2 Sequences and Series of Functions

If (f_n) is a sequence of functions, what should we mean by $\lim f_n$? This question is of great relevance to the history of calculus; Issac Newton's work in the late 1600's made great use of *power series*, which are naturally constructed as limits of sequences of polynomials.

For instance, for each $n \in \mathbb{N}_0$, we might consider the polynomial function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n$$

This is easy to work with, to differentiate and integrate using the power law. What, however, are we to make of the following *series*?

$$f(x) := \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

Does this make sense? What is its domain? Does it equal the limit of the sequence (f_n) in a meaningful way? Is it continuous, differentiable or integrable? Can we compute its limit/derivative/integral term-by-term in the obvious way; for instance, is it legitimate to write

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots ?$$

To many in Newton's time, these questions were of diminished importance when compared to the burgeoning applications of calculus to the natural sciences. However, for the 18$^{\text{th}}$ and 19$^{\text{th}}$ century mathematicians who followed, the widespread application of calculus only increased the imperative to rigorously address these issues.

23 Power Series

First we recall some of the important definitions, examples and results regarding infinite series.

Definition 2.1. Let $(b_n)_{n=m}^{\infty}$ be a sequence of real numbers. The *(infinite) series* $\sum b_n$ is the limit of the sequence (s_n) of *partial sums*,

$$s_n = \sum_{k=m}^{n} b_n = b_m + b_{m+1} + \cdots + b_n, \qquad \sum_{n=m}^{\infty} b_n = \lim_{n\to\infty} s_n$$

The series $\sum b_n$ converges, diverges to infinity or diverges by oscillation[1]if the sequence (s_n) does so. $\sum b_n$ is *absolutely convergent* if $\sum |b_n|$ converges. A convergent series that is not absolutely convergent is *conditionally convergent*.

[1]Recall that every sequence (s_n) has subsequences tending to each of

$$\limsup s_n = \lim_{N\to\infty} \sup\{x_n : n > N\} \quad \text{and} \quad \liminf s_n = \lim_{N\to\infty} \inf\{x_n : n > N\}$$

If (s_n) converges, or diverges to $\pm\infty$, then $\lim s_n = \limsup s_n = \liminf s_n$. The remaining case, divergence by oscillation, is when $\liminf s_n \neq \limsup s_n$: there exist (at least) two subsequences tending to different limits.

Examples 2.2. These examples form the standard reference dictionary for analysis of more complex series. Make sure you are familiar with them![2]

1. (Geometric series) Let r be a constant, then $s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$. It follows that

$$\sum_{n=0}^{\infty} r^n \begin{cases} \text{converges (absolutely) to } \frac{1}{1-r} & \text{if } -1 < r < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \\ \text{diverges by oscillation} & \text{if } r \leq -1 \end{cases}$$

2. (Telescoping series) If $b_n = \frac{1}{n(n+1)}$, then $s_n = \sum_{k=1}^{n} b_n = 1 - \frac{1}{n+1} \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is (absolutely) convergent. In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, though checking this explicitly is tricky.
4. (Harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent to ∞.
5. (Alternating harmonic series) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 2.3 (Root Test). *Given a series $\sum b_n$, let $\beta = \limsup |b_n|^{1/n}$,*

- *If $\beta < 1$ then the series converges absolutely.*
- *If $\beta > 1$ then the series diverges.*

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[2] We give sketch proofs, and/or refer you to a standard 'test.' Review these if you are unfamiliar.

1. $s_n - rs_n = 1 + r + \cdots + r^n - (r + \cdots + r^n + r^{n+1}) = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$.
2. $b_n = \frac{1}{n} - \frac{1}{n+1} \implies s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.
3. Use the comparison or integral tests. Alternatively: For each $n \geq 2$, we have $\frac{1}{n^2} < \frac{1}{n(n-1)}$. By part 2,

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=1}^{n} \frac{1}{k(k-1)} \leq 2$$

   Since (s_n) is a monotone up sequence, bounded above by 2, we conclude that $\sum \frac{1}{n^2}$ is convergent.
4. Use the integral test. Alternatively, observe that

$$s_{2^{n+1}} - s_{2^n} = \sum_{k=2^n-1}^{2^{n+1}} \frac{1}{k} \geq \frac{2^n}{2^{n+1}} = \frac{1}{2} \implies s_{2^n} \geq \frac{n}{2} \xrightarrow[n\to\infty]{} \infty$$

   Since $s_n = \sum_{k=1}^{n} \frac{1}{k}$ defines an increasing sequence we conclude that $s_n \to \infty$.
5. Use the alternating series test, or explicitly check that both the even and odd partial sums (s_{2n}) and (s_{2n+1}) are convergent (monotone and bounded) to the same limit.

Root Test: $\beta < 1 \implies \exists \epsilon > 0$ such that $|b_n|^{1/n} \leq 1 - \epsilon$ (for large n) $\implies \sum |b_n|$ converges by comparison with the convergent geometric series $\sum(1-\epsilon)^n$.

$\beta > 1 \implies$ a subsequence of $(|b_n|^{1/n})$ converges to $\beta > 1$, whence $b_n \not\to 0 \implies \sum b_n$ diverges (n^{th}-term test).

The root test is inconclusive if $\beta = 1$. Some simple inequalities[3] yield a simpler test.

Corollary 2.4 (Ratio Test). *Given a series* $\sum b_n$,

- *If* $\limsup \left|\frac{b_{n+1}}{b_n}\right| < 1$ *then* $\sum b_n$ *converges absolutely.*
- *If* $\liminf \left|\frac{b_{n+1}}{b_n}\right| > 1$ *then* $\sum b_n$ *diverges.*

We can now properly define and analyze our main objects of interest.

Definition 2.5. A *power series centered at* $c \in \mathbb{R}$ is a formal expression

$$\sum_{n=m}^{\infty} a_n (x-c)^n$$

where $(a_n)_{n=m}^{\infty}$ is a sequence of real numbers and x is considered a variable.

