V$. For instance

$$f(x) = \frac{x^2(x-7)}{(x-2)(x^2-9)} \quad \text{assigns to } x = 1 \text{ the value} \quad f(1) = \frac{1(-6)}{(-1)(-8)} = \frac{3}{4}$$

Domain $\operatorname{dom} f = U$ is the set of *inputs* to f. When f is defined by a formula, its *implied domain* is the largest set on which the formula is defined: the above example has implied domain $\operatorname{dom} f = \mathbb{R} \setminus \{2, 3, -3\}$. In examples, the domain is most often a union of intervals of positive length.

Codomain $\operatorname{codom} f = V$ is the set of *possible outputs*. In real analysis, we often take $V = \mathbb{R}$ by default.

Range $\operatorname{range} f = f(U) = \{f(x) : x \in U\}$ is the set of *realized outputs*, and is a subset of $V = \operatorname{codom} f$.

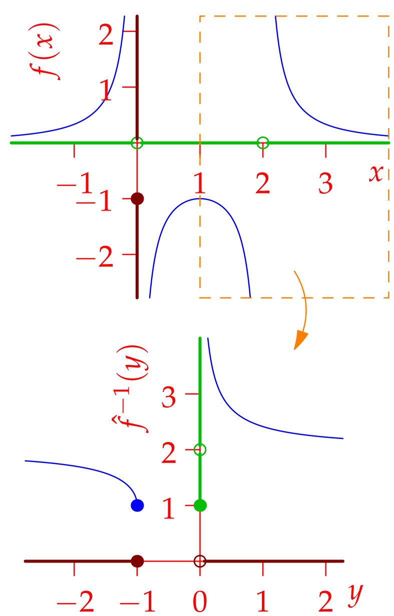
Injectivity f is *injective/one-to-one* if distinct inputs produce distinct outputs. This is usually stated in the contrapositive: $f(x) = f(u) \implies x = u$.

Surjectivity f is *surjective/onto* if every possible output is realized: that is $f(U) = V$.

Inverses f is *bijective/invertible* if it is both injective and surjective. Equivalently, f has an *inverse function* $f^{-1} : V \to U$ defined as follows:

- Given $y \in V$, f surjective $\implies \exists x \in U$ such that $f(x) = y$.
- Since f is injective, $f(x) = f(u) \implies x = u$, so x is unique. We *define* $f^{-1}(y) = x$.

Example 4.1. The function defined by $f(x) = \frac{1}{x(x-2)}$ has implied



$$\operatorname{dom} f = \mathbb{R} \setminus \{0, 2\} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

$$\operatorname{range} f = (-\infty, -1] \cup (0, \infty)$$

The function is neither injective (e.g., $f(3) = f(-1)$) nor surjective (e.g., $0 \notin \operatorname{range} f$).

We can remedy both issues by restricting the domain and codomain. For instance, the same rule/formula but with

$$\operatorname{dom} \hat{f} = [1, 2) \cup (2, \infty)$$
$$\operatorname{codom} \hat{f} = (-\infty, -1] \cup (0, \infty)$$

defines a bijection with inverse function

$$\hat{f}^{-1}(y) = \begin{cases} 1 + y^{-1}\sqrt{y+1} & \text{if } y > 0 \\ 1 - y^{-1}\sqrt{y+1} & \text{if } y \le -1 \end{cases}$$

Observe that $\operatorname{dom} \hat{f}^{-1} = \operatorname{codom} \hat{f}$ and $\operatorname{codom} \hat{f}^{-1} = \operatorname{dom} \hat{f}$.

4.17 Continuous Functions

To introduce continuity, consider two common naïve notions.

The graph of f can be drawn without removing one's pen from the page This is intuitive but unusable: *drawn* is poorly defined, so how might we *calculate* or *prove* anything with this concept? It moreover cannot reasonably be extended to other situations or higher dimensions where *drawing a graph* is meaningless.

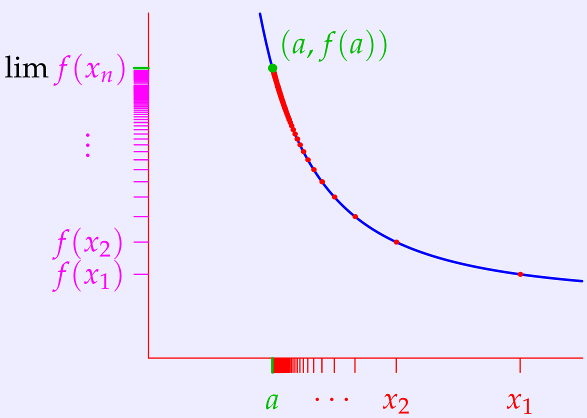
If x is close to a, then $f(x)$ is close to $f(a)$ This is better and admits generalization. The major issue is the unclear meaning of *close*. Our formal definition of continuity addresses this using *sequences* and *limits*.

Definition 4.2 (Sequential continuity). A real-valued function $f : U \to V$ is *continuous at $a \in U$* if,

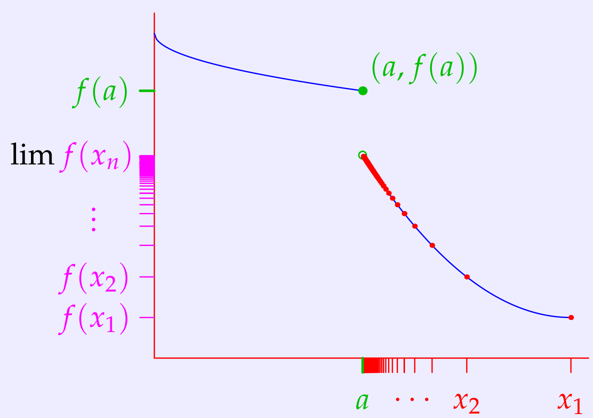
$$\forall (x_n) \subseteq U,\ \lim x_n = a \implies \lim f(x_n) = f(a)$$

f is *continuous (on U)* if it is continuous at every point $a \in U$.
We say that f is *discontinuous* at $a \in U$ if,

$$\exists (x_n) \subseteq U, \text{ such that } \lim x_n = a \text{ and } \big(f(x_n)\big) \text{ does not converge to } f(a)$$



Continuity at a: *every sequence* with $\lim x_n = a$ has $\lim f(x_n) = f(a)$



Discontinuous at a: *at least one sequence* with $\lim x_n = a$ has $\lim f(x_n) \neq f(a)$

Examples 4.3. 1. $f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$ is continuous (at every $a \in \mathbb{R}$). To see this, suppose (x_n) converges to a, then, by the limit laws,

$$\lim f(x_n) = \lim x_n^2 = (\lim x_n)^2 = a^2 = f(a)$$

2. The function with $g(x) = 1 + \frac{4}{x^2}$ is continuous. Choose any $a \in \operatorname{dom} g = \mathbb{R} \setminus \{0\}$ and any $(x_n) \subseteq \operatorname{dom} g$ with $\lim x_n = a$. Again, by the limit laws,

$$\lim g(x_n) = \lim \left(1 + \frac{4}{x_n^2}\right) = 1 + \frac{4}{(\lim x_n)^2} = 1 + \frac{4}{a^2} = f(a)$$

This example (with $a = 1$ and $x_n = 1 + \frac{2}{n}$) is the first picture in the above definition.

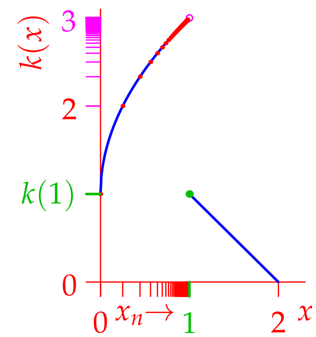
3. $h : [0, \infty) \to \mathbb{R} : x \mapsto 3x^{1/4}$ is continuous. Again, everything follows from the limit laws. If $x_n \to a$ where $x_n \geq 0$ and $a \geq 0$, then

$$\lim h(x_n) = \lim 3x_n^{1/4} = 3(\lim x_n)^{1/4} = 3a^{1/4} = h(a)$$

4. The function defined by

$$k(x) = \begin{cases} 1 + 2\sqrt{x} & \text{if } x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$

is discontinuous at $a = 1$. This seems obvious from the picture, but we need to use the definition. The sequence with $x_n = (1 - \frac{1}{n})^2$ converges to 1 from below, however the limit laws tells us that



$$\lim k(x_n) = \lim \left(1 + 2\left(1 - \frac{1}{n}\right)\right) = 3 \neq 1 = k(1)$$

Basic Examples and Combinations of Continuous Functions

By appealing to the limit laws for sequences (Theorem 2.15), continuous functions may be combined in natural ways. For instance, if f, g are continuous at a, then

$$\lim x_n = a \implies \lim f(x_n) + g(x_n) = \lim f(x_n) + \lim g(x_n) = f(a) + g(a)$$

whence $f + g$ is continuous at a. Here is a general summary.

