Math 140B - Notes

Neil Donaldson

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1 Continuity

The primary goal of this course is to make elementary calculus rigorous. We begin with a review of some basic concepts and conventions.

Sets & Functions We are concerned with functions $f : U \to V$ where both U, V are subsets of the real numbers $\mathbb{R}$:

Domain $\text{dom}(f) = U$; the *inputs* to f. Often implied to be the largest set on which a formula is defined. In calculus examples, the domain is typically a union of intervals of *positive length.*

Codomain $\text{codom}(f) = V$. We often take $V = \mathbb{R}$ by default.

Range $\text{range}(f) = f(U) = \{f(x) : x \in U\}$; the *outputs* of f and a subset of V.

Injectivity f is *injective/one-to-one* if $f(x) = f(y) \implies x = y$.

Surjectivity f is *surjective/onto* if $f(U) = V$.

Inverses f is *bijective/invertible* if it is injective and surjective. Equivalently, $\exists f^{-1} : V \to U$ satisfying

$$\forall u \in U,\ f^{-1}\big(f(u)\big) = u \quad \text{and} \quad \forall v \in V,\ f\big(f^{-1}(v)\big) = v$$

Example 1.1. The function defined by $f(x) = \frac{1}{x(x-2)}$ has implied

$$\text{dom}(f) = \mathbb{R} \setminus \{0, 2\} = (-\infty, 0) \cup (0, 2) \cup (2, \infty)$$

$$\text{range}(f) = (-\infty, -1] \cup (0, \infty)$$

The function is neither injective nor surjective.
By restricting the domain/codomain, we obtain a bijection:

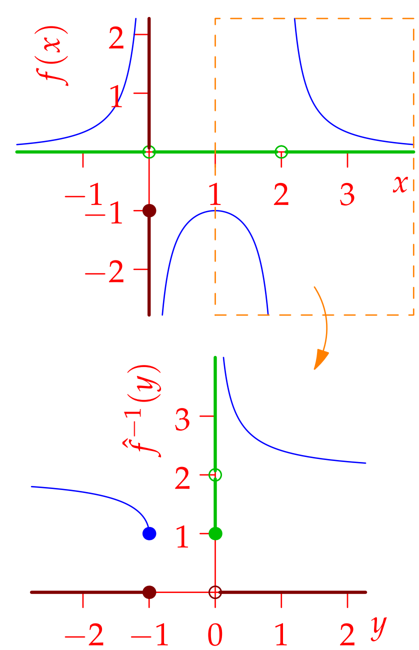
$$\text{dom}(\hat{f}) = [1, 2) \cup (2, \infty)$$

$$\text{codom}(\hat{f}) = (-\infty, -1] \cup (0, \infty)$$

with inverse

$$\hat{f}^{-1}(y) = \begin{cases} 1 + y^{-1}\sqrt{y+1} & \text{if } y > 0 \\ 1 - y^{-1}\sqrt{y+1} & \text{if } y \leq -1 \end{cases}$$

Now $\text{dom}(\hat{f}^{-1}) = \text{codom}(\hat{f})$ and $\text{codom}(\hat{f}^{-1}) = \text{dom}(\hat{f})$.



Suprema and Infima A set $U \subseteq \mathbb{R}$ is *bounded above* if it has an *upper bound* M:

$$\exists M \in \mathbb{R} \text{ such that } \forall u \in U,\ u \le M$$

Axiom 1.2 (Completeness). If $U \subseteq \mathbb{R}$ is non-empty and bounded above then it has a *least upper bound*, the *supremum* of U

$$\sup U = \min\{M \in \mathbb{R} : \forall u \in U,\ u \le M\}$$

By convention, $\sup U = \infty$ if U is unbounded above and $\sup \emptyset = -\infty$; now every subset of $\mathbb{R}$ has a supremum. Similarly, the *infimum* of U is its *greatest lower bound*:

$$\inf U = \begin{cases} \max\{m \in \mathbb{R} : \forall u \in U,\ u \ge m\} & \text{if } U \neq \emptyset \text{ is bounded below} \\ -\infty & \text{if } U \neq \emptyset \text{ is unbounded below} \\ \infty & \text{if } U = \emptyset \end{cases}$$

Examples 1.3. Here are four sets with their suprema and infima stated. You should be able to verify these assertions directly from the definitions.

U	$\{1,2,3,4\}$	$(0,5)$	$(-\infty,\pi]$	$\mathbb{R}$	$\{\frac{1}{n} : n \in \mathbb{N}\}$
$\sup U$	4	5	π	∞	1
$\inf U$	1	0	$-\infty$	$-\infty$	0

Note how the supremum/infimum might or might not lie in the set itself.

Interiors, closures, boundaries and neighborhoods These last concepts might not be review, but they will be used repeatedly.

Definition 1.4. Let $U \subseteq \mathbb{R}$. A value $a \in \mathbb{R}$ is *interior* to U if it lies in some open subinterval of U:

$$\exists \delta > 0 \text{ such that } (a-\delta, a+\delta) \subseteq U$$

A *neighborhood* of a is any set to which a is interior: the interval $(a-\delta, a+\delta)$ is an *open δ-neighborhood* of a. A *punctured neighborhood* of a is a neighborhood with a deleted.

The set of points interior to U is denoted U°.

A *limit point* of U is the limit of some sequence $(x_n) \subseteq U$. The *closure* $\overline{U}$ is the set of limit points.

The *boundary* is the set $\partial U = \overline{U} \setminus U^\circ$.

Examples 1.5. 1. If $U = [1,3)$, then $U^\circ = (1,3)$, $\overline{U} = [1,3]$ and $\partial U = \{1,3\}$.

2. $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}$.

3. $(-3,5) \cup (5,7]$ is a punctured neighborhood of 5.

17 Continuity of Functions

Everything in this section[1] *should* be review.

Definition 1.6. A function $f : U \to \mathbb{R}$ is *continuous at* $u \in U$ if either of the following hold:

1. For all sequences $(x_n) \subseteq U$ converging to u, the sequence $(f(x_n))$ converges to $f(u)$.
2. $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in U, |x - u| < \delta \implies |f(x) - f(u)| < \epsilon$.

A function f is *continuous on* U if it is continuous at every point $u \in U$.

Examples 1.7. 1. We prove that $f(x) = x^3$ is continuous at $u = 2$.