It is common to refer simply to a *series*, and modify by infinite/power when clarity requires. We almost always have $m = 0$ or 1, and it is common for examples to be centered at $c = 0$.

Example 2.6. Using the geometric series formula, we see that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-4)^n = \frac{1}{1-\frac{-(x-4)}{2}} = \frac{2}{x-2} \quad \text{whenever} \quad \left|-\frac{x-4}{2}\right| < 1 \iff 2 < x < 6$$

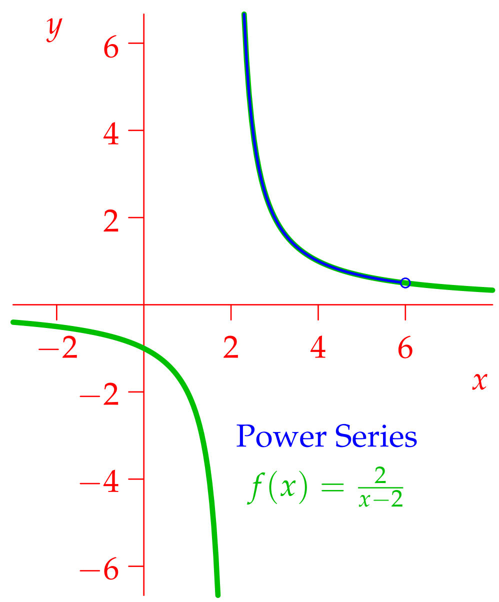
The series is valid (converges) only on a small subinterval of the implied domain of the function $x \mapsto \frac{2}{x-2}$. The behavior of both as $x \to 2^+$ should not be a surprise; evaluating the power series results in the divergent infinite series

$$\sum 1 = +\infty$$

By contrast, as $x \to 6^-$, we see that limits and infinite series do not interact the way we might expect,

$$\lim_{x\to 6^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-4)^n = \lim_{x\to 6^-} \frac{2}{x-2} = \frac{1}{2}$$
$$\sum_{n=0}^{\infty} \lim_{x\to 6^-} \frac{(-1)^n}{2^n}(x-4)^n = \sum (-1)^n = \text{DNE}$$

with the last divergent by oscillation.



As the example shows, we cannot take limits inside an infinite sum; understanding when we can do this is one of our primary goals.

[3] $\liminf \left|\frac{b_{n+1}}{b_n}\right| \le \liminf |b_n|^{1/n} \le \limsup |b_n|^{1/n} \le \limsup \left|\frac{b_{n+1}}{b_n}\right|$

Radius and Interval of Convergence

At any real number x, a series may converge absolutely, converge conditionally, diverge to $\pm\infty$, or diverge by oscillation. A series defines a *function* whose implied domain is the set on which the series converges. In the previous example, the domain was an *interval* $(2, 6)$. By applying the root test (Theorem 2.3), we can show that this holds for *every* series.

Theorem 2.7 (Root Test for Power Series). *Given a series* $\sum_{n=0}^{\infty} a_n(x-c)^n$, *define*[4]

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Exactly one of the following is true:

$R = \infty$	*the series converges absolutely for all* $x \in \mathbb{R}$
$R = 0$	*the series converges only when* $x = c$
$R \in \mathbb{R}^+$	*the series converges absolutely when* $\lvert x-c\rvert < R$ *and diverges when* $\lvert x-c\rvert > R$

Proof. For each fixed $x \in \mathbb{R}$, let $b_n = a_n(x-c)^n$ and apply the root test to $\sum b_n$, noting that

$$\limsup |b_n|^{1/n} = \begin{cases} 0 & \text{if } x = c \text{ or } R = \infty \\ \infty & \text{if } x \neq c \text{ and } R = 0 \\ \limsup |a_n|^{1/n} |x-c| = \frac{1}{R}|x-c| & \text{otherwise} \end{cases}$$

In the final situation, $\limsup |b_n|^{1/n} < 1 \iff |x-c| < R$, etc. ∎

Definition 2.8. The *radius of convergence* is the value R defined in Theorem 2.7. The *interval of convergence* is the set of values x for which the series converges; the implied domain.

Radius of convergence	Interval of convergence
∞	$\mathbb{R} = (-\infty, \infty)$
0	$\{c\}$
R	$(c-R, c+R)$, $(c-R, c+R]$, $[c-R, c+R)$, or $[c-R, c+R]$

In the third case convergence/divergence at the endpoints of the interval of convergence must be tested separately.

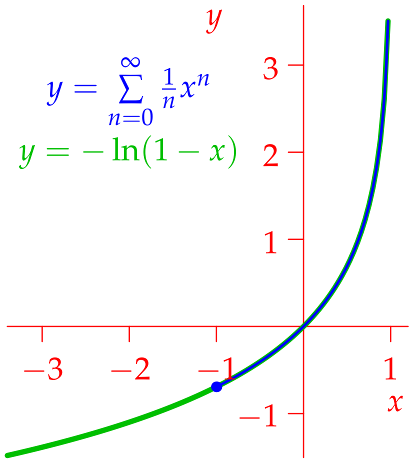
By applying Corollary 2.4, we obtain a more user-friendly result.

Corollary 2.9 (Ratio Test for Power Series). *If the limit exists,* $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

[4]Since $|a_n| \geq 0$, we here adopt the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. With similar caveats, it is also reasonable to write $R = \liminf |a_n|^{-1/n}$.

Examples 2.10. 1. The series $\sum_{n=1}^{\infty} \frac{1}{n} x^n$ is centered at 0. The ratio test tells us that

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1$$



Test the endpoints of the interval of convergence separately:

$x = 1$ $\sum \frac{1}{n} = \infty$ diverges

$x = -1$ $\sum \frac{(-1)^n}{n}$ converges (conditionally)

We conclude that the interval of convergence is $[-1, 1)$.