Theorem 4.4. 1. *Suppose f, and g are continuous and that k is constant. Then the following functions are continuous (on their domains):*

$$kf, \quad |f|, \quad f+g, \quad f-g, \quad fg, \quad \frac{f}{g}, \quad \max(f,g), \quad \min(f,g)$$

2. *If $n \in \mathbb{N}$ then $f : x \mapsto x^{1/n}$ is continuous on its domain.*

3. *Compositions of continuous functions are continuous: if g is continuous at a and f is continuous at $g(a)$, then $f \circ g$ is continuous at a.*

4. *Algebraic functions are continuous (includes all polynomials and rational functions).*

Proof. Parts 1, 2 are the limit laws; for the maximum and minimum, see Exercise 2. For part 3:

$$\lim x_n = a \overset{g \text{ cont}}{\implies} \lim g(x_n) = g(a) \overset{f \text{ cont}}{\implies} \lim f(g(x_n)) = f(g(a))$$

Part 4 follows by combining parts 1, 2 and 3. ∎

Example 4.5. The following algebraic function is continuous on its domain

$$f : (7, \infty) \to \mathbb{R} : x \mapsto \sqrt{\frac{3x^{5/2} + 7x^2 + 4}{(x-7)^{1/3}}}$$

Theorem 4.6 (Squeeze theorem). *Suppose $f(x) \le g(x) \le h(x)$ for all $x \neq a$, that f, h are continuous at a, and that $f(a) = g(a) = h(a)$. Then g is continuous at a.*

Proof. This is simply the squeeze theorem (2.12) for sequences: if $\lim x_n = a$, then

$$f(x_n) \le g(x_n) \le h(x_n) \implies \lim g(x_n) = g(a)$$

∎

To provide more interesting examples, we state the following without proof.

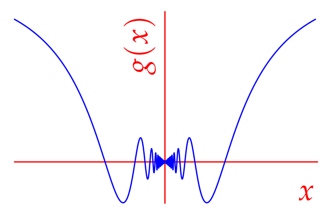
Theorem 4.7. *The common trigonometric, exponential and logarithmic functions are continuous.*

It is possible, though slow and ugly, to address some of this now. We won't do this since it is cleaner to define these functions using power series,[27] which makes their continuity (and differentiability/integrability!) come for free.

Examples 4.8. 1. $f(x) = \frac{\sqrt{x}}{\sin e^x}$ is continuous on its domain $\mathbb{R} \setminus \{\ln(n\pi) : n \in \mathbb{N}_0\}$.

2. If $g(x) = x \sin \frac{1}{x}$ when $x \neq 0$, and $g(0) = 0$, then g is continuous on $\mathbb{R}$. When $x \neq 0$, this follows from Theorems 4.4 and 4.7, while at $a = 0$ we rely on the squeeze theorem:

$$x \neq 0 \implies -x \le x \sin \frac{1}{x} \le x$$



The ϵ-δ Definition of Continuity

The sequential definition of continuity uses limits *twice*. By stating each of these using the ϵ-definition of limit, we can reformulate continuity without mentioning sequences!

To motivate this, consider $f(x) = x^2$ at $a = 2$. By continuity, if (x_n) is a sequence with $\lim x_n = 2$, then $\lim f(x_n) = 4$. We restate each of these using the definition of limit:

(a) $(\lim x_n = 2)$ $\forall \delta > 0, \exists M$ such that $n > M \implies |x_n - 2| < \delta$

(b) $(\lim x_n^2 = 4)$ $\forall \epsilon > 0, \exists N$ such that $n > N \implies \left|x_n^2 - 4\right| < \epsilon$

Here is a short argument that shows how (a) ⇒ (b) (we'll revisit this formally in a moment).

Assume (a) and suppose $\epsilon > 0$ is given. *Define* $\delta = \min(1, \frac{\epsilon}{5})$. Since $\lim x_n = 2$, $\exists M$ such that

$$\begin{aligned} n > M \implies \left|x_n^2 - 4\right| &= |x_n - 2|\,|x_n + 2| < \delta\,|(x_n - 2) + 4| && \text{(by (a))} \\ &\le \delta\big(|x_n - 2| + 4\big) && \text{($\triangle$-inequality)} \\ &< \delta(\delta + 4) \le 5\delta \le \epsilon && \text{((a) again)} \end{aligned}$$

Let $N = M$ to conclude (b).

It turns out not to be very important that (x_n) be a *sequence*. In fact we can dispense with it entirely...

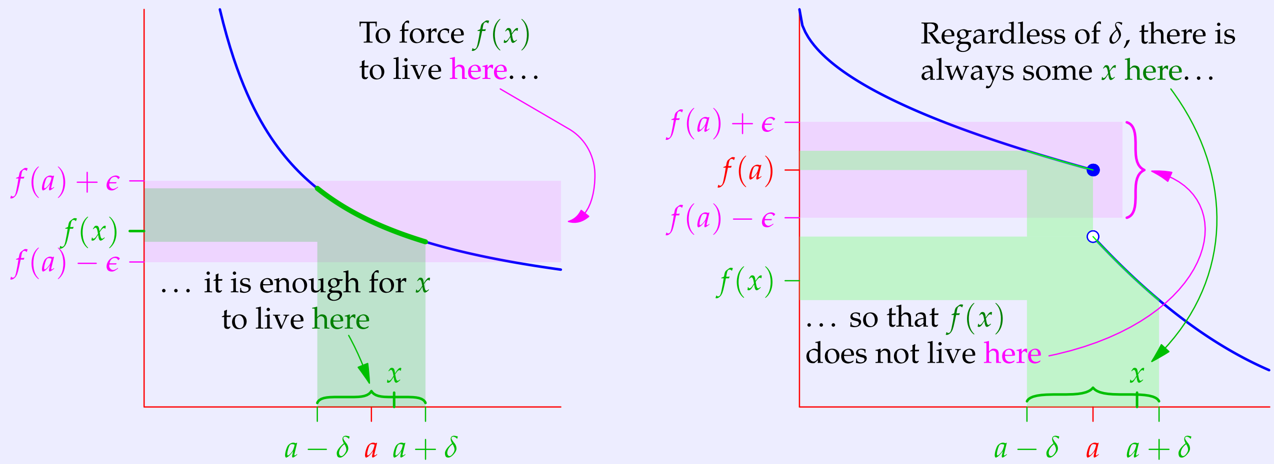
[27] For instance via Maclaurin series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ and $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$

Definition 4.9 (ϵ-δ continuity). A real-valued function $f : U \to V$ is continuous at $a \in U$ if[28]

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x \in U)\ |x-a| < \delta \implies |f(x) - f(a)| < \epsilon \tag{$*$}$$

We say that f is *discontinuous* at $a \in U$ if,

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x \in U \text{ with } |x-a| < \delta \text{ and } |f(x) - f(a)| \geq \epsilon \tag{$\dagger$}$$



Continuity at a — Discontinuous at a

This fits with the intuitive interpretation of continuity: if x is close to a, then $f(x)$ is close to $f(a)$; ϵ and δ are our measures of *closeness*. Many mathematicians consider the ϵ-δ version to be *the* definition of continuity. Thankfully, it doesn't matter which you prefer...

Theorem 4.10. *The sequential and ϵ-δ definitions of continuity (4.2 & 4.9) are equivalent.*

Examples (4.3, cont). Before seeing a proof, we repeat our earlier examples using the ϵ-δ definition. As with ϵ-N arguments for limits, it is often useful to do some scratch work first.

1. Suppose $f(x) = x^2$ and $a \in \mathbb{R}$. Our goal is to control the size of $\left|x^2 - a^2\right|$ whenever $|x-a|$ is small. To keep things simple, assume $|x-a| < 1$, then,

$$\begin{aligned}\left|x^2 - a^2\right| &= |x-a|\,|x+a| = |x-a|\,\big|(x-a)+2a\big| \\ &\overset{\triangle}{\leq} |x-a|\,\big(|x-a| + 2\,|a|\big) = |x-a|\,\big(1+2\,|a|\big)\end{aligned}$$

   Now let $\epsilon > 0$ be given and define $\delta = \min(1, \frac{\epsilon}{1+2|a|})$. Then

$$|x-a| < \delta \implies |f(x) - f(a)| = \left|x^2 - a^2\right| < \delta\big(1 + 2\,|a|\big) \leq \epsilon$$

   Thus f is continuous at a. This is simply a general version of the argument on page 63 with all mention of sequences removed!

[28] The bracketed $\forall x \in U$ is often omitted in ($*$) since the implication requires that x be universally quantified. It is important that $x \in U = \operatorname{dom} f$ rather than merely $x \in \mathbb{R}$! By contrast, the expression $\exists x \in U$ in ($\dagger$) is *always* written.