(a) (Limit method) Let $x_n \to 2$. By the *limit laws* (i.e. $\lim(x_n^k) = (\lim x_n)^k$),

$$\lim_{x_n \to 2} f(x_n) = \lim_{x_n \to 2} x_n^3 = \left(\lim_{x_n \to 2} x_n\right)^3 = 2^3 = f(2)$$

(b) (ϵ–δ method) Let $\epsilon > 0$ be given and let $\delta = \min\left(1, \frac{\epsilon}{19}\right)$.

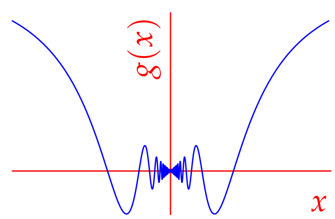
$$|x - 2| < \delta \implies |x - 2| < 1 \implies 1 < x < 3$$

from which

$$\left|x^3 - 2^3\right| = |x - 2| \left|x^2 + 2x + 2^2\right| < 19\,|x - 2| \leq \epsilon$$

where we used the triangle inequality.

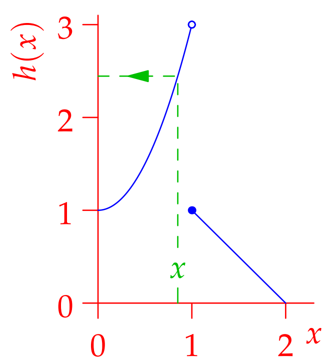
2. Let $g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$



Then g is continuous at $x = 0$. Again this can be done with limits or an ϵ–δ argument; both are essentially the *squeeze theorem.*

3. The function defined by

$$h(x) = \begin{cases} 1 + 2x^2 & \text{if } x < 1 \\ 2 - x & \text{if } x \geq 1 \end{cases}$$



is discontinuous at $x = 1$.

(a) The sequence with $x_n = 1 - \frac{1}{n}$ converges to 1, yet

$$\lim h(x_n) = 3 \neq 1 = h(1)$$

(b) Choose $\epsilon = 1$ and suppose $\delta > 0$ is given. Now choose $x = \max\{1 - \frac{\delta}{2}, \frac{1}{\sqrt{2}}\}$ to see that

$$|x - 1| < \delta \quad \text{and} \quad |h(x) - h(1)| \geq 1 = \epsilon$$

[1] Section numbers are identical to those in the official textbook.

Theorem 1.8. *The two parts of Definition 1.6 are equivalent.*

Proof. (1 ⇒ 2) We prove the contrapositive. Suppose condition 2 is *false*; that is,

$$\exists \epsilon > 0, \text{ such that } \forall \delta > 0,\ \exists x \in U \text{ with } |x-u| < \delta \text{ and } |f(x) - f(u)| \geq \epsilon$$

In particular, for any $n \in \mathbb{N}$ we may let $\delta = \frac{1}{n}$ to obtain

$$\exists \epsilon > 0, \text{ such that } \forall n \in \mathbb{N},\ \exists x_n \in U \text{ with } |x_n - u| < \frac{1}{n} \text{ and } |f(x_n) - f(u)| \geq \epsilon$$

The sequence (x_n) shows that condition 1 is *false*:

- $\forall n,\ |x_n - u| < \frac{1}{n}$ whence $x_n \to u$.
- $\forall n,\ |f(x_n) - f(u)| \geq \epsilon > 0$, whence $f(x_n)$ *does not converge to* $f(u)$.

(2 ⇒ 1) Suppose condition 2 is true, that $(x_n) \subseteq U$ converges to u and that $\epsilon > 0$ is given. Then

$$\exists \delta > 0 \text{ such that } |x-u| < \delta \implies |f(x) - f(u)| < \epsilon$$

However, by the definition of convergence ($x_n \to u$),

$$\exists N \in \mathbb{N} \text{ such that } n > N \implies |x_n - u| < \delta \implies |f(x_n) - f(u)| < \epsilon$$

Otherwise said, $f(x_n) \to f(u)$. ∎

Rather than use these definitions every time, it is helpful to have a working dictionary.

Theorem 1.9 (Common Continuous Functions).

1. *Suppose f and g are continuous at u, that h is continuous at $f(u)$ and that k is constant. Then the following are continuous at u (if defined):*

$$f+g, \quad f-g, \quad fg, \quad \frac{f}{g}, \quad |f|, \quad kf, \quad \max(f,g), \quad \min(f,g), \quad h \circ f$$

2. *Algebraic[2] functions are continuous.*
3. *The common transcendental functions are continuous:* exp, ln, sin, *etc.*

Example 1.10. $f(x) = \sin \frac{\sqrt[3]{x^2+7}}{x-2} + \cos \frac{1}{e^x - 1}$ is continuous on its domain $(-\infty, 0) \cup (0,1) \cup (1, \infty)$.

These claims are tedious to prove using elementary definitions. The first two require many uses of the limit laws, while the transcendental claim is easier to defer until we can define the common functions using power series, after which continuity comeS for free.

[2]Constructed using finitely many addition/subtraction, multiplication/division and n^{th} root operations

Exercises 17 1. Give examples to show that $g \circ f$ being continuous can happen with:

(a) f continuous and g discontinuous.

(b) g continuous and f discontinuous.

(c) Both f, g discontinuous.

You may use pictures, but make sure they clearly describe the functions f, g.

2. (a) Prove that the function $f(x) = x^3$ is continuous at $x = -2$ using an ϵ–δ argument.

   (b) Prove that $f(x) = x^3$ is continuous at $x = u$ using an ϵ–δ argument.

3. Prove that the following are discontinuous at $x = 0$: use *both* definitions of continuity.

   (a) $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.

   (b) $g(x) = \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$.

4. Suppose f and g are continuous at u. Prove the following using ϵ–δ arguments.

   (a) $f - g$ is continuous at u.

   (b) If h is continuous at $f(u)$, then $h \circ f$ is continuous at u.

5. Contrary to our standing assumption, suppose $f : U \to \mathbb{R}$ is a function whose domain U contains an *isolated point* a: i.e. $\exists r > 0$ such that $(a - r, a + r) \cap U = \{a\}$. Prove that f is continuous at a.

6. Refresh your prerequisites by giving formal proofs of the following:

   (a) (Suprema and sequences) If $M = \sup U$, then $\exists (x_n) \subseteq U$ such that $x_n \to M$. (*Remember that this has to work even if* $M = \infty \ldots$)

   (b) (Limit of a bounded sequence) If $(x_n) \subseteq [a, b]$ and $x_n \to x$, then $x \in [a, b]$.