It can be seen that the series converges to $-\ln(1-x)$ on its interval of convergence. As in Example 2.6, this function has a larger domain $(-\infty, -1)$, than that of the series.

2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2} x^n$ similarly has

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{n^2} = 1$$

Since $\sum \frac{1}{n^2}$ is absolutely convergent, we conclude that the power series also converges absolutely at $x = \pm 1$; the interval of convergence is $[-1, 1]$.

3. The series $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$ converges absolutely for all $x \in \mathbb{R}$, since

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \lim_{n\to\infty} (n+1) = \infty$$

You should recall from elementary calculus that this series converges to the natural exponential function $\exp(x) = e^x$ everywhere on $\mathbb{R}$; indeed this is one of the common *definitions* of the exponential!

4. The series $\sum_{n=0}^{\infty} n! x^n$ has $R = 0$, and thus only converges at its center $x = 0$.

5. Let $a_n = \left(\frac{2}{3}\right)^n$ if n is even and $\left(\frac{3}{2}\right)^n$ if n is odd. If we try to apply the ratio test to the series $\sum_{n=0}^{\infty} a_n x^n$, we see that

$$\left| \frac{a_n}{a_{n+1}} \right| = \begin{cases} \left(\frac{2}{3}\right)^{2n+1} & \text{if } n \text{ even} \\ \left(\frac{3}{2}\right)^{2n+1} & \text{if } n \text{ odd} \end{cases} \implies \limsup \left| \frac{a_n}{a_{n+1}} \right| = \infty \neq 0 = \liminf \left| \frac{a_n}{a_{n+1}} \right|$$

The ratio test therefore fails. However, by the root test,

$$|a_n|^{1/n} = \begin{cases} \frac{2}{3} & \text{if } n \text{ even} \\ \frac{3}{2} & \text{if } n \text{ odd} \end{cases} \implies R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{3/2} = \frac{2}{3}$$

It is easy to check that the series diverges at $x = \pm\frac{2}{3}$; the interval of convergence is $(-\frac{2}{3}, \frac{2}{3})$.

With the help of the root test, we can understand the domain of a power series. The issues of limits, continuity, differentiability and integrability are more delicate. We will return to these once we've developed some of the ideas around *convergence* for *sequences of functions.*

Exercises 23 1. For each power series, find the radius and interval of convergence:

(a) $\sum \frac{(-1)^n}{n^2 4^n} x^n$ (b) $\sum \frac{(n+1)^2}{n^3}(x-3)^n$ (c) $\sum \sqrt{n}x^n$

(d) $\sum \frac{1}{n\sqrt{n}}(x+7)^n$ (e) $\sum (x-\pi)^{n!}$ (f) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

2. For each $n \in \mathbb{N}$ let $a_n = \left(\frac{4+2(-1)^n}{5}\right)^n$

   (a) Find $\limsup |a_n|^{1/n}$, $\liminf |a_n|^{1/n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.

   (b) Do the series $\sum a_n$ and $\sum(-1)^n a_n$ converge? Why?

   (c) Find the interval of convergence of the power series $\sum a_n x^n$.

3. Suppose that $\sum a_n x^n$ has radius of convergence R. If $\limsup |a_n| > 0$, prove that $R \le 1$.

4. On the interval $(-\frac{2}{3}, \frac{2}{3})$, express the series in Example 2.10.5 as a simple function.

   (*Hints: Use geometric series formulæ and the fact that the value of an absolutely convergent series is independent of rearrangements*)

5. Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{3^n n}(x-7)^{5n+1} = \frac{1}{3}(x-7) + \frac{1}{18}(x-7)^6 + \frac{1}{81}(x-7)^{11} + \cdots$$

   Since only one in five of the terms are non-zero, it is a little tricky to analyze using a naïve application of our standard tests.

   (a) Explain why the ratio test for power series (Corollary 2.9) does not apply.

   (b) Writing the series as $\sum a_m (x-7)^m$, observe that

$$a_m = \begin{cases} \frac{5}{3^{\frac{m-1}{5}}(m-1)} & \text{if } m \equiv 1 \mod 5 \\ 0 & \text{otherwise} \end{cases}$$

   Use the root test (Theorem 2.7) and your understanding of elementary limits to directly compute the radius of convergence.

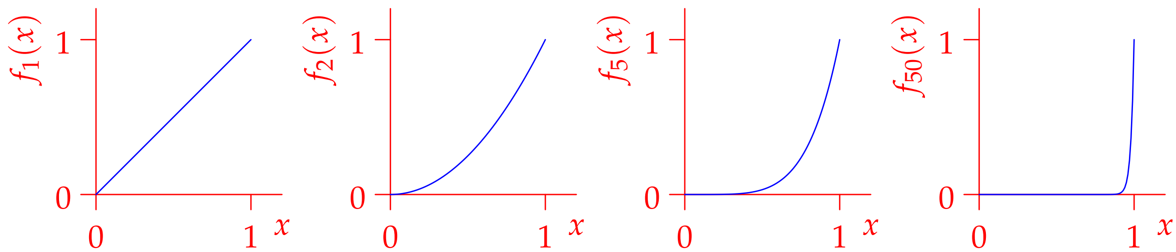
   (c) Alternatively, write $\sum \frac{1}{3^n n}(x-7)^{5n+1} = \sum b_n$. Apply the ratio test for *infinite* series (Corollary 2.4): what do you observe? Use your observation to compute the radius of convergence of the original series in a simpler manner than part (a).

   (d) Finally, check the endpoints to determine the interval of convergence.

24 Uniform Convergence

In this section we consider sequences (f_n) of *functions* $f_n : U \to \mathbb{R}$.

Example 2.11. For each $n \in \mathbb{N}$, consider $f_n : (0,1) \to \mathbb{R} : x \mapsto x^n$.