2. Let $g(x) = 1 + \frac{4}{x^2}$ and $a \neq 0$. The first challenge is to keep away from zero so that $\frac{1}{x}$ behaves. To do this, we insist that $\delta \leq \frac{|a|}{2}$, so that

$$|x-a| < \delta \implies \frac{|a|}{2} < |x| < \frac{3|a|}{2} \implies \frac{1}{|x|} < \frac{2}{|a|} \tag{$*$}$$

Now consider the required difference. If $|x-a| < \delta$, then

$$\begin{aligned}|g(x)-g(a)| &= \left|1+\frac{4}{x^2}-1-\frac{4}{a^2}\right| = \frac{4\left|a^2-x^2\right|}{a^2x^2} = \frac{4|a+x|}{a^2x^2}|x-a| < \frac{4|a+x|}{a^2x^2}\delta \\ &\overset{\triangle}{\leq} 4\left(\frac{1}{|a|x^2}+\frac{1}{a^2|x|}\right)\delta \overset{(*)}{<} 4\left(\frac{4}{|a|^3}+\frac{2}{|a|^3}\right)\delta = \frac{24}{|a|^3}\delta\end{aligned}$$

Given $\epsilon > 0$, it suffices to let $\delta = \min(\frac{1}{2}|a|, \frac{1}{24}|a|^3\epsilon)$. Then $|x-a| < \delta \implies |g(x)-g(a)| < \epsilon$.

3. For $h(x) = 3x^{1/4}$ there are two cases. Suppose $\epsilon > 0$ is given.

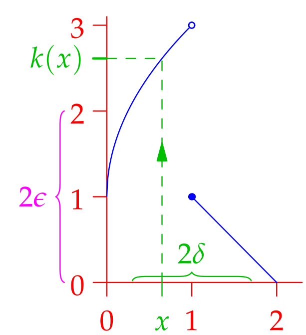
   - If $a = 0$, let $\delta = \left(\frac{\epsilon}{3}\right)^4$, then[29]

   $$|x-a| < \delta \implies 0 \leq x < \delta \implies |h(x)-h(a)| = 3x^{1/4} < 3\delta^{1/4} = \epsilon$$

   - If $a > 0$, let $\delta = \frac{1}{3}a^{3/4}\epsilon$. Then, if $|x-a| < \delta$,

   $$|h(x)-h(a)| = 3\left|x^{1/4}-a^{1/4}\right| = \frac{3|x-a|}{x^{\frac{3}{4}}+a^{\frac{1}{4}}x^{\frac{2}{4}}+a^{\frac{2}{4}}x^{\frac{1}{4}}+a^{\frac{3}{4}}} \leq \frac{3|x-a|}{a^{3/4}} < \frac{3\delta}{a^{3/4}} = \epsilon$$

4. We could establish the discontinuity statement (†) directly, but it is typically easier to argue by contradiction.
Suppose k is continuous at 1 and let $\epsilon = 1$. Then $\exists\delta > 0$ for which



$$\begin{aligned}|x-1| < \delta &\implies |k(x)-k(1)| = |k(x)-1| < 1 \\ &\implies 0 < k(x) < 2\end{aligned}$$

However, $x = \max(\frac{1}{4}, 1-\frac{\delta}{2})$ satisfies $|x-1| \leq \frac{\delta}{2} < \delta$ and $k(x) \geq k(\frac{1}{4}) = 1 + \frac{2}{2} = 2$. Contradiction. Think this last bit through!

The basic rules for combining continuous functions may also be proved using ϵ-δ arguments. E.g.,

ϵ-δ proof of the squeeze theorem. Given $\epsilon > 0$, we know there exist $\delta_1, \delta_2 > 0$ for which

$$|x-a| < \delta_1 \implies |f(x)-f(a)| < \epsilon \quad \text{and} \quad |x-a| < \delta_2 \implies |h(x)-h(a)| < \epsilon$$

Let $\delta = \min(\delta_1, \delta_2)$, then

$$|x-a| < \delta \implies |g(x)-g(a)| \leq \max\big(|f(x)-f(a)|, |h(x)-h(a)|\big) < \epsilon$$

whence g is continuous at 0. ∎

[29]Remember the hidden quantifier: $|x-a| < \delta$ *for all* $x \in \operatorname{dom} f = [0,\infty)$, thus $x \geq 0$ for the duration of this example.

Several other arguments are in the exercises. Finally, here is the promised proof of equivalence.

Proof of Theorem 4.10. (sequential $\Rightarrow \epsilon$–δ) We prove the contrapositive. Suppose a is an ϵ-δ discontinuity (†) and let $\delta = \frac{1}{n}$. Then there exists $x_n \in U$ such that

$$|x_n - a| < \frac{1}{n} \quad \text{and} \quad |f(x_n) - f(a)| \geq \epsilon$$

Repeating for all $n \in \mathbb{N}$ plainly produces a sequence (x_n) for which $\lim x_n = a$ and $\lim f(x_n) \neq f(a)$: otherwise said, a is a sequential discontinuity.

(ϵ-$\delta \Rightarrow$ sequential) Assume $(*)$, let $(x_n) \subseteq U$ and suppose $\lim x_n = a$; we must prove that $\lim f(x_n) = f(a)$. Let $\epsilon > 0$ be given so that a suitable δ satisfying $(*)$ exists. Since $\lim x_n = a$,

$$\begin{aligned} \exists N \text{ such that } n > N &\implies |x_n - a| < \delta && \text{(since } x_n \to a \text{ and } \delta > 0 \text{ is given)} \\ &\implies |f(x_n) - f(a)| < \epsilon && \text{(by } (*)) \end{aligned}$$

We conclude that $\lim f(x_n) = f(a)$, as required. ■

Examples 4.11. We finish with a couple of esoteric examples on the same theme.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be the *indicator function* for the rational numbers:

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Suppose f is continuous at a and let $\epsilon = 1$. Then $\exists\delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < 1 \tag{‡}$$

There are two cases; both rely on the fact that any interval contains both rational and irrational numbers (Corollary 1.23, etc.).

(a) If $a \in \mathbb{Q}$, then $f(a) = 1$. There exists an irrational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |0 - 1| = 1 \not< 1$.

(b) If $a \notin \mathbb{Q}$, then $f(a) = 0$. There exists a rational number $x \in (a - \delta, a + \delta)$, whence $|f(x) - f(a)| = |1 - 0| = 1 \not< 1$.

Either way, we have contradicted (‡). We conclude that f is *nowhere continuous.*

2. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Since $0 \leq |g(x)| \leq |x|$, the squeeze theorem tells us that g is continuous at $x = 0$.

Now suppose g is continuous at $a \neq 0$ and let $\epsilon = |a|$. Then $\exists\delta$ such that

$$|x - a| < \delta \implies |f(x) - f(a)| < |a|$$

The same two cases as in the previous example provide contradictions. We conclude that g *is continuous at precisely one point!*

Exercises 4.17. *Key concepts: Sequential and ϵ-δ continuity definitions/equivalence, ϵ-δ examples*

1. Consider the function with $f(x) = \frac{1}{\sqrt{x^2+2x-3}}$.
   (a) The implied domain of f has the form $\operatorname{dom} f = (-\infty, a) \cup (b, \infty)$. Find a and b.
   (b) What is the range of f?
   (c) Show that $f : (b, \infty) \to \operatorname{range} f$ is *bijective* and compute its inverse function.
   (d) Find the inverse function when we instead restrict the domain to $(-\infty, a)$.
   (e) Briefly explain why f is continuous on its domain.
2. Let f and g be continuous functions at a.
   (a) Show that $\max(f, g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ and deduce that $\max(f, g)$ is continuous at a.
   (b) How might you show continuity of $\min(f, g)$?
3. Use ϵ-δ arguments to prove the following.
   (a) $f(x) = x^2 - 3x$ is continuous at $x = 1$.
   (b) $g(x) = x^3$ is continuous at $x = a$.
   (c) $h : [0, \infty) \to \mathbb{R} : x \mapsto \sqrt{x}$ is continuous.
   (d) $j(x) = 3x^{-1}$ is continuous on $\mathbb{R} \setminus \{0\}$.
4. Rephrase Example 4.3.4's ϵ-δ argument by directly justifying the discontinuity definition (†).
5. Prove that each function is discontinuous at $x = 0$; use *both* sequential and ϵ-δ formulations.
   (a) $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.
   (b) $g(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $g(0) = 0$.
6. Suppose f and g are continuous at a. Prove the following using ϵ-δ arguments.
   (a) $f - g$ is continuous at a.
   (b) If h is continuous at $f(a)$, then $h \circ f$ is continuous at a.
7. Suppose $f : U \to V \subseteq \mathbb{R}$ is a function whose domain U contains an *isolated point* a: i.e. $\exists r > 0$ such that $(a - r, a + r) \cap U = \{a\}$. Prove that f is continuous at a.
8. In Example 4.11.2, provide the details of the required contradiction.
9. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function for which $f(x) = 0$ whenever $x \in \mathbb{Q}$. Prove that $f(x) = 0$ for all $x \in \mathbb{R}$.
   (b) Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions such that $f(x) = g(x)$ for all rational x. Prove that $f = g$.
10. (Hard) Consider $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{q} & \text{whenever } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q > 0 \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

    For instance, $f(1) = f(2) = f(-7) = 1$, and $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, etc.
    (a) Prove that f is discontinuous at each rational number r.
    (b) Prove that f is continuous at each irrational number i.
    (*Hint: given* $\epsilon > 0$, *let* $q = \lceil \frac{1}{\epsilon} \rceil$, $A = \{r \in \mathbb{Q} : f(r) \geq \frac{1}{q}\}$ *and let* $\delta = \min_{r \in A} |i - r| \ldots$)

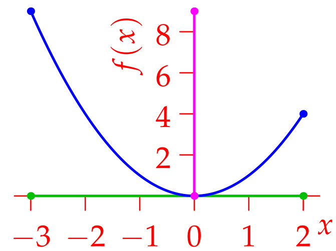
4.18 Properties of Continuous Functions

In this section we consider how continuous functions transform *intervals*.