   (c) (Bolzano–Weierstraß) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (*Hint: If* $(x_n) \subseteq [a, b]$, *explain why there exists a family of nested intervals* $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ *such that* infinitely many *of the terms* (x_n) *lie in each interval* I_k. *Hence obtain a subsequence* (x_{n_k}) *and prove that it is* Cauchy.[3])

7. (Hard) Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{q} & \text{whenever } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q > 0 \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

For example, $f(1) = f(2) = f(-7) = 1$, and $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, etc. Prove that f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.

[3]This is a good moment to review the notion of a Cauchy sequence

$$\forall \epsilon > 0, \exists N \text{ such that } m, n > N \implies |x_m - x_n| < \epsilon$$

and the discussion of Cauchy completeness: $(x_n) \subseteq \mathbb{R}$ is convergent if and only if it is Cauchy.

18 Properties of Continuous Functions

The goal of this section is to describe the behavior of a continuous function on an interval. We first consider the special case when the domain is a closed bounded interval $[a,b]$.

Theorem 1.11 (Extreme Value Theorem). *A continuous function on a closed, bounded interval is bounded and attains its bounds. Otherwise said, if $f:[a,b]\to\mathbb{R}$ is continuous, then*

$$\exists x,y\in[a,b] \text{ such that } f(x)=\sup\operatorname{range}(f) \text{ and } f(y)=\inf\operatorname{range}(f)$$

In particular, the supremum and infimum are finite.

Proof. Suppose f is continuous with domain $[a,b]$ and let $M=\sup\{f(x):x\in[a,b]\}$. We invoke the three parts of Exercise 17.6:

- (Part a) There exists a sequence $(x_n)\subseteq[a,b]$ such that $f(x_n)\to M$.
- (Part c) There exists a convergent subsequence (x_{n_k}) with limit x.
- (Part b) $x\in[a,b]$.

Since f is continuous, we now have $f(x)=\lim\limits_{k\to\infty}f(x_{n_k})=M$. This shows that M is *finite* and that f attains its least upper bound. For the lower bound, apply this to $-f$. ∎

It is worth considering how the result can fail when one of the hypotheses is weakened. For example:

f *discontinuous* $\quad f:[0,1]\to\mathbb{R}:x\mapsto\begin{cases}x & \text{if } x\neq 1\\ 0 & \text{if } x=1\end{cases}$ is bounded but does not attain its bounds.

$\operatorname{dom}(f)$ *not closed* $\quad f:[0,1)\to\mathbb{R}:x\mapsto x$ is bounded but does not attain its bounds.

$\operatorname{dom}(f)$ *not bounded* $\quad f:[0,\infty)\to\mathbb{R}:x\mapsto x$ is unbounded.

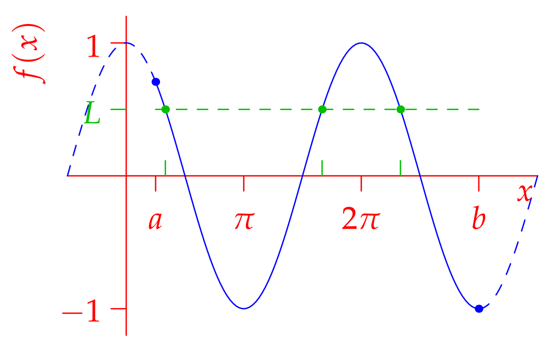
We now generalize to functions on arbitrary intervals. Our next result should be familiar from elementary calculus and is intuitively obvious from the naïve notion of continuity: graph such a function without taking your pen from the page.

Theorem 1.12 (Intermediate Value Theorem). *Let $f:I\to\mathbb{R}$ be continuous on an interval I. Suppose $a,b\in I$ with $a<b$ and that $f(a)\neq f(b)$. If L lies between $f(a)$ and $f(b)$, then $\exists\xi\in(a,b)$ such that $f(\xi)=L$.*

Example 1.13. Let $f(x)=\cos x$ with $a=\frac{\pi}{4}$, $b=3\pi$ and $L=\frac{1}{2}$; then

$$f(\xi)=L \iff \xi\in\left\{\frac{\pi}{3},\frac{5\pi}{3},\frac{7\pi}{3}\right\}$$

There may therefore be several suitable values of ξ. It is even possible (see Exercise 18.2) for there to be *infinitely many*.

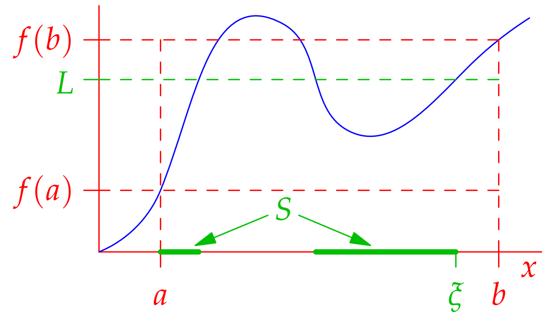


Proof. Suppose WLOG that $f(a) < L < f(b)$ and let

$$S = \{x \in [a,b] : f(x) < L\}$$

Plainly $S \subseteq [a,b)$ is non-empty, hence $\xi := \sup S$ exists and $\xi \in [a,b]$. It remains to show that ξ satisfies the required properties.

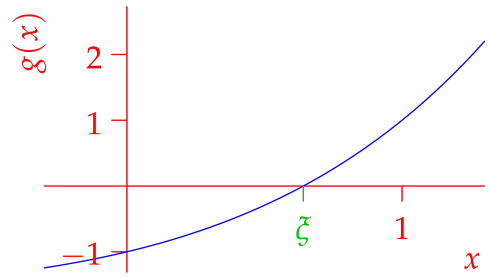
By Exercise 6, $\exists (s_n) \subseteq S$ with $\lim s_n = \xi$. Since f is continuous, $f(\xi) = \lim f(s_n) \le L$. In particular, $\xi \neq b$.



To finish the proof, we can play a similar game with the sequence defined by $t_n = \min\{b, \xi + \frac{1}{n}\}$; this is left to Exercise 4. ∎

Example 1.14. The intermediate value theorem is particularly useful for demonstrating the existence of solutions to equations. For example, we can use the following steps to show that the equation $x2^x = 1$ has a solution.

- $g(x) = x2^x - 1$ is continuous.
- $g(0) = -1 < 0$.
- $g(1) = 1 > 0$.
- By the intermediate value theorem $\exists \xi \in (0,1)$ such that $g(\xi) = 0$: that is $\xi \cdot 2^\xi = 1$.