There turn out to be several good notions of convergence for sequences of functions; the simplest it where, for each x, $(f_n(x))$ is treated as a separate sequence of real numbers.

Definition 2.12. Suppose a function f and a sequence of functions (f_n) are given. We say that (f_n) *converges pointwise to f on U* if,

$$\forall x \in U, \lim_{n\to\infty} f_n(x) = f(x)$$

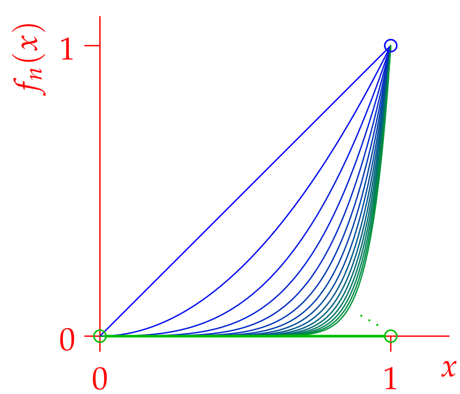
It is common to write '$f_n \to f$ pointwise.' For reference, we state two equivalent rephrasings:

1. $\forall x \in U, |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$;
2. $\forall x \in U, \forall \epsilon > 0, \exists N$ such that $n > N \implies |f_n(x) - f(x)| < \epsilon$.

As we'll see shortly, the relative positions of the quantifiers ($\forall x$ and $\exists N$) is crucial: in this definition, the value of N is permitted to depend on x as well as ϵ.

Example (2.11, mk. II). The sequence (f_n) converges pointwise on the domain $(0,1)$ to

$$f : (0,1) \to \mathbb{R} : x \mapsto 0$$



We prove this explicitly as a sanity check. First observe that

$$|f_n(x) - f(x)| = x^n$$

Suppose $x \in (0,1)$, that $\epsilon > 0$ is given, and let $N = \frac{\ln \epsilon}{\ln x}$. Then

$$n > N \implies n \ln x < \ln \epsilon \implies x^n < \epsilon$$

where the inequality switches sign since $\ln x < 0$.

The example is nice in that a sequence of continuous functions converges pointwise to a continuous function. Unfortunately, this desirable situation is not universal.

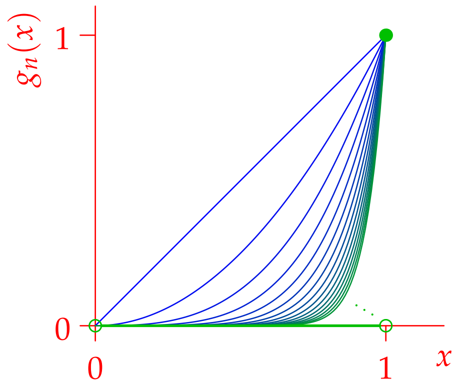
Example (2.11, mk. III). Extend the domain to include $x = 1$; define

$$g_n : (0, 1] \to \mathbb{R} : x \mapsto x^n$$

Each g_n is a continuous function, however its pointwise limit

$$g(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

has a *jump discontinuity* at $x = 1$.

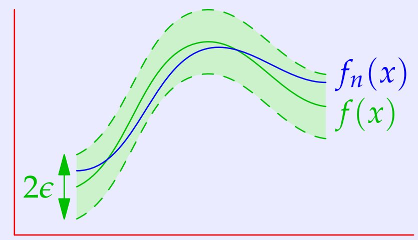


With the goal of having convergence of functions preserve continuity, we make a tighter definition.

Definition 2.13. (f_n) *converges uniformly* to f on U if either

1. $\sup_{x \in U} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$, or,
2. $\forall \epsilon > 0$, $\exists N$ such that $\forall x \in U, n > N \implies |f_n(x) - f(x)| < \epsilon$

A common notation is $f_n \rightrightarrows f$, though we won't use it.



Whenever $n > N$, the graph of $f_n(x)$ must lie between those of $f(x) \pm \epsilon$.
We'll show that statements 1 and 2 are equivalent momentarily. For the present, compare with the corresponding statements for pointwise convergence:

- As with *continuity* versus *uniform continuity*, the distinction comes in the *order of the quantifiers*: in uniform convergence, x is quantified *after* N and so *the same* N *works for all* x.
- Uniform convergence implies pointwise convergence.

For the last time, we revisit our main example.

Example (2.11, mk. IV). If $f_n : (0, 1) \to \mathbb{R} : x \mapsto x^n$ and f are defined as before, then the pointwise convergence $f_n \to f$ is *non-uniform.* We show this using both criteria.

1. For *every* n,

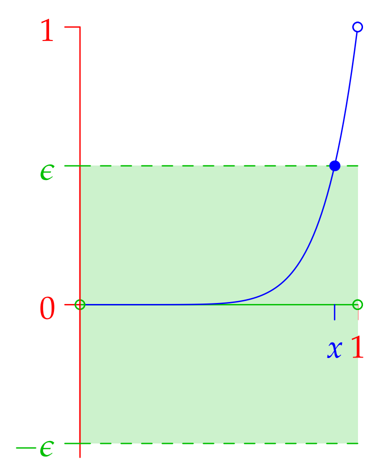
$$\sup_{x \in (0,1)} |f_n(x) - f(x)| = \sup\{x^n : 0 < x < 1\} = 1 \not\to 0$$

2. Suppose the convergence were uniform and let $\epsilon = \frac{1}{2}$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall x \in (0, 1),\ n > N \implies x^n < \frac{1}{2}$$

Since $N \in \mathbb{N}$, a simple choice results in a contradiction;

$$x = \left(\frac{1}{2}\right)^{\frac{1}{N+1}} \in (0, 1) \implies x^{N+1} = \frac{1}{2}$$



Theorem 2.14. *The criteria for uniform convergence in Definition 2.13 are equivalent.*

Proof. (1 ⇒ 2) This follows from the fact that

$$\forall x \in U, \ |f_n(x) - f(x)| \leq \sup_{x \in U} |f_n(x) - f(x)|$$

(2 ⇒ 1) Suppose $\epsilon > 0$ is given. Then

$$\exists N \in \mathbb{R} \text{ such that } \forall x \in U, \ n > N \implies |f_n(x) - f(x)| < \frac{\epsilon}{2}$$

But then

$$n > N \implies \sup_{x \in U} |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$$

■

Somewhat amazingly, the subtle change of definition results in the preservation of continuity.