Example 4.12. $f(x) = x^2$ maps $[-3, 2]$ onto $[0, 9]$. In particular:

- f transforms an interval into another.
- f transforms a closed bounded set into another.

Our goal is to see that these are general properties exhibited by *any* continuous function.



First recall a couple of definitions.

Definition 4.13. Suppose $f : U \to V$ where $U, V \subseteq \mathbb{R}$.

1. (a) U is *bounded* if $\exists M$ such that $\forall x \in U,\ |x| \le M$.

   (b) f is *bounded* if its range is a bounded set: $\exists M$ such that $\forall x \in U,\ |f(x)| \le M$.

2. (Definition 2.46) U is *closed* if every convergent sequence in U has its limit in U:

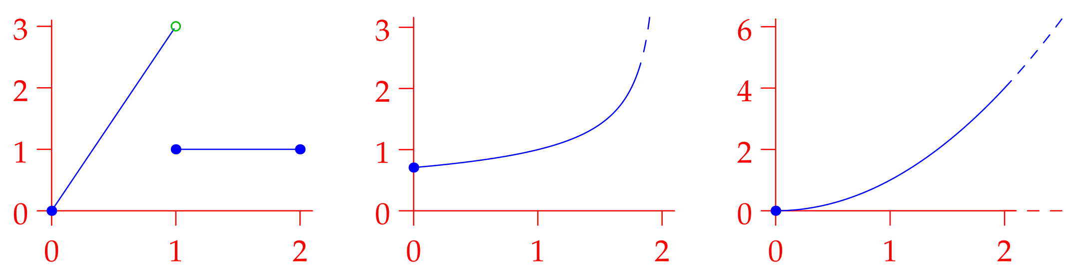
$$\forall (x_n) \subseteq U,\ \lim x_n = s\ (\in \mathbb{R}) \implies s \in U$$

Theorem 4.14 (Extreme Value Theorem). *Suppose $f : U \to V$ is continuous where U is closed and bounded. Then $f(U)$ is closed and bounded. In particular, f is bounded and attains its bounds:*

$$\exists s, i \in U \quad \textit{such that} \quad f(s) = \sup f(U) \quad \textit{and} \quad f(i) = \inf f(U)$$

Examples 4.15. 1. (Example 4.12) If $f(x) = x^2$ on $U = [-3, 2]$, then $f(U) = [0, 9]$ is closed and bounded. Moreover, $\sup f(U) = f(-3)$ and $\inf f(U) = f(0)$ (i.e., $s = -3$ and $i = 0$).

2. Before seeing the proof, here are three examples where we weaken one of the hypotheses of the extreme value theorem and see that the conclusion fails.



(a) f discontinuous (b) U not closed (c) U not bounded

(a) If $U = [0, 2]$, $f(x) = 3x$ when $x < 1$ and $f(x) = 1$ when $x \ge 1$, then $f(U) = [0, 3)$. In particular, $\sup f(U) = 3$ is not attained.

(b) If $f(x) = \frac{1}{\sqrt{2-x}}$ and $U = [0, 2)$, then $f(U) = [\frac{1}{\sqrt{2}}, \infty)$ is unbounded.

(c) If $f(x) = x^2$ and $U = [0, \infty)$, then $f(U) = [0, \infty)$ is unbounded.

The strategy of the proof is to show that every limit point of $f(U) = \operatorname{range} f$ lies in $f(U)$. We break things into simple steps; observe where each **hypothesis** is used.

Proof. 1. Suppose M is a limit point of $f(U)$: that is, $M = \lim f(x_n)$ for some sequence $(x_n) \subseteq U$. *A priori*, M need not be finite, but $M = \sup f(U)$ or $\inf f(U)$ are certainly possible.[30]

2. Since $(x_n) \subseteq U$ is **bounded**, Bolzano–Weierstraß (Theorem 2.41) says it has a convergent subsequence, $\lim\limits_{k\to\infty} x_{n_k} = x$.

3. Since U is **closed**, we have $x \in U$. This means $f(x)$ can be evaluated (it is *finite*).

4. Since f is **continuous**, $\lim f(x_{n_k}) = f(x)$.

5. Finally, $M = f(x)$ since all subsequences of a convergent (or divergent to $\pm\infty$) sequence tend to the same limit (Lemma 2.37). It follows that all limit points M are *finite* and lie in $f(U)$: otherwise said, $f(U)$ is closed and bounded.

Choosing $M = \sup f(U)$ yields $x = s \in U$ (similarly $\inf f(U)$ leads to $i \in U$). ∎

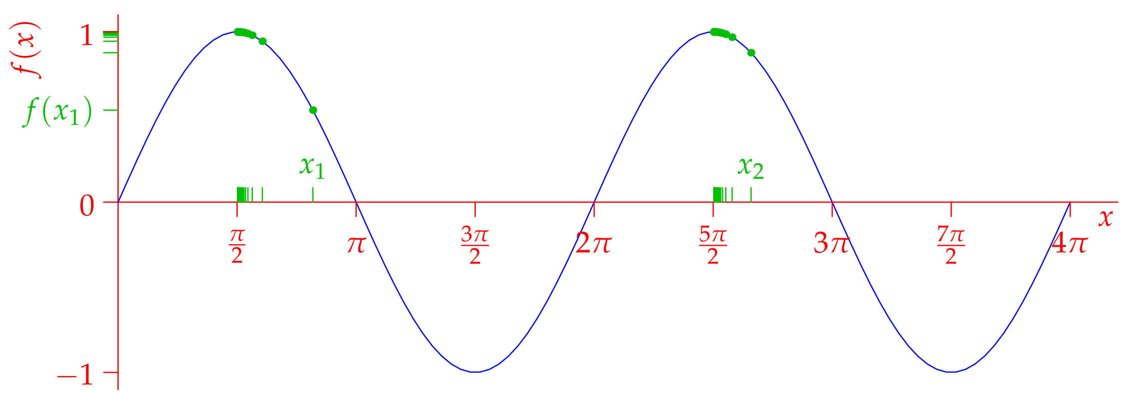
Example 4.16. It is worth considering why we needed a *subsequence* in the proof. The reason is that the bounds of f might be attained multiple times. For example, suppose

$$f : [0, 4\pi] \to \mathbb{R} : x \mapsto \sin x$$

This satisfies the hypotheses of the extreme value theorem: $U = [0, 4\pi]$ is closed and bounded and f is continuous. Indeed $\max f(U) = 1$ is attained at *both* $x = \frac{\pi}{2}$ and $\frac{5\pi}{2}$. The sequence defined by

$$x_n = \begin{cases} \frac{\pi}{2} + \frac{1}{n} & \text{if } n \text{ is odd} \\ \frac{5\pi}{2} + \frac{1}{n} & \text{if } n \text{ is even} \end{cases} \quad \text{has} \quad f(x_n) = \sin\left(\frac{\pi}{2} + \frac{1}{n}\right) \xrightarrow[n\to\infty]{} 1 = \sup f(U)$$

and therefore satisfies step 1 of the proof. However, (x_n) itself is *divergent by oscillation*. Bolzano–Weierstraß is used to force the existence of a convergent subsequence; in this case the subsequence of odd terms $(x_{n_k}) = (x_{2k-1})$ satisfies the remaining steps.



[30]If $M = \sup f(U)$, then a suitable (x_n) might be constructed as follows:

- If $M \in \mathbb{R}$, then for each $n \in \mathbb{N}$, $\exists x_n \in U$ such that $M - \frac{1}{n} < f(x_n) \le M$ (Lemma 1.20).
- If $M = \infty$, then for each $n \in \mathbb{N}$, $\exists x_n \in U$ such that $f(x_n) \ge n$.