It is inefficient, but one can home in on ξ by repeatedly halving the size of the interval: for instance,

$$g(\tfrac{1}{2}) = \tfrac{\sqrt{2}}{2} - 1 < 0, \quad g(\tfrac{3}{4}) = \tfrac{3}{4} \cdot 2^{3/4} - 1 \approx 0.26 > 0 \ldots \implies \tfrac{1}{2} < \xi < \tfrac{3}{4}$$

We finish with a useful corollary.

Corollary 1.15. *Continuous functions map intervals to intervals (or points).*

Proof. An interval I is characterized by the following property

$$\forall x_1, x_2 \in I,\ x \in \mathbb{R},\ x_1 < x < x_2 \implies x \in I$$

Let $f : I \to \mathbb{R}$ be continuous and suppose its range $f(I)$ is not a single point. If $f(a) < L < f(b)$, then $\exists \xi$ between a, b such that $f(\xi) = L$. Otherwise said, $L \in f(I)$ and so $f(I)$ is an interval. ∎

More generally, if $\text{dom}(f) = \bigcup I_n$ is written as a union of disjoint intervals and f is continuous, then

$$\text{range}(f) = \bigcup f(I_n)$$

is also a union of intervals, though these need not be disjoint: a continuous function can bring intervals together, but cannot break an interval apart.

For example, $f(x) = \sqrt{x^2 - 4}$ has domain $(-\infty, -2] \cup [2, \infty)$ and range $[0, \infty)$: both original intervals get mapped to the same interval by f.

A more general statement from topology says that if $f : U \to V$ is continuous between topological spaces and a, b lie in the same *component* of U, then $f(a)$ and $f(b)$ lie in the same component of $f(U)$. In single-variable real analysis each component is an interval.

Exercises 18

1. Give examples of the following:
   (a) An unbounded discontinuous function on a closed bounded interval.
   (b) An unbounded continuous function on a non-closed bounded interval.
   (c) A bounded continuous function on a closed unbounded interval which fails to attain its bounds.

2. Consider the function $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
   (a) Explain why f is continuous on any interval I.
   (b) Suppose $a < 0 < b$ and that $f(a), f(b)$ have opposite signs. If $L = 0$, show that the intermediate value theorem is satisfied by *infinitely many* distinct values ξ.

3. Use the intermediate value theorem to prove that the equation $8x^3 - 12x^2 - 2x + 1 = 0$ has at least 3 real solutions (and thus, by the fundamental theorem of algebra, exactly 3).

4. Complete the proof of the intermediate value theorem by defining $t_n = \min(b, \xi + \frac{1}{n})$.

5. (a) Suppose $f : U \to \mathbb{R}$ is continuous and that $U = \bigcup_{k=1}^{n} I_k$ is the union of a finite sequence (I_k) of closed bounded intervals. Prove that f is bounded and attains its bounds.
   (b) Let $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for each $n \in \mathbb{N}$. Give an example of a continuous function $f : U \to \mathbb{R}$ which is either unbounded or does not attain its bounds. Explain.

19 Uniform Continuity

Recall the ϵ–δ definition of continuity: $f : U \to \mathbb{R}$ is continuous at all points[4] $y \in U$, we require

$$\forall y \in U,\ \forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x \in U)\ |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

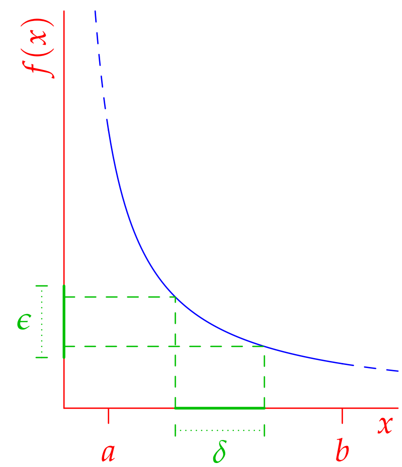
Note the order of the quantifiers: δ is permitted to depend on *both* y and ϵ. In the naïve sense of continuity (x close to $y \implies f(x)$ close to $f(y)$), the meaning of *close* is seen to depend on the *location* y. Uniform continuity is a stronger condition where the meaning of 'close' is *independent* of location.

Definition 1.16. $f : U \to \mathbb{R}$ is *uniformly continuous* if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x, y \in U)\ |x-y| < \delta \implies |f(x) - f(y)| < \epsilon$$

We've included the (typically) hidden quantifiers $(\forall x, y)$ in both definitions to make clear that ϵ and δ are independent of x and y. Note also that the definition is now symmetric in x and y.

Example 1.17. Consider $f(x) = \frac{1}{x}$.



1. If $0 < a < b \leq \infty$, then f is uniformly continuous on $[a, b)$.
   Let $\epsilon > 0$ be given and let $\delta = a^2\epsilon$. Then $\forall x, y \in [a, b)$,

$$|x-y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y-x}{xy}\right| < \frac{\delta}{xy} \leq \frac{\delta}{a^2} = \epsilon$$

2. If $0 < b \leq \infty$, then f is *not* uniformly continuous on $(0, b)$.

   Let $\epsilon = 1$ and suppose $\delta > 0$ is given.
   Let $x = \min(\delta, 1, \frac{b}{2})$ and $y = \frac{x}{2}$.
   Certainly $x, y \in (0, b)$ and $|x - y| = \frac{x}{2} \leq \frac{\delta}{2} < \delta$. However,

$$|f(x) - f(y)| = \frac{1}{x} \geq 1 = \epsilon$$

Think about how ϵ and δ must relate as one slides the intervals in the picture up/down and left/right.

Some intuition will help make sense of the above examples.

Bounded/unbounded gradient In part 1, $\epsilon = \delta a^2$, where $\frac{1}{a^2} = |f'(a)|$ bounds the gradient of f. By contrast, the slope of f is *unbounded* in part 2.

Extendability In part 1 (if $b \neq \infty$), the domain of f may be extended: $g : [a, b] \to \mathbb{R} : x \mapsto \frac{1}{x}$ is continuous. In part 2, this is impossible: there is no continuous function $g : [0, b) \to \mathbb{R}$ such that $g(x) = \frac{1}{x}$ whenever $x > 0$.

If the gradient of a continuous function is bounded or if you can 'fill in the holes' at the endpoints of its domain, then the function is uniformly continuous. While the utility of uniform continuity is often in proofs when the independence of ϵ and location are critical, it is often one of the above properties that is being invoked. The remainder of this section involves making thse observations watertight.

[4]To promote the symmetry in the coming definition, we use y instead of u for a generic point of $\operatorname{dom}(f)$.

Theorem 1.18. *Let $f : I \to \mathbb{R}$ be differentiable on an interval I. If the derivative f' is bounded on the interior I°, then f is uniformly continuous on I.*

The proof depends on the mean value theorem, which we'll prove later in the term.