Theorem 2.15. *Suppose that (f_n) is a sequence of continuous functions. If $f_n \to f$ uniformly, then f is continuous.*

Proof. We demonstrate the continuity of f at $a \in U$. Let $\epsilon > 0$ be given.

- Since $f_n \to f$ uniformly,

$$\exists N \text{ such that } \forall x \in U, \ n > N \implies |f(x) - f_n(x)| < \frac{\epsilon}{3}$$

- Choose any $n > N$. Since f_n is continuous at a,

$$\exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3} \tag{†}$$

Simply put these together with the triangle inequality to see that

$$\begin{aligned} |x - a| < \delta \implies |f(x) - f(a)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

■

We need not have fixed a at the start of the proof. Rewriting (†) to become

$$\exists \delta > 0 \text{ such that } \forall x, a \in U, \ |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

proves a related result.

Corollary 2.16. *If $f_n \to f$ uniformly where each f_n is* uniformly *continuous, then f is uniformly continuous.*

Examples 2.17. 1. Let $f_n(x) = x + \frac{1}{n}x^2$. This is continuous on $\mathbb{R}$ for all x, and converges pointwise to the continuous function $f : x \mapsto x$.

(a) On any bounded interval $[-M, M]$ the convergence $f_n \to f$ is uniform,

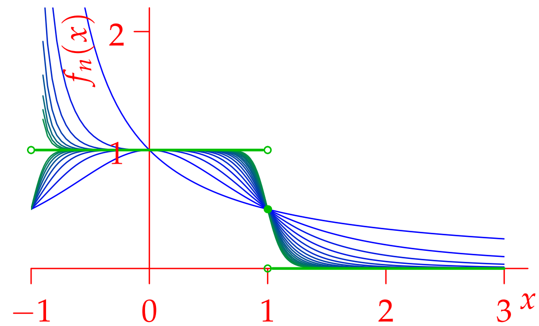
$$\sup_{x\in[-M,M]} |f_n(x) - f(x)| = \sup\left\{\frac{1}{n}x^2 : x \in [-M, M]\right\} = \frac{M^2}{n} \xrightarrow[n\to\infty]{} 0$$

(b) On any unbounded interval, $\mathbb{R}$ say, the convergence is non-uniform,

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup\left\{\frac{1}{n}x^2 : x \in \mathbb{R}\right\} = \infty$$

2. Consider $f_n(x) = \frac{1}{1+x^n}$; this is continuous on $(-1, \infty)$ and converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } -1 < x < 1 \end{cases}$$



We consider the convergence $f_n \to f$ on several intervals.

(a) On $[2, \infty)$, the pointwise limit is continuous. Moreover, $f_n(x)$ is decreasing, whence

$$\sup_{x\in[2,\infty)} |f_n(x) - 0| = \frac{1}{1+2^n} \xrightarrow[n\to\infty]{} 0$$

and the convergence is uniform. Alternatively; if $\epsilon \in (0, 1)$, let $N = \log_2(\epsilon^{-1} - 1)$, then

$$\forall x \geq 2,\ n > N \implies |f_n(x) - 0| = \frac{1}{1+x^n} \leq \frac{1}{1+2^n} < \frac{1}{1+2^N} = \epsilon$$

The same argument shows that $f_n \to f$ uniformly on any interval $[a, \infty)$ where $a > 1$.

(b) On $[1, \infty)$ the convergence is not uniform, since the pointwise limit is discontinuous,

$$f(x) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$$

(c) The convergence is not even uniform on the open interval $(1, \infty)$,

$$\sup_{x\in[1,\infty)} |f_n(x) - f(x)| = \sup\left\{\frac{1}{1+x^n} : x > 1\right\} = \frac{1}{2} \not\xrightarrow[n\to\infty]{} 0$$

(d) Similarly, for any $a \in (0, 1)$, the convergence $f_n \to f$ is uniform on $[0, a]$, this time to the (continuous) constant function $f(x) = 1$,

$$\sup_{x\in[0,a]} |f_n(x) - 1| = \left|1 - \frac{1}{1+a^n}\right| = \frac{a^n}{1+a^n} \xrightarrow[n\to\infty]{} 0$$

(e) Finally, on $(-1, 1)$ the convergence is not uniform,

$$\sup_{x\in[0,1)} |f_n(x) - f(x)| = \sup\left\{\frac{x^n}{1+x^n} : x \in [0, 1)\right\} = \frac{1}{2} \not\xrightarrow[n\to\infty]{} 0$$

Exercises 24 1. For each sequence of functions defined on $[0, \infty)$:

(i) Find the pointwise limit $f(x)$ as $n \to \infty$.

(ii) Determine whether $f_n \to f$ uniformly on $[0, 1]$.

(iii) Determine whether $f_n \to f$ uniformly on $[1, \infty)$.

(a) $f_n(x) = \dfrac{x}{n}$ (b) $f_n(x) = \dfrac{x^n}{1 + x^n}$ (c) $f_n(x) = \dfrac{x^n}{n + x^n}$

(d) $f_n(x) = \dfrac{x}{1 + nx^2}$ (e) $f_n(x) = \dfrac{nx}{1 + nx^2}$

2. Let $f_n(x) = \left(x - \frac{1}{n}\right)^2$. If $f(x) = x^2$, we clearly have $f_n \to f$ pointwise on any domain.

   (a) Prove that the convergence is uniform on $[-1, 1]$.