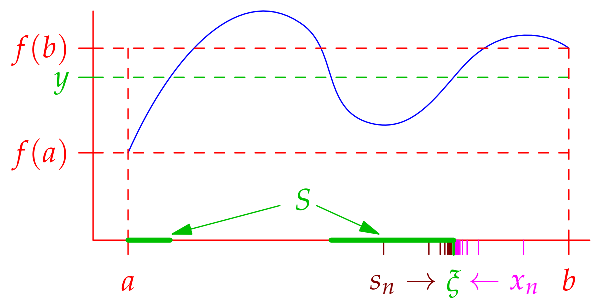
The Intermediate Value Theorem and its Consequences

This result should be familiar from elementary calculus, even if its proof is not. It should also be intuitive: like the Grand Old Duke of York, if you march up a hill, then at some point you must be half-way up…

Theorem 4.17 (Intermediate Value Theorem (IVT)). *Suppose f is continuous on $[a,b]$ and that y lies strictly between $f(a)$ and $f(b)$. Then $\exists \xi \in (a,b)$ such that $f(\xi) = y$.*

Being an existence result, it should be no surprise that *completeness* is used in the proof.

Proof. WLOG assume $f(a) < y < f(b)$. Let $S = \{x \in [a,b] : f(x) < y\}$ and *define* $\xi := \sup S$.



Since S is non-empty ($a \in S$) and bounded above (by b), we see that ξ exists and is finite. It remains to prove that $f(\xi) = y$ and $\xi \neq a, b$.

First choose any $(s_n) \subseteq S$ such that $\lim s_n = \xi$. Continuity forces $\lim f(s_n) = f(\xi)$. Moreover

$$f(s_n) < y \implies f(\xi) \leq y$$

Since $f(b) > y$, this also shows that $\xi \neq b$.

We now play a similar game from the other side: define $x_n := \min(\xi + \frac{1}{n}, b)$, then $\lim x_n = \xi$ and

$$\begin{aligned} x_n > \xi = \sup S &\implies x_n \notin S \implies f(x_n) \geq y \\ &\implies f(\xi) = \lim f(x_n) \geq y \end{aligned}$$

again via the continuity of f and the convergence properties of bounded sequences. Since $y > f(a)$, we also conclude that $\xi \neq a$.

Putting it all together, $f(\xi) = y$ and $\xi \in (a,b)$. ∎

Note how the value of ξ in the proof is always the *largest* of potentially several choices.

Examples 4.18. In elementary calculus, the intermediate value theorem is typically applied to demonstrate the existence of solutions to equations.

1. We show that the equation $x^7 + 3x = 1 + 4\cos(\pi x)$ has a solution.

   The trick is to express the equation in the form $f(x) = y$ where f is continuous, then choose suitable a, b to fit the theorem. In this case,

   $$f(x) = x^7 + 3x - 4\cos(\pi x) \quad \text{and} \quad y = 1$$

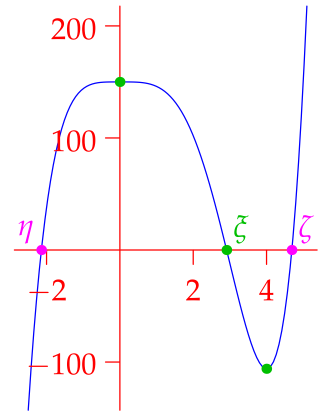
   are suitable choices. Now observe

   $$f(0) = -4 < y \quad \text{and} \quad f(1) = 1 + 3 + 4 = 8 > y \qquad \text{(i.e., } a = 0 \text{ and } b = 1\text{)}$$

   whence $\exists \xi \in (0,1)$ such that $f(\xi) = y = 1$. Otherwise said, ξ is a solution to the original equation.

   The function f is plainly continuous on $\mathbb{R}$, a much larger interval than $[a,b]$, but no matter.

2. The existence of a root ξ of the (continuous) polynomial



$$f(x) = x^5 - 5x^4 + 150$$

follows from the intermediate value theorem by observing that

$$f(0) = 150 > 0 \quad \text{and} \quad f(4) = -256 + 150 = -106 < 0$$

We conclude that such a root ξ exists satisfying $\xi \in (0, 4)$.
As the graph suggests, there are other roots (η, ζ), the existence of which may be shown by observing, say,

$$f(-3) = -798 < 0 \quad \text{and} \quad f(5) = 150 > 0$$

With an eye on generalizing, here is an alternative approach. Define sequences (s_n), (t_n) via

$$s_n := \frac{f(-n)}{n^5} = -1 - \frac{5}{n} + \frac{150}{n^5} \qquad t_n := \frac{f(n)}{n^5} = 1 - \frac{5}{n} + \frac{150}{n^5}$$

Since $\lim s_n = -1$ and $\lim t_n = 1$, we see that

$$\exists a \text{ such that } s_a < -\frac{1}{2} \implies f(-a) = a^5 s_a < -\frac{1}{2}a^5 < 0$$
$$\exists b \text{ such that } t_b > \frac{1}{2} \implies f(b) = b^5 t_b > \frac{1}{2}b^5 > 0$$

Applying the intermediate value theorem on $[-a, b]$ shows the existence of a root.

The second approach in Example 4.16.2 may be applied to prove a general result.

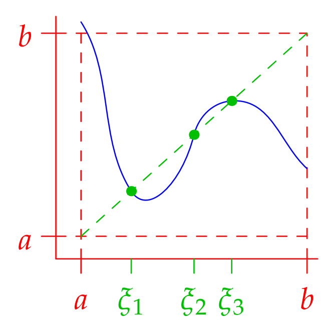
Corollary 4.19. *A polynomial function of odd degree has at least one real root.*

The proof is an exercise. An even simpler exercise shows the existence of a *fixed point* for a particular type of continuous function.

Corollary 4.20 (Fixed Point Theorem). *Suppose a and b are finite and that $f : [a, b] \to [a, b]$ is continuous. Then f has a fixed point:*

$$\exists \xi \in [a, b] \text{ such that } f(\xi) = \xi$$

As the picture shows, a function could have several fixed points.



This is the most basic fixed-point theorem in analysis: if you continue your studies you'll meet several more. Many important consequences flow from such results, including a common fractal construction and the standard existence/uniqueness result for differential equations.

For a final corollary, first note a straightforward characterization that helps us consider all types of interval simultaneously: $U \subseteq \mathbb{R}$ is an interval precisely when

$$a, b \in U \text{ and } a < y < b \implies y \in U \qquad (*)$$

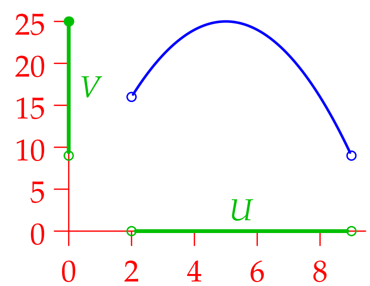
Corollary 4.21 (Preservation of Intervals). *Suppose U is an interval of positive length, and that $f : U \to V$ is continuous and surjective ($V = f(U)$).*

1. *V is an interval or a point.*
2. *If f is strictly increasing (decreasing), then:*
   (a) *V is an interval of positive length, f is* injective*, and therefore bijective.*
   (b) *The inverse function $f^{-1} : V \to U$ is also continuous and strictly increasing (decreasing).*

Example 4.22. The interval V *need not be of the same type* as U. For instance, if $f(x) = 10x - x^2$, then f maps the *open* interval $U = (2, 9)$ to the *half-open* interval $V = (9, 25]$.

The extreme value theorem, however, guarantees that if U is closed and bounded, then V is also. For instance,

$$f([2, 9]) = [9, 25]$$



Proof. 1. If V is not a point, then $\exists a, b \in U$ such that $f(a) < f(b)$. If y lies between these, IVT says $\exists \xi$ between a and b such that $y = f(\xi)$. That is, $y \in f(U)$. By ($*$), $V = f(U)$ is an interval.

2. (a) If f is strictly increasing, then $\forall a, b \in U$, $a < b \implies f(a) < f(b)$. Plainly f is injective and V contains at least two points; by part 1 it is an interval of positive length.

   (b) Let $y_1 < y_2$ where both lie in V, and define $x_i = f^{-1}(y_i)$ for $i = 1, 2$. Since f is increasing,

$$x_2 \le x_1 \implies y_2 = f(x_2) \le f(x_1) = y_1$$

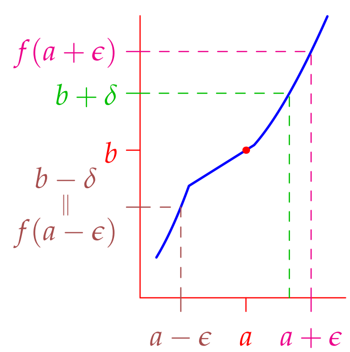
is a contradiction. Thus $x_1 < x_2$ and f^{-1} is also strictly increasing.