Proof. Suppose $|f'(x)| \le M$ on I°. Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{M}$ and suppose $x, y \in I$ with $x > y$. Then

$$|x-y| < \delta \implies \exists \xi \in I^\circ \text{ such that } f'(\xi) = \frac{f(x)-f(y)}{x-y} \qquad \text{(MVT)}$$
$$\implies |f(x)-f(y)| = \left|f'(\xi)\right| |x-y| < M\delta = \epsilon$$

■

Theorem 1.18 isn't a biconditional: for instance, Exercise 19.5 shows that $f(x) = \sqrt{x}$ on $[0, \infty)$ and $g(x) = x^{1/3}$ on $\mathbb{R}$ are both uniformly continuous even though they have unbounded slope.
We now discuss the idea of extendability and how uniform continuity relates to continuity on closed sets. First we see that for closed bounded sets, uniform continuity is nothing new.

Theorem 1.19. *If $g : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.*

Proof. Suppose g is continuous but not uniformly so. Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x, y \in [a, b] \text{ for which } |x-y| < \delta \text{ and } |g(x)-g(y)| \ge \epsilon \qquad (*)$$

For each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ to see that there exists sequences $(x_n), (y_n) \subseteq [a, b]$ satisfying the above. By Bolzano–Weierstraß, the bounded sequence (x_n) has a convergent subsequence $x_{n_k} \to x \in [a, b]$. Clearly

$$|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \to 0 \implies y_{n_k} \to x$$

But then $|g(x_{n_k}) - g(y_{n_k})| \to 0$ which contradicts $(*)$.

■

Now we build to a partial converse of this.

Lemma 1.20. *If $f : U \to \mathbb{R}$ is uniformly continuous and $(x_n) \subseteq U$ is a Cauchy sequence, then $(f(x_n))$ is also Cauchy.*

Proof. Let $\epsilon > 0$ be given. Then:

- (Uniform Continuity) $\exists \delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < \epsilon$.
- (Cauchy) $\exists N \in \mathbb{N}$ such that $m, n > N \implies |x_m - x_n| < \delta$.

Putting these together, we see that

$$\exists N \in \mathbb{N} \text{ such that } m, n > N \implies |f(x_m) - f(x_n)| < \epsilon$$

Otherwise said, $(f(x_n))$ is Cauchy.

■

We now see that a function $f : I \to \mathbb{R}$ is uniformly continuous on a bounded interval if and only it is has a *continuous extension* $g : \overline{I} \to \mathbb{R}$ defined on the closure of its domain.

Theorem 1.21. *Suppose $f : I \to \mathbb{R}$ is continuous where I is a bounded interval with endpoints $a < b$. Define $g : [a, b] \to \mathbb{R}$ via*

$$g(x) = \begin{cases} f(x) & \text{if } x \in I \\ \lim f(x_n) & \text{whenever } (x_n) \subseteq I \text{ and } x_n \to a \\ \lim f(x_n) & \text{whenever } (x_n) \subseteq I \text{ and } x_n \to b \end{cases}$$

Then f is uniformly continuous if and only g is well-defined (g is continuous, if *well-defined).*

Proof. ($\Rightarrow$) Suppose f is uniformly continuous on I and that $a \notin I$. Let $(x_n), (y_n) \subseteq I$ be sequences converging to a. To show that g is well-defined, we must prove that $(f(x_n))$ and $(f(y_n))$ are convergent, and to the same limit.

Define a sequence

$$(u_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$$

Since (x_n) and (y_n) have the same limit a, we see that $u_n \to a$. But then (u_n) is Cauchy; by Lemma 1.20, $(f(u_n))$ is also Cauchy and thus convergent. Since $(f(x_n))$ and $(f(y_n))$ are subsequences of a convergent sequence, they must also converge to the same (finite!) limit.

The argument when $b \notin I$ is identical.

($\Leftarrow$) Certainly if g is well-defined then it is continuous. By Theorem 1.19 it is uniformly so. Since $f = g$ on a subset of $\text{dom}(g)$, the same choice of δ will work for f as for g: f is therefore uniformly continuous. ∎

Examples 1.22. 1. $f : x \mapsto x^2$ is uniformly continuous on $(-3, 10)$ since its derivative $f'(x) = 2x$ is bounded ($|f'(x)| = 2|x| \leq 20$) on its domain. It has the obvious continuous extension $g(x) = x^2$ on $[-3, 10]$.

2. Neither argument works for $f(x) = x^2$ on the domain $(-3, \infty)$: both f' and the domain $(-3, \infty)$ are unbounded, so neither Theorem 1.18 nor 1.21 applies.

   Instead, note that if $\epsilon = 1$, then for any $\delta > 0$, we can choose $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$. Clearly

$$|x - y| = \frac{\delta}{2} < \delta \text{ and } \left|x^2 - y^2\right| = 1 + \frac{\delta^2}{4} > 1 = \epsilon$$

   whence f is not uniformly continuous.

3. $f(x) = x \sin \frac{1}{x}$ is continuous on the interval $(0, \infty)$. Strictly, neither Theorem 1.18 nor 1.21 applies since the derivative

$$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

   is unbounded as is the domain. However, by breaking the domain into two pieces...

- On $(1,\infty)$, the derivative is bounded: $|f'(x)| \leq 1 + \frac{1}{x^2} \leq 2$ by the triangle inequality. Theorem 1.18 says f is uniformly continuous on $(1,\infty)$.
- f is continuous on $(0,1]$ and, by the squeeze theorem

  $$x_n \to 0^+ \implies \lim f(x_n) = 0$$

  Extending f so that $f(0) = 0$ defines a continuous extension. By Theorem 1.21, f is uniformly continuous on $(0,1]$.

Putting this together, f is uniformly continuous on $(0,\infty)$. Indeed the function

$$h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is uniformly continuous on $\mathbb{R}$.

Exercises 19 1. Which of the following functions are uniformly continuous on the specified set? Justify your answers.

(a) $f(x) = x^4$ on $[-1,1]$.

(b) $f(x) = x^4$ on $(-1,1]$.

(c) $f(x) = x^{-4}$ on $(0,2]$.

(d) $f(x) = x^{-4}$ on $(1,2]$.

(e) $f(x) = x^2 \sin \frac{1}{x}$ on $(0,1]$.

2. Prove that each of the following functions is uniformly continuous on the indicated set by verifying the ϵ–δ property.

   (a) $f(x) = 2x - 14$ on $\mathbb{R}$.