   (b) Prove that the convergence is non-uniform on $\mathbb{R}$.

3. For each sequence, find the pointwise limit and decide if the convergence is uniform.

   (a) $f_n(x) = \frac{1+2\cos^2(nx)}{\sqrt{n}}$ for $x \in \mathbb{R}$.

   (b) $f_n(x) = \cos^n(x)$ on $[-\pi/2, \pi/2]$.

4. For each $n \in \mathbb{N}$, consider the continuous function

   $$f_n : [0,1] \to \mathbb{R} : x \mapsto nx^n(1 - x)$$

   (a) Given $0 \le x < 1$, let $a \in (x, 1)$. Explain why $\exists N$ such that

   $$n > N \implies |f_{n+1}(x)| \le a\,|f_n(x)|$$

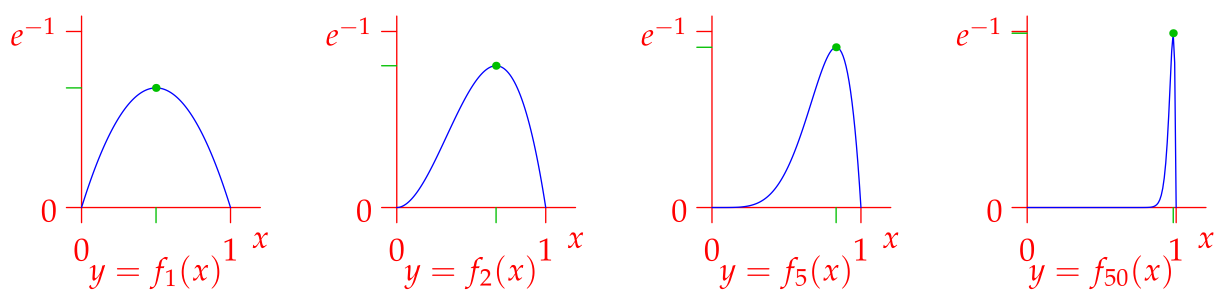
   Hence conclude that the pointwise limit of (f_n) is the zero function.

   (b) Use elementary calculus ($f_n'(x) = 0 \iff \ldots$) to prove that the maximum value of f_n is located at $x_n = \frac{n}{1+n}$. Hence compute

   $$\sup_{x \in [0,1]} |f_n(x) - f(x)|$$

   and use it to show that the convergence $f_n \to 0$ is non-uniform.

This shows that the converse to Theorem 2.15 is false, even on a bounded interval: the continuous sequence (f_n) converges non-uniformly *to a continuous function. Sketches of several f_n are below.*



5. Explain where the proof of Theorem 2.15 fails if $f_n \to f$ *non-uniformly.*

25 More on Uniform Convergence

While we haven't yet developed calculus, our familiarity with basic differentiation and integration makes it natural to pause to consider the interaction of these operations with sequences of functions.

We also consider a Cauchy-criterion for uniform convergence, which leads to the useful Weierstraß M-test.

Example 2.18. Recall that $f_n(x) = x^n$ converges uniformly to $f(x) = 0$ on any interval $[0, a]$ where $a < 1$. We easily check that

$$\int_0^a f_n(x)\,\mathrm{d}x = \frac{1}{n+1}a^{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^a f(x)\,\mathrm{d}x$$

In fact the sequence of derivatives converge here also

$$\frac{\mathrm{d}}{\mathrm{d}x} f_n(x) = nx^{n-1} \xrightarrow[n\to\infty]{} 0 = f'(x)$$

It is perhaps surprising that integration interacts more nicely with uniform limits than does differentiation. We therefore consider integration first.

Theorem 2.19. *Let $f_n \to f$ uniformly on $[a, b]$ where the functions f_n are integrable. Then f is integrable on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

Proof. Given $\epsilon > 0$, note that $\int_a^b \frac{\epsilon}{2(b-a)}\,\mathrm{d}x = \frac{\epsilon}{2}$. Since $f_n \to f$ uniformly, $\exists N$ such that[5]

$$\begin{aligned}
\forall x \in [a, b],\ n > N &\implies |f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)} \\
&\implies f_n(x) - \frac{\epsilon}{2(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{2(b-a)} \\
&\implies \int_a^b f_n(x)\,\mathrm{d}x - \frac{\epsilon}{2} \le \int_a^b f(x)\,\mathrm{d}x \le \int_a^b f_n(x)\,\mathrm{d}x + \frac{\epsilon}{2} \\
&\implies \left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| \le \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

■

The appearance of uniform convergence in the proof is subtle. If $N = N(\epsilon)$ were allowed to depend on x, then the integral $\int_a^b f_n(x)\,\mathrm{d}x$ would be meaningless: Which n would we consider? Larger than $N(x, \epsilon)$ for *which* x? Taking n 'larger' than *all* the $N(x, \epsilon)$ might produce the absurdity $n = \infty$!

[5] This assumes f is already integrable. Once we've properly defined (Riemann) integrability at the end of the course, we can insert the following

$$\int_a^b f_n(x)\,\mathrm{d}x - \frac{\epsilon}{2} \le L(f) \le U(f) \le \int_a^b f_n(x)\,\mathrm{d}x + \frac{\epsilon}{2} \implies 0 \le U(f) - L(f) \le \epsilon \implies U(f) = L(f)$$

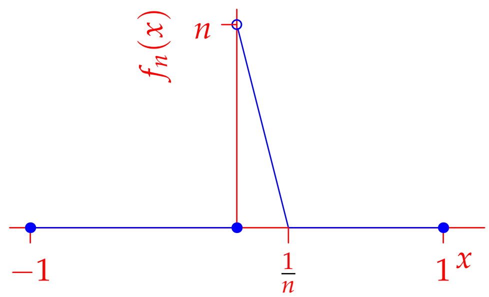
where $U(f)$ and $L(f)$ are the upper and lower Darboux integrals of f; their equality shows that f is integrable on $[a, b]$.