If $a \in U$, it remains to show that f^{-1} is continuous at $b = f(a)$. Assume first that a is not an endpoint of U and let $\epsilon > 0$ be given such that $[a - \epsilon, a + \epsilon] \subseteq U$,. Now define

$$\delta := \min\big(b - f(a - \epsilon), f(a + \epsilon) - b\big)$$

This is *positive* since f is strictly increasing. But now

$$\begin{aligned} |y - b| < \delta &\implies f(a - \epsilon) - b < y - b < f(a + \epsilon) - b \\ &\implies f(a - \epsilon) < y < f(a + \epsilon) \\ &\implies a - \epsilon < f^{-1}(y) < a + \epsilon \\ &\implies \left| f^{-1}(y) - f^{-1}(b) \right| = \left| f^{-1}(y) - a \right| < \epsilon \end{aligned}$$

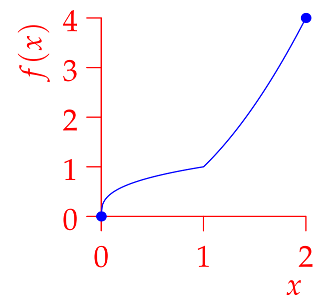


where ($\implies$) used the fact that f is strictly increasing.

If a is an endpoint of U, instead use $[a - \epsilon, a] \subseteq U$ or $[a, a + \epsilon] \subseteq U$ and only the corresponding half of the expression defining δ.

■

Example 4.23. The function $f : [0,2] \to [0,4]$ defined by



$$f(x) = \begin{cases} \sqrt[3]{x} & \text{if } 0 \le x \le 1 \\ x^2 & \text{if } 1 < x \le 2 \end{cases}$$

is continuous, surjective and strictly increasing. It therefore has a continuous inverse $f^{-1} : [0,4] \to [0,2]$. Compare this with the familiar statement from elementary calculus: $f' > 0 \implies f$ injective. We cannot apply this here since f is not differentiable!

Exercises 4.18. *Key concepts: Extreme/Intermediate Value Theorems, Cont functions preserve intervals*

1. (a) Give an example of a *discontinuous* function $f : [0,1] \to \mathbb{R}$ which is *not bounded.*
   (b) State a *continuous* function with domain $(1,\infty)$ whose range is *bounded* but *not closed.*

2. Let $a < b$ be given. Give examples of *continuous* functions $g, h : (a,b) \to \mathbb{R}$ such that:
   (a) g is *not bounded.*
   (b) h is bounded but *does not attain its bounds.*

3. Compute the inverse of the function f in Example 4.23.

4. Let $S \subseteq \mathbb{R}$ and suppose there exists a sequence (x_n) in S converging to some $x_0 \notin S$. Show that there exists an unbounded continuous function on S.

5. Prove that $x = \cos x$ for some $x \in (0, \frac{\pi}{2})$.

6. Suppose that f is a real-valued continuous function on $\mathbb{R}$ and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove that there exists some x between a, b such that $f(x) = 0$.

7. Suppose f is continuous on $[0,2]$ and that $f(0) = f(2)$. Prove that there exist $x, y \in [0,2]$ such that $|y - x| = 1$ and $f(x) = f(y)$.
   (*Hint: consider* $g(x) = f(x+1) - f(x)$ *on* $[0,1]$)

8. (a) Prove the fixed point theorem (Corollary 4.20).
   (*Hint: If neither a nor b are fixed points, consider* $g(x) = f(x) - x$)
   (b) Prove Corollary 4.19 for a general odd-degree monic polynomial $f(x) = x^{2m+1} + \sum_{k=0}^{2m} \alpha_k x^k$.

9. Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x \sin \frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$.
   (a) Explain why f is continuous on any interval U.
   (b) Suppose $a < 0 < b$ and that $f(a), f(b)$ have opposite signs. If $y = 0$, show that the intermediate value theorem is satisfied by *infinitely many* distinct values ξ.

10. (a) Suppose $f : U \to \mathbb{R}$ is continuous and that $U = \bigcup_{k=1}^{n} I_k$ is the union of a finite sequence (I_k) of closed bounded intervals. Prove that f is bounded and attains its bounds.
    (b) Let $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for each $n \in \mathbb{N}$. Give an example of a continuous function $f : U \to \mathbb{R}$ which is either unbounded or does not attain its bounds. Explain.
    (*This relates to the idea that* ***finite*** *unions of closed sets are closed, but* ***infinite*** *unions need not be*)

4.19 Uniform Continuity

Suppose $f: U \to V$ is continuous. By the ϵ-δ definition (4.9),

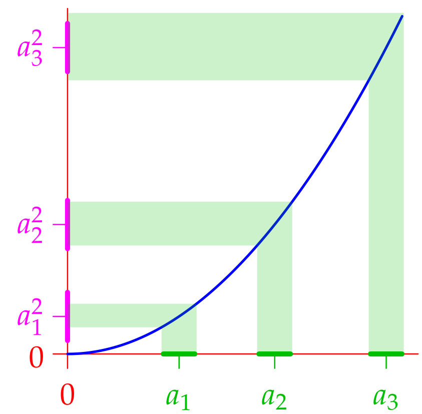
$$\forall a \in U,\ \forall \epsilon > 0,\ \exists \delta(a,\epsilon) > 0 \text{ such that } (\forall x \in U)\ |x-a| < \delta \implies |f(x) - f(a)| < \epsilon \qquad (*)$$

We write $\delta(a,\epsilon)$ to stress that δ can depend both on the *location* a and the *distance* ϵ. The goal of this section is to understand if/when it is possible to choose δ *independently of the location* a.

Example 4.24. We start with an example where our desire cannot be satisfied.

Consider $f(x) = x^2$ with domain $U = [0,\infty)$. Since f is continuous, given $\epsilon > 0$ and $a_1 \in U$, there exists δ such that

$$|x - a_1| < \delta \implies |f(x) - f(a_1)| = |x^2 - a_1^2| < \epsilon$$



On page 64 we saw that $\delta = \min(1, \frac{\epsilon}{1+2a_1})$ was suitable, but this *depends on the location* a_1. Of course other expressions for δ will also work...
Visualize what happens if we attempt to use the *same* ***constant*** δ for different a_i: imagine sliding the fixed-width δ-interval along the x-axis while simultaneously sliding the ϵ-interval vertically. As a_i increases, the image of the δ-interval eventually becomes too large for the ϵ-interval to contain: if δ is constant, then

$$\text{length}\big(f(a_i - \delta, a_i + \delta)\big) = (a_i + \delta)^2 - (a_i - \delta)^2 = 4a_i\delta$$

increases unboundedly with a_i. For fixed ϵ, as a increases, the *increasing gradient* of f means that we need to choose a *smaller* δ.

By contrast, if $f(x) = x^2$ on a *finite* domain $[0,b]$, then any δ that demonstrates continuity at $x = b$ will also do so everywhere else on $[0,b]$. We'll check this explicitly in a moment.

To obtain a formal definition, we rewrite $(*)$ with the extra assumption that δ may be chosen independently of the location a; this amounts to moving the quantifier $\forall a \in U$ after δ.

Definition 4.25. A function $f: U \to V \subseteq \mathbb{R}$ is *uniformly continuous* if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x, y \in U)\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon \qquad (\dagger)$$

We use y instead of a for symmetry. Observe how δ, being quantified *before* x, y, now depends only on ϵ. As before, the quantifiers for x, y are usually hidden. Note also how uniform continuity is only relevant on the entire domain U; it makes no sense to speak of uniform continuity at a single point.

For the sake of tidiness, we make one more observation before seeing some examples.

Lemma 4.26. *If f is uniformly continuous on U, then it is continuous on U.*

This is trivial: $(\dagger)$ is the ϵ-δ continuity of f at $y \in U$, *for all y simultaneously*! The special feature of the definition is that the same δ works for all y.

Examples 4.27. 1. We re-analyze $f(x) = x^2$ in view of the definition. Recall first that

$$|f(x) - f(y)| = \left|x^2 - y^2\right| = |x - y|\,|x + y|$$

where $|x - y|$ is easily controlled by δ. We consider the behavior of $|x + y|$ in two cases.

Bounded domain If $U = \operatorname{dom} f \subseteq [-T, T]$ for some $T > 0$, we show that f is uniformly continuous. This will follow because $|x + y| \le 2T$ is also easily controlled.
Let $\epsilon > 0$ be given and define $\delta = \frac{\epsilon}{2T}$, then

$$|x - y| < \delta \implies |f(x) - f(y)| < \delta \cdot 2T = \epsilon$$

Compare with Example 4.24. Our approach works for *this* function because the gradient (and therefore the discrepancy between $x^2 - y^2$ and $x - y$) is greatest at the endpoints of the interval. The same approach may not work for other functions!

Unbounded domain We show that f is not uniformly continuous when $\operatorname{dom} f = [0, \infty)$.
For contradiction, assume f is uniformly continuous; let $\epsilon = 1$ and suppose $\delta > 0$ satisfies the definition. Taking $x - y = \frac{\delta}{2}$, we see that

$$|x + y| = 2y + \frac{\delta}{2} \implies |f(x) - f(y)| = \frac{\delta}{2}\left(2y + \frac{\delta}{2}\right) = \delta\left(y + \frac{\delta}{4}\right) > \delta y$$

Let $y = \frac{1}{\delta}$ for the contradiction $|f(x) - f(y)| > 1 = \epsilon$ (large y are the problem!).