   (b) $f(x) = x^3$ on $[1,5]$.

   (c) $f(x) = x^{-1}$ on $(1,\infty)$.

   (d) $f(x) = \frac{x+1}{x+2}$ on $[0,1]$.

3. Prove that $f(x) = x^4$ is not uniformly continuous on $\mathbb{R}$.

4. (a) Suppose that f is uniformly continuous on a bounded interval I. Prove that f is bounded on I.

   (b) Use part (a) to write down a bounded interval on which the function $f(x) = \tan x$ is defined, but *not* uniformly continuous.

5. (a) Let $f(x) = \sqrt{x}$ with domain $[0,\infty)$. Show that $f'(x)$ is unbounded, but that f is still uniformly continuous on $[0,\infty)$.
   (*Hint: try* $\delta = \epsilon^2$ *and WLOG assume* $0 \leq y \leq x$. *Now compute* $(\sqrt{y} + \epsilon)^2 \ldots$)

   (b) Prove that $g(x) = x^{1/3}$ is uniformly continuous on $\mathbb{R}$.
   (*Hint: try* $\delta = (\frac{\epsilon}{2})^3$ *and consider the cases* $x \geq y \geq 0$, $x \leq y \leq 0$ *and* $x > 0 > y$ *separately*)

20 Limits of Functions

You've likely seen many calculations of the following form in elementary calculus:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6$$

Our next goal is to make this notation precise and to tie it to our earlier notion of limit.

Definition 1.23. Suppose $f:U\to\mathbb{R}$, that $S\subseteq U$, and that a is the limit of a sequence[5] in S. We write $\lim\limits_{x\to a^S} f(x)=L$ and say that L is the *limit of $f(x)$ as x tends to a along S*, provided

$$\forall (x_n)\subseteq S,\ \lim x_n=a \implies \lim f(x_n)=L$$

We can now define one-sided and two-sided limits:

Right-hand limit: $\lim\limits_{x\to a^+} f(x)=L$ means $\exists S=(a,b)\subseteq U$ for which $\lim\limits_{x\to a^S} f(x)=L$

Left-hand limit: $\lim\limits_{x\to a^-} f(x)=L$ means $\exists S=(c,a)\subseteq U$ for which $\lim\limits_{x\to a^S} f(x)=L$

Two-sided limit: $\lim\limits_{x\to a} f(x)=L$ means $\exists S=(c,a)\cup(a,b)\subseteq U$ for which $\lim\limits_{x\to a^S} f(x)=L$

- The one-sided definitions apply when $a=\pm\infty$, though we omit the $\pm$ modifiers: for instance,

  $$\lim_{x\to\infty} f(x)=L \iff \lim_{x\to\infty^S} f(x)=L \text{ for some } S=(c,\infty)\subseteq U$$

- The subtlety in the definition is that for $\lim\limits_{x\to a} f(x)$ to be defined, the domain U of f must contain a *punctured neighborhood* S of a: i.e. $a\in U^\circ$. The one-sided limits similarly require a *one-sided punctured neighborhood*. These conditions are always satisfied if U is a disjoint union of intervals of positive length, in which case $\lim\limits_{x\to a^{(\pm)}} f(x)=L$ if and only if

  $$\lim f(x_n)=L,\ \forall (x_n)\subseteq U\setminus\{a\} \text{ tending to } a \text{ (from above/below)}$$

  In this situation, Definition 1.6 recovers the familiar idea from elementary calculus:

  $$f \text{ is continuous at } a\in U \iff f(a)=\begin{cases}\lim\limits_{x\to a} f(x) & \text{when } a\in U^\circ\\ \lim\limits_{x\to a^\pm} f(x) & \text{when } a\in U\setminus\partial U\end{cases} \qquad (*)$$

- By modifying the proof of Theorem 1.8 in the case that $a,L\in\mathbb{R}$ are finite, the above can be written in ϵ-language. For example $\lim\limits_{x\to a} f(x)=L$ means

  $$\forall\epsilon>0,\ \exists\delta>0 \text{ such that } (\forall x\in\mathbb{R})\ 0<|x-a|<\delta \implies |f(x)-L|<\epsilon$$

  If a and/or L is infinite, use the language of unboundedness: e.g. $\lim\limits_{x\to a} f(x)=\infty$ means

  $$\forall M>0,\ \exists\delta>0 \text{ such that } 0<|x-a|<\delta \implies f(x)>M$$

  There are *fifteen* distinct combinations: *three* two-sided and *six* each of the one-sided limits!

[5] I.e. $a\in\overline{S}$ or perhaps $a=\pm\infty$ if S is unbounded.

Examples 1.24. 1. Let $f(x) = \frac{2+x}{x}$ where $\text{dom}(f) = U = \mathbb{R} \setminus \{0\} = (-\infty, 0) \cup (0, \infty)$

The following should be clear:

$$\lim_{x \to 3} f(x) = \frac{5}{3} \qquad \lim_{x \to \infty} f(x) = 1$$

To compute the first, for instance, we could choose $S = (0,3) \cup (3,\infty)$; if $(x_n) \subseteq S$ and $x_n \to 3$, then the limit laws justify the first claim

$$\lim_{n \to \infty} f(x_n) = \frac{2+3}{3} = \frac{5}{3}$$

as does the fact that f is continuous at $x = 3$. The second claim can be checked similarly.

We can take one-sided limits at $x = 0$:

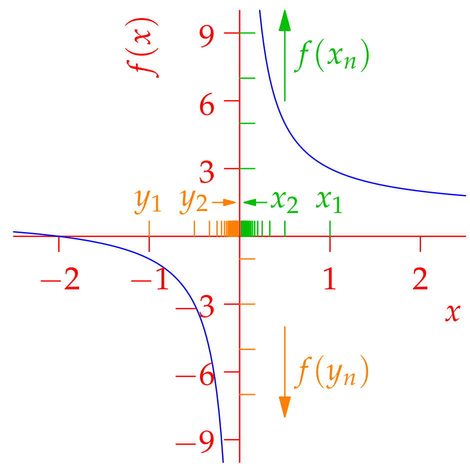
$$\lim_{x \to 0^+} f(x) = \infty \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -\infty$$

For instance, let $(x_n) \subseteq (0, \infty)$ satisfy $x_n \to 0$. Again, the limit laws show that $\lim_{n \to \infty} f(x_n) = \infty$, which is enough to justify the first claim.