Examples 2.20. 1. Uniform convergence is not required for the integrals to converge as we'd like. For instance, recall that extending the previous example to the domain $[0, 1]$ results in non-uniform convergence; however, we still have

$$\int_0^1 f_n(x)\,\mathrm{d}x = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^1 f(x)\,\mathrm{d}x$$

2. To obtain a sequence of functions $f_n \to f$ for which $\int f_n \not\to \int f$ requires a bit of creativity. Consider the sequence

$$f_n : [-1,1] \to \mathbb{R} : x \mapsto \begin{cases} n - n^2 x & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$



If $0 < x < 1$, then for large $n \in \mathbb{N}$ we have

$$x \geq \frac{1}{n} \implies f_n(x) = 0$$

We conclude that $f_n \to 0$ pointwise. Since the area under f_n is a triangle with base $\frac{1}{n}$ and height n, the integral is constant and *non-zero*;

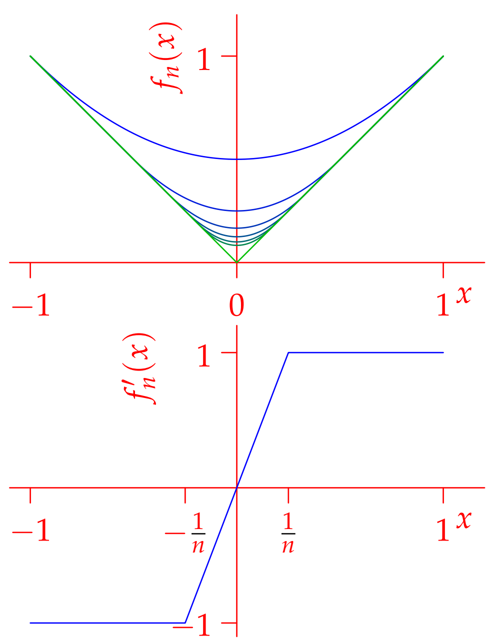
$$\int_{-1}^1 f_n(x)\,\mathrm{d}x = \frac{1}{2} \neq 0 = \int_{-1}^1 f(x)\,\mathrm{d}x$$

It should be obvious why the convergence $f_n \to 0$ is non-uniform; why?

Derivatives and Uniform Limits We've already seen that a uniform limit of differentiable functions *might* be differentiable (Example 2.18), but this shouldn't be expected in general since even uniform limits of differentiable functions can have corners!

Example 2.21. For each $n \in \mathbb{N}$, consider the function

$$f_n : [-1,1] \to \mathbb{R} : x \mapsto \begin{cases} |x| & \text{if } |x| \geq \frac{1}{n} \\ \frac{n}{2}x^2 + \frac{1}{2n} & \text{if } |x| < \frac{1}{n} \end{cases}$$



- f_n converges pointwise to $f(x) = |x|$.
- $f_n \to f$ uniformly since

$$\sup_{x\in[-1,1]} |f_n(x) - f(x)| = \frac{1}{2n} \to 0$$

- Each f_n is differentiable: $f_n'(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ nx & \text{if } |x| < \frac{1}{n} \\ -1 & \text{if } x \leq -\frac{1}{n} \end{cases}$
- The uniform limit f is *not differentiable* at $x = 0$.

Transferring differentiability to the limit of a sequence of functions is a bit messy.

Theorem 2.22. *Suppose (f_n) is a sequence and g is a function on $[a,b]$ for which:*

- *$f_n \to f$ pointwise;*
- *Each f_n is differentiable with continuous derivative;*[6]
- *$f_n' \to g$ uniformly.*

Then $f_n \to f$ uniformly on $[a,b]$ and f is differentiable with derivative g.

The issue in the previous example is that the *pointwise limit* of the derived sequence (f_n') is discontinuous at $x = 0$ and therefore $f_n' \to g$ isn't uniform!

Proof. For any $x \in [a,b]$, the fundamental theorem of calculus tells us that

$$\int_a^x f_n'(t)\,\mathrm{d}t = f_n(x) - f_n(a)$$

By Theorem 2.19, the left side converges to $\int_a^x g(t)\,\mathrm{d}t$, while the right converges to $f(x) - f(a)$. Since $f_n' \to g$ uniformly, we see that g is continuous and we can apply the fundamental theorem again: $\int_a^x g(t)\,\mathrm{d}t = f(x) - f(a)$ is differentiable with derivative g.
The uniformity of the convergence $f_n \to f$ follows from Exercise 10. ■

Uniformly Cauchy Sequences and the Weierstraß M-Test

Recall that one may use Cauchy sequences to demonstrate convergence *without knowing the limit in advance.* An analogous discussion is available for sequences of functions.

Definition 2.23. A sequence of functions (f_n) is *uniformly Cauchy* on U if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall x \in U,\ m,n > N \implies |f_n(x) - f_m(x)| < \epsilon$$

Example 2.24. Let $f_n(x) = \sum_{k=1}^{n} \frac{1}{k^2} \sin k^2 x$ be defined on $\mathbb{R}$. Given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$, then

$$\begin{aligned} m > n > N \implies |f_m(x) - f_n(x)| &= \left| \sum_{k=n+1}^{m} \frac{1}{k^2} \sin k^2 x \right| \le \sum_{k=n+1}^{m} \frac{1}{k^2} \le \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} = \frac{1}{n} - \frac{1}{m} < \frac{1}{N} = \epsilon \end{aligned}$$

whence (f_n) is uniformly Cauchy.

[6]Without the continuity assumption, the fundamental theorem of calculus doesn't apply and the proof requires an alternative approach. One can also weaken the hypotheses: if $f_n' \to g$ uniformly and that $(f_n(x))$ converges for *at least one* $x \in [a,b]$, then there exists f such that $f_n \to f$ is uniform and $f' = g$.

As with sequences of real numbers, uniformly Cauchy sequences converge; in fact uniformly!