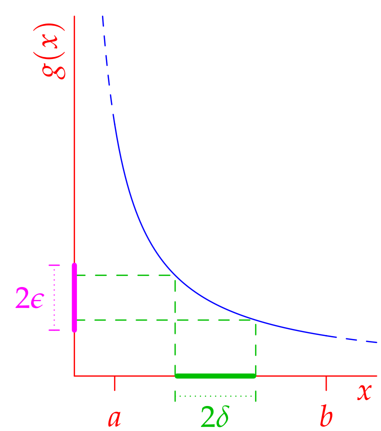
2. Let $g(x) = \frac{1}{x}$; we again consider two domains.

Uniform continuity on $[a, b)$ whenever $0 < a < b \le \infty$.

Let $\epsilon > 0$ be given and let $\delta = a^2\epsilon$. Then,

$$|x - y| < \delta \implies |g(x) - g(y)| = \left|\frac{y - x}{xy}\right| < \frac{\delta}{xy} \le \frac{\delta}{a^2} = \epsilon$$

where the last inequality follows because $x, y \ge a$.



Non-uniform continuity on $(0, b)$ whenever $0 < b \le \infty$.

As before, let $\epsilon = 1$ and suppose $\delta > 0$ is given. Let

$$x = \min\left(\delta, 1, \frac{b}{2}\right) \quad \text{and} \quad y = \frac{x}{2}$$

Certainly $x, y \in (0, b)$ and $|x - y| = \frac{x}{2} \le \frac{\delta}{2} < \delta$. However,

$$|f(x) - f(y)| = \frac{1}{x} \ge 1 = \epsilon$$

Think about how ϵ and δ must relate as one slides the intervals in the picture up/down and left/right. In this case, large values of x, y are not the problem, it's the vertical asymptote at zero that causes trouble.

General Conditions for Uniform Continuity

For the remainder of this section, we develop a few general ideas related to uniform continuity. The first is a little out of order since it depends on differentiation and the mean value theorem.

Theorem 4.28. *Suppose f is continuous on an interval U (finite or infinite) and differentiable except perhaps at its endpoints. If f' is bounded, then f is uniformly continuous on U.*

Proof. Suppose $|f'(x)| \le M$. Let $\epsilon > 0$ and $\delta = \frac{\epsilon}{M}$. Then

$$|x-y| < \delta \implies |f(x)-f(y)| = \left|f'(\xi)\right| |x-y| < M\delta = \epsilon$$

for some ξ between x and y. The existence of ξ follows from the mean value theorem.[31] ■

Examples 4.29. 1. Compare the arguments in the previous exercise. For instance, if $\operatorname{dom} f \subseteq [-T,T]$,

$$f(x) = x^2 \implies f'(x) = 2x \implies \left|f'(x)\right| \le 2T$$

The derivative is bounded, whence f is uniformly continuous on $[-T,T]$.

2. Any polynomial is uniformly continuous on any bounded interval.
3. The function $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$ since $f'(x) = \cos x$ is bounded (by 1).
4. Consider $f(x) = \frac{1}{x} - \frac{5}{x^2}$ on $(1,\infty)$. We have

$$f'(x) = -\frac{1}{x^2} + \frac{10}{x^3} \implies \left|f'(x)\right| \le 11$$

We conclude that f is uniformly continuous on $(1,\infty)$.

The approach is often useful when you are asked to show *using the definition* that a function is uniformly continuous; provided f' is bounded by M, you may always choose $\delta = \frac{\epsilon}{M}$ to obtain an argument. For instance, with our function:

Given $\epsilon > 0$, let $\delta = \frac{\epsilon}{11}$. If $x,y \in (1,\infty)$ and $|x-y| < \delta$, then

$$\begin{aligned}
|f(x)-f(y)| &= \left|\frac{1}{x} - \frac{1}{y} + \frac{5}{y^2} - \frac{5}{x^2}\right| = |x-y| \left|\frac{5(x+y)}{x^2y^2} - \frac{1}{xy}\right| \\
&= |x-y| \left|\frac{5}{xy^2} + \frac{5}{x^2y} - \frac{1}{xy}\right| \\
&< 11\,|x-y| \qquad \text{($\triangle$-inequality, since $x,y > 1$)} \\
&< 11\delta = \epsilon
\end{aligned}$$

As we'll see very shortly, the above result isn't a biconditional: non-differentiable functions and functions with unbounded derivatives can be uniformly continuous.

[31] If $x < y$ then $\exists \xi \in (x,y)$ such that $f'(\xi) = \dfrac{f(x)-f(y)}{x-y}$.

Our remaining conditions are variations on a theme: uniform continuity on a bounded interval U is roughly the same thing as continuity on its *closure* $\overline{U}$ (Definition 2.46).

Theorem 4.30. *A continuous function on a closed bounded domain is uniformly continuous.*

Proof. Assume f is continuous, but not uniformly so, on a closed bounded domain U. Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x, y \in U \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon \qquad (*)$$

Let $\delta = \frac{1}{n}$ for each $n \in \mathbb{N}$ to obtain sequences $(x_n), (y_n) \subseteq U$ satisfying $(*)$.[32]
Since $(x_n) \subseteq U$ is bounded, Bolzano–Weierstraß says there exists a convergent subsequence (x_{n_k}) which, since U is closed, converges to some $x_0 \in U$.
Since $|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \leq \frac{1}{k}$, we see that $\lim_{k\to\infty} y_{n_k} = x_0$. Finally, the continuity of f contradicts $(*)$:

$$\epsilon \leq \lim |f(x_{n_k}) - f(y_{n_k})| = |f(x_0) - f(x_0)| = 0$$

■

Both hypotheses on the domain are crucial: Examples 4.27 provide counter-examples if either is weakened.

Example 4.31. $f(x) = \sqrt{x}$ is uniformly continuous on $[0, 1]$ since it is already continuous! This **cannot** be concluded from Theorem 4.28, since the derivative $f'(x) = \frac{1}{2}x^{-1/2}$ is unbounded on $(0, 1)$.

We now develop a partial converse, for which we first need a lemma.

Lemma 4.32. *If f is uniformly continuous on U and $(x_n) \subseteq U$ is Cauchy, then $\big(f(x_n)\big)$ is also Cauchy.*

To apply the result, consider a convergent (Cauchy) sequence in U whose limit is *not* itself in U.

Example (4.27.2, just easier!). Let $f(x) = \frac{1}{x}$ have $U = \operatorname{dom} f = (0, \infty)$ and consider the Cauchy sequence defined by $x_n = \frac{1}{n}$; note crucially that its limit 0 *does not lie in U*. Moreover,

$$\lim f(x_n) = \lim n = \infty$$

Plainly $\big(f(x_n)\big)$ is not Cauchy, whence f is not uniformly continuous.

Proof. Let $\epsilon > 0$ be given. Since f is uniformly continuous,

$$\exists \delta > 0 \text{ such that } \forall x, y \in U,\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Now use this δ in the definition of (x_n) being Cauchy:

$$\exists N \text{ such that } m, n > N \implies |x_n - x_m| < \delta \implies |f(x_n) - f(x_m)| < \epsilon$$

Otherwise said, $\big(f(x_n)\big)$ is Cauchy.

■

The Cauchy condition is critical: we cannot apply uniform continuity directly to a convergent sequence ($|x_n - x| < \delta \ldots$) if we do not already know that its limit (x) lies in U!

[32] These arguments should feel familiar: compare this line to the proof of Theorem 4.10 and the rest to Theorem 4.14.

We apply the Lemma to show that a continuous function on a *bounded* interval is uniformly continuous if and only if has a *continuous extension.*

Theorem 4.33. *Suppose f is continuous on a bounded interval (a,b). Define $g:[a,b]\to\mathbb{R}$ via*

$$g(x) := \begin{cases} f(x) & \text{if } x\in(a,b) \\ \lim f(x_n) & \text{whenever } (x_n)\subseteq(a,b) \text{ and } \lim x_n = a \text{ or } b \end{cases}$$

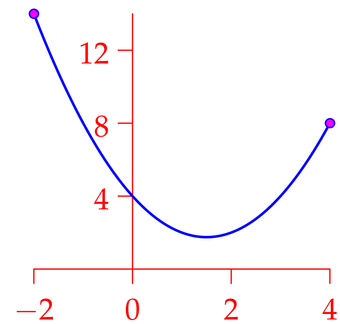
Then f is uniformly continuous if and only if g is well-defined; in such a case g is automatically continuous.

Examples 4.34.