Finally, the sequences defined by $x_n = \frac{1}{n}$ and $y_n = -\frac{1}{n}$ both lie in $S = \mathbb{R} \setminus \{0\}$ and converge to zero, yet

$$\lim_{n \to \infty} f(x_n) = \infty \neq -\infty = \lim_{n \to \infty} f(y_n)$$

It follows that the two-sided limit $\lim_{x \to 0} f(x)$ does not exist.



2. Let $f(x) = \frac{1}{x^2}$ whenever $x \neq 0$ and additionally let $f(0) = 0$. Here the two-sided limit exists

$$\lim_{x \to 0} f(x) = \infty$$

However the value of the function at $x = 0$ does not equal this limit: clearly f is discontinuous at $x = 0$.

3. We revisit our motivating example. Let $f(x) = \frac{x^2-9}{x-3}$ have domain $U = \mathbb{R} \setminus \{3\}$. Whenever $x_n \neq 3$, we see that

$$f(x_n) = \frac{(x_n - 3)(x_n + 3)}{x_n - 3} = x_n + 3$$

By the limit laws, we conclude that $\lim f(x_n) = 3 + 3 = 6$ and so

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = 6$$

Since we referenced the limit laws so often in the above examples, it is appropriate to update them to this new context. We do so without proof.

Corollary 1.25 (Limit Laws). *Suppose $f, g : U \to \mathbb{R}$ satisfy $L = \lim_{x\to a} f(x)$ and $M = \lim_{x\to a} g(x)$ exist. Then,*

1. $\lim_{x\to a}(f+g)(x) = L + M$.
2. $\lim_{x\to a}(fg)(x) = LM$.
3. $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$ *(requires $M \neq 0$).*
4. *If $L \in \mathbb{R}$ and h is continuous at L, then $\lim_{x\to a}(h\circ f)(x) = h(L)$.*
5. *(Squeeze Theorem) If $L = M$ and $f(x) \le h(x) \le g(x)$ for all $x \in U$, then $\lim_{x\to a} h(x) = L$.*

The corresponding results for one-sided limits also hold.

As with the original limit laws for sequences, parts 1–3 apply provided the limits are not *indeterminate forms* (e.g. $\infty - \infty$, $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$). We'll see later how l'Hôpital's rule may be applied to such cases.

Examples 1.26. 1. Since $f(x) = \frac{x^2+5}{3x^2-2}$ is a rational function (continuous at all points of its domain), we quickly conclude that

$$\lim_{x\to 2}\frac{x^2+5}{3x^2-2} = f(2) = \frac{9}{10}$$

Alternatively, we may tediously invoke the other parts of the theorem:

$$\lim_{x\to 2}\frac{x^2+5}{3x^2-2} \overset{(3)}{=} \frac{\lim(x^2+5)}{\lim(3x^2-2)} \overset{(1)}{=} \frac{\lim x^2 + \lim 5}{\lim 3x^2 - \lim 2} \overset{(2)}{=} \frac{(\lim x)^2+5}{(\lim 3)(\lim x)^2 - 2}$$
$$= \frac{2^2+5}{3\cdot 2^2-2} = \frac{9}{10}$$

2. As $x\to\infty$, the simplistic approach results in a nonsense indeterminate form:

$$\lim_{x\to\infty}\frac{x^2+5}{3x^2-2} \overset{?}{=} \frac{\lim(x^2+5)}{\lim(3x^2-2)} \overset{?}{=} \frac{\infty}{\infty}$$

However, a little pre-theorem algebra quickly yields[6]

$$\lim_{x\to\infty}\frac{x^2+5}{3x^2-2} = \lim_{x\to\infty}\frac{1+5x^{-2}}{3-2x^{-2}} = \frac{\lim(1+5x^{-2})}{\lim(3-2x^{-2})} = \frac{1}{3}$$

[6]Be careful! The expressions $\frac{x^2+5}{3x^2-2}$ and $\frac{1+5x^{-2}}{3-2x^{-2}}$ do not describe the same function, yet their *limits* at ∞ are equal. Being able easily to equate these limits is one of the advantages of the '$\exists S$' formulation of Definition 1.23. Think about why; what is a suitable set S in this context?

Classification of Discontinuities

We finish this section by considering the ways in which a function can fail to be continuous.

Definition 1.27. Suppose that a function is continuous on an interval except at finitely many values: we call these *isolated discontinuities.*

Examples 1.28. 1. $f(x) = \frac{1}{x}$ has a discontinuity at $x = 0$ since it is continuous on the interval $\mathbb{R}$, except at one point $x = 0$. Note that a function need not be defined at a discontinuity!

2. $f(x) = \frac{1}{\sin \frac{1}{x}}$ has a *non-isolated discontinuity* at $x = 0$: on any interval containing zero, f has infinitely many discontinuities: $x = \frac{1}{\pi n}$ where $|n| \in \mathbb{N}$.

The next result helps us classify isolated discontinuities.

Theorem 1.29. *Let $f : U \to \mathbb{R}$ and suppose $a \in U^\circ$ is an interior point. Then*

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$$

Proof. ($\Rightarrow$) Let $S = (c, a) \cup (a, b)$ satisfy the definition for $\lim_{x\to a} f(x) = L$. Since any sequence (say) in S^+ is also in S, plainly $S^+ = (a, b)$ and $S^- = (c, a)$ satisfy the one-sided definitions.

($\Leftarrow$) Suppose $S^- = (c, a)$ and $S^+ = (a, b)$ satisfy the one-sided definitions and denote $S = S^- \cup S^+$. Let $(x_n) \subseteq S$ be such that $x_n \to a$. Clearly (x_n) is the disjoint union of two subsequences $(x_n) \cap S^+$ and $(x_n) \cap S^-$, both of which[7] converge to a. There are three cases:

L finite: Let $\epsilon > 0$ be given. Because of the one-sided limits,

- $\exists N_1$ such that $n > N_1$ and $x_n > a \implies |f(x_n) - L| < \epsilon$
- $\exists N_2$ such that $n > N_2$ and $x_n < a \implies |f(x_n) - L| < \epsilon$

Now let $N = \max(N_1, N_2)$ in the definition of limit to see that $\lim f(x_n) = L$. Since this holds for all sequences $(x_n) \subseteq S$ converging to a, we conclude that $\lim_{x\to a} f(x) = L$.

$L = \pm\infty$: This is an exercise.