1. $f(x) = x^2-3x+4$ is uniformly continuous on $(-2,4)$ since it has a continuous extension

$$g:[-2,4]\to\mathbb{R}:x\mapsto x^2-3x+4$$

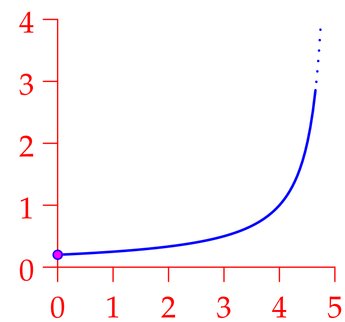
It should be obvious what is happening from the picture: to create the extension g, we simply fill in the holes at the endpoints of the graph.



2. The function $f(x)=\frac{1}{5-x}$ is continuous, but not uniformly, on the interval $(0,5)$. This follows since

$$\lim f\left(5-\frac{1}{n}\right) = \lim n = \infty$$

means we cannot define $g(5)$ unambiguously. Again the picture is helpful; while we can fill in the hole at the left endpoint ($a=0$), the vertical asymptote at $b=5$ means that there is no hole to fill in and thereby extend the function.



Proof. ($\Leftarrow$) Suppose g is well-defined; we leave the claim that it is continuous as an exercise, but by Theorem 4.30 it is uniformly so. Since $f=g$ on a subset $(a,b)\subseteq \operatorname{dom} g$, the same choice of δ will work for f as it does for g: f is therefore uniformly continuous.

($\Rightarrow$) Suppose f is uniformly continuous on (a,b). Let $(x_n),(y_n)\subseteq(a,b)$ be sequences converging to a. To show that $g(a)$ is unambiguously defined, we must prove that $\big(f(x_n)\big)$ and $\big(f(y_n)\big)$ are convergent, and to the same limit.

Define a sequence

$$(u_n) = (x_1,y_1,x_2,y_2,x_3,y_3,\ldots)$$

Plainly $\lim u_n = a$ since (x_n) and (y_n) have the same limit. But then (u_n) is Cauchy; by Lemma 4.32, $\big(f(u_n)\big)$ is also Cauchy and thus convergent. Since $\big(f(x_n)\big)$ and $\big(f(y_n)\big)$ are subsequences of a convergent sequence, they also converge to the same (finite!) limit.

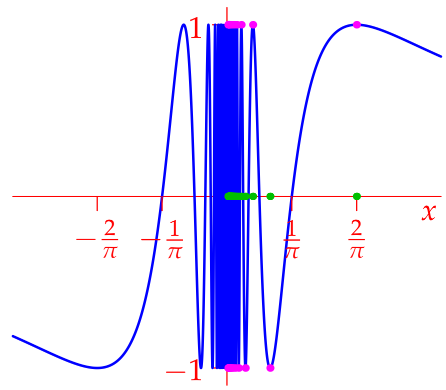
The case for $g(b)$ is similar. ∎

Examples 4.35. We finish with three related examples of continuous functions $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$; these will appear repeatedly as you continue to study analysis.

1. $f(x) = \sin \frac{1}{x}$ is continuous but *not uniformly so*. To see this, note that $x_n = \frac{1}{(n+\frac{1}{2})\pi}$ defines a Cauchy sequence ($\lim x_n = 0$), and yet

$$f(x_n) = \sin\left(n + \frac{1}{2}\right)\pi = (-1)^n$$

is not Cauchy (it diverges by oscillation). Consequently, we cannot extend f to a continuous function on any interval containing $x = 0$.



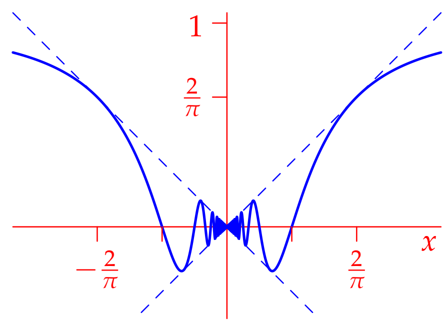
2. $f(x) = x \sin \frac{1}{x}$ is *uniformly* continuous. One way to see this is to extend the function to the origin by defining

$$g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

By the squeeze theorem, $\lim x_n = 0 \implies \lim f(x_n) = 0$, so g is well-defined and continuous on $\mathbb{R}$. By Theorem 4.33, f is uniformly continuous on any bounded interval. Moreover, the derivative

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

is bounded whenever x is large; together with Exercise 6 we conclude that $f(x)$ is uniformly continuous on $\mathbb{R} \setminus \{0\}$. We cannot use the derivative argument on the whole domain $\mathbb{R} \setminus \{0\}$, since $f'(x)$ is unbounded when x is small ($\lim f'\left(\frac{1}{2\pi n}\right) = \lim(-2\pi n) = -\infty$).

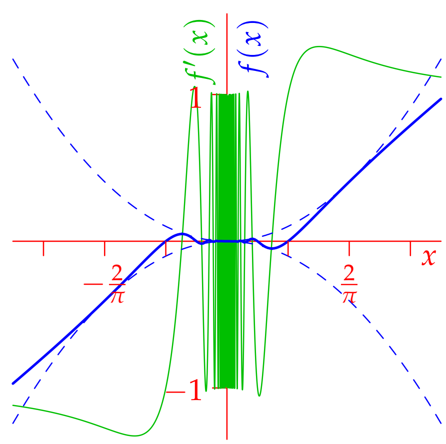


3. $f(x) = x^2 \sin \frac{1}{x}$ is also *uniformly* continuous: again extend by $g(0) = 0$. This time however, we could argue that the derivative is bounded

$$\left|f'(x)\right| = \left|2x \sin \frac{1}{x} - \cos \frac{1}{x}\right| \leq 3$$

since $|\sin y| \leq |y|$ and $|\cos y| \leq 1$ for all y.

Something stranger is going on. As you may verify (see Exercise 3), the extended function g is *everywhere differentiable* with $g'(0) = 0$, and yet the derivative $g'(x)$ itself is *discontinuous* at $x = 0$!



Exercises 4.19. *Key concepts: Order of quantifiers! Bounded derivative $\Rightarrow$ Unif cont $\Leftrightarrow$ Cont extension*

1. Decide whether each f is uniformly continuous. Explain your answers.

   (a) $f(x) = x^3$ on $[-2,4]$
   (b) $f(x) = x^3$ on $(-2,4)$
   (c) $f(x) = x^{-3}$ on $(0,4]$
   (d) $f(x) = x^{-3}$ on $(1,4]$
   (e) $f(x) = e^x$ on $(-\infty,100)$
   (f) $f(x) = e^x$ on $\mathbb{R}$

2. Prove that each f is uniformly continuous by verifying the ϵ-δ property.

   (a) $f(x) = 3x + 11$ on $\mathbb{R}$
   (b) $f(x) = x^2$ on $[0,3]$
   (c) $f(x) = \frac{1}{x^2}$ on $[\frac{1}{2},\infty)$
   (d) $f(x) = \frac{x+2}{x+1}$ on $[0,1]$

3. Verify the claim in Example 4.35.3 that the function $g(x)$ is differentiable at zero[33] but that the derivative $g'(x)$ is discontinuous there.

4. (a) If f is uniformly continuous on a bounded set U, prove that f is bounded on U. (*Hint: for contradiction, assume* $\exists(x_n) \subseteq U$ *for which* $|f(x_n)| \to \infty \ldots$)
   (b) Use (a) to give another proof that $\frac{1}{x^2}$ is not uniformly continuous on $(0,1)$.
   (c) Give an example to show that a uniformly continuous function on an *unbounded* set U could be unbounded.

5. Suppose g is defined on U and $a \in U$. Give *very brief* (one line!) arguments for the following.

   (a) Prove that g is continuous at a provided

   $$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |g(x) - g(a)| < \epsilon$$

   (b) Prove that g is continuous at a provided

   $$\forall (x_n) \subseteq U \setminus \{a\},\ \lim x_n = a \implies \lim g(x_n) = g(a)$$

   (c) Verify that the function g defined in Theorem 4.33 is indeed continuous whenever it is well-defined.

6. (a) Suppose f is uniformly continuous on intervals U_1, U_2 for which $U_1 \cap U_2$ is non-empty. Prove that f is uniformly continuous on $U_1 \cup U_2$. (*Hint: if x, y do not lie in the same interval U_i, choose some $a \in U_1 \cap U_2$ between x and y*)
   (b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0,\infty)$.
   (c) More generally, prove that any root function $f(x) = x^{1/n}$ ($n \in \mathbb{N}$) is uniformly continuous on its domain ($\mathbb{R}$ if n is odd and $[0,\infty)$ if n is even).
   (d) (Hard) Given $f(x) = x^{1/n}$, show that $\delta = \epsilon^n$ demonstrates uniform continuity when n is even and $\delta = \left(\frac{\epsilon}{2}\right)^n$ when n is odd. (*Hint: use the binomial theorem to prove that* $0 \le y < x + \delta \implies y^{1/n} < x^{1/n} + \delta^{1/n}$)

[33] Use the definition $g'(0) = \lim\limits_{x\to 0} \frac{g(x)-g(0)}{x-0}$. Limits of functions are covered formally in the next section (course!), but you should be familiar with the idea from elementary calculus.