■

Example 1.30. Recalling elementary calculus, we show that the following is continuous at $x = 1$:

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x \geq 1 \\ 3 - 5x & \text{if } x < 1 \end{cases}$$

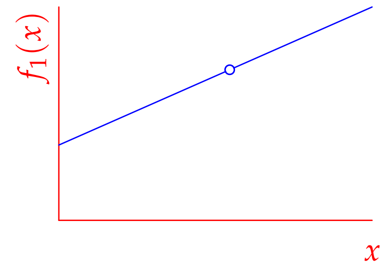
Step 1: Compute the left- and right-handed limits and check that these are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3 - 5x = -2, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 - 3 = -2$$

Step 2: Check that the value of the limits equals that of the function: $f(1) = 1^2 - 3 = -2$.

[7] It is possible for *one* of these subsequences to be finite; say if $x_n > a$ for all large n. This is of no concern; one of the ϵ-N conditions would be empty and thus vacuously true.

Recalling $(*)$ on page 13, we describe the different types of isolated discontinuity at some point a.



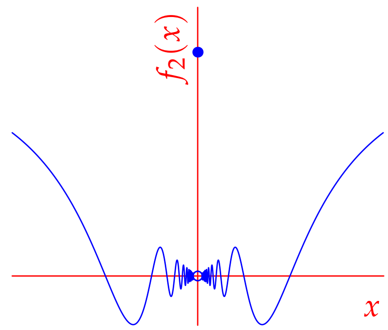
Removable discontinuity The two-sided limit $\lim_{x\to a} f(x) = L$ is finite, and either:

$$f(a) \neq L \text{ or } f(a) \text{ is undefined.}$$

The term comes from the fact that we can remove the discontinuity by changing the behavior of f only at $x = a$:

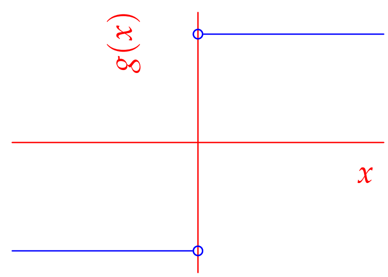
$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \neq a \\ \lim_{x\to a} f(x) & \text{if } x = a \end{cases}$$

is now continuous at $x = a$. In the pictures,



$$f_1(x) = \frac{x^2 - 9}{x - 3} \quad \text{and} \quad f_2(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

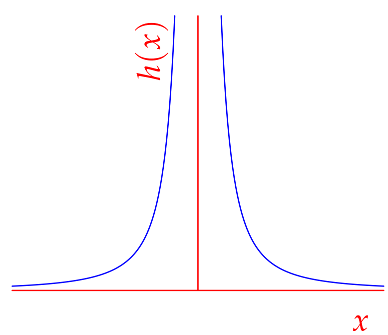
have removable discontinuities at $x = 3$ and 0 respectively.



Jump Discontinuity The one-sided limits are finite but *not equal*. A jump discontinuity cannot be removed by changing or inserting a value at $x = a$. The picture shows

$$g(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

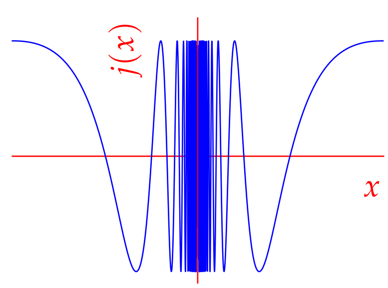
with a jump discontinuity at $x = 0$.



Infinite discontinuity The one-sided limits exist but at least one is infinite. We call the line $x = a$ a *vertical asymptote*. The picture shows

$$h(x) = \frac{1}{x^2}$$

with an infinite discontinuity $x = 0$. The fact that the one-sided limits of h are equal (and infinite) is irrelevant.



Essential discontinuity At least one of the one-sided limits does not exist. The picture shows $j(x) = \sin \frac{1}{x}$ for which neither of the limits $\lim_{x\to 0^\pm} j(x)$ exist.

It is also reasonable to refer to removable, infinite or essential discontinuities at interval endpoints.

Exercises 20 1. For the function $f(x) = \frac{x^3}{|x|}$, determine the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0} f(x)$, if they exist.

2. Evaluate the following limits *using the methods of this section*

   (a) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$ (b) $\lim_{x\to a} \frac{x^{-3/2}-a^{-3/2}}{x-a}$

   (c) $\lim_{x\to 0} \frac{\sqrt{1+3x^2}-1}{x^2}$ (d) $\lim_{x\to -\infty} \frac{\sqrt{4+3x^2}-2}{x}$

3. Suppose that the limits $L = \lim_{x\to a^+} f(x)$ and $M = \lim_{x\to a^+} g(x)$ exist.

   (a) Suppose $f(x) \le g(x)$ for all x in some interval (a,b). Prove that $L \le M$.

   (b) Do we have the same conclusion if we have $f(x) < g(x)$ on (a,b), or can we conclude that $L < M$? Prove your assertion, or give a counter-example.

4. Suppose that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Using *only* this information, which of the following can you evaluate? Prove your assertions in each case.

   (a) $\lim_{x\to\infty} (f+g)(x)$ (b) $\lim_{x\to\infty} (f-g)(x)$

   (c) $\lim_{x\to\infty} (fg)(x)$ (d) $\lim_{x\to\infty} (f/g)(x)$

5. Complete the proof of Theorem 1.29 by considering the $L = \pm\infty$ cases.

6. Graph $f : \mathbb{R} \to \mathbb{R}$, find and identify the types of its discontinuities.

$$f(x) = \begin{cases} 0 & x = 0, \pm 1 \\ \frac{x}{|x|} & 0 < |x| < 1 \\ x^2 & |x| > 1 \end{cases}$$

7. Find the discontinuities and identify their types for the following function

$$f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x < 0 \text{ or } x > 1 \\ \frac{1}{x} & \text{if } 0 < x \le 1 \end{cases}$$

8. Let $a \in U^\circ$. Verify the claim following Definition 1.23: $\lim_{x\to a} f(x) = L$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$$

9. Recall Exercise 17.5, where we saw that a function $f : U \to \mathbb{R}$ is continuous at any isolated point $a \in U$.

   (a) Any function with domain $\text{dom}(f) = \mathbb{Z}$ is continuous everywhere! Explain why we cannot define any limits $\lim_{x\to a^{(\pm)}} f(x)$ for such a function.

   (*Hint: Being unable to define a limit is different from saying* $\lim f(x) = DNE$*: see page 13.*)

   (b) Suppose $g(x) = x^2 h(x)$ has $\text{dom}(g) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}\}$, where h is any function taking values in the interval $[-1,1]$. Explain why g is continuous at every point of its domain.

(*These awkward examples of continuity can be avoided if we follow our usual approach where a domain is a union of intervals of positive length. This restriction is essentially baked in to the Definition 1.23.*)