Theorem 2.25. *A sequence (f_n) is uniformly Cauchy on U if and only if it converges uniformly to some $f : U \to \mathbb{R}$.*

Proof. ($\Rightarrow$) Let (f_n) be uniformly Cauchy on U. For each $x \in U$, the sequence $(f_n(x)) \subseteq \mathbb{R}$ is Cauchy and thus convergent. Define $f : U \to \mathbb{R}$ via

$$f(x) := \lim_{n\to\infty} f_n(x)$$

We claim that $f_n \to f$ uniformly. Let $\epsilon > 0$ be given, then

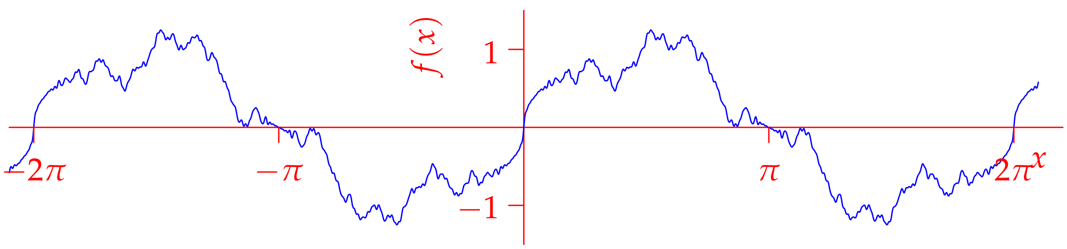
$$\begin{aligned} \exists N \in \mathbb{N} \text{ such that } m > n > N &\implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2} \\ &\implies f_n(x) - \frac{\epsilon}{2} < f_m(x) < f_n(x) + \frac{\epsilon}{2} \\ &\implies f_n(x) - \frac{\epsilon}{2} \le f(x) \le f_n(x) + \frac{\epsilon}{2} \qquad \text{(take limits as } m \to \infty) \\ &\implies |f_n(x) - f(x)| \le \frac{\epsilon}{2} < \epsilon \end{aligned}$$

($\Leftarrow$) This is Exercise 2. ∎

Example (2.24, mk. II). Since (f_n) is uniformly Cauchy on $\mathbb{R}$, it converges uniformly to some $f : \mathbb{R} \to \mathbb{R}$. It seems reasonable to write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 x$$

The graph of this function looks somewhat bizarre:



Since each f_n is (uniformly) continuous, Theorem 2.15 says that f is also (uniformly) continuous. By Theorem 2.19, $f(x)$ is integrable, indeed

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} -k^{-4} \cos k^2 x \Big|_a^b = \sum_{n=1}^{\infty} \frac{1}{n^4}(\cos n^2 a - \cos n^2 b)$$

which converges (comparison test) for all a, b. By contrast, the derived sequence

$$f_n'(x) = \sum_{k=1}^{n} \cos k^2 x$$

does not converge for *any* x since $\cos n^2 x \underset{k\to\infty}{\not\longrightarrow} 0$. We should thus expect (though we offer no proof) that f is nowhere differentiable.

The example generalizes. Suppose (g_k) is a sequence of functions on U and define the *series* $\sum g_k(x)$ as the pointwise limit of the sequence (f_n) of partial sums

$$\sum_{k=k_0}^{\infty} g_k(x) := \lim_{n\to\infty} f_n(x) \quad \text{where} \quad f_n(x) = \sum_{k=k_0}^{n} g_k(x)$$

whenever the limit exists. The series is said to converge uniformly whenever (f_n) does so. Theorems 2.15, 2.19 and 2.22 immediately translate.

Corollary 2.26. *Let $\sum g_k$ be a series of functions converging uniformly on U. Then:*

1. *If each g_k is (uniformly) continuous then $\sum g_k$ is (uniformly) continuous.*
2. *If each g_k is integrable, then $\int \sum g_k(x)\,\mathrm{d}x = \sum \int g_k(x)\,\mathrm{d}x$.*
3. *If each g_k is continuously differentiable, and the sequence of derived partial sums f_n' converges uniformly, then $\sum g_k$ is differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}\sum g_k(x) = \sum g_k'(x)$.*

As an application of the uniform Cauchy criterion, we obtain an easy test for uniform convergence.

Theorem 2.27 (Weierstraß M-test). *Suppose (g_k) is a sequence of functions on U. Moreover assume:*

1. *(M_k) is a non-negative sequence such that $\sum M_k$ converges.*
2. *Each g_k is bounded by M_k; that is $|g_k(x)| \le M_k$.*

Then $\sum g_k(x)$ converges uniformly on U.

Proof. Let $f_n(x) = \sum_{k=k_0}^{n} g_k(x)$ define the sequence of partial sums. Since $\sum M_k$ converges, its sequence of partial sums is Cauchy (the *Cauchy criterion* for infinite series); given $\epsilon > 0$,

$$\exists N \text{ such that } m > n > N \implies \sum_{k=n+1}^{m} M_k < \epsilon$$

However, by assumption,

$$m > n > N \implies |f_m(x) - f_n(x)| = \left|\sum_{k=n+1}^{m} g_k(x)\right| \le \sum_{k=n+1}^{m} |g_k(x)| \le \sum_{k=n+1}^{m} M_k < \epsilon$$

∎

The sequence of partial sums is uniformly Cauchy and thus uniformly convergent.

Example 2.28. Given the series $\sum_{n=1}^{\infty} \frac{1+\cos^2(nx)}{n^2}\sin(nx)$, we clearly have

$$\left|\frac{1+\cos^2(nx)}{n^2}\sin(nx)\right| \le \frac{2}{n^2} \text{ for all } x \in \mathbb{R}$$

Since $\sum \frac{2}{n^2}$ converges, the M-test shows that the original series converges uniformly on $\mathbb{R}$.

Exercises 25 1. For each $n \in \mathbb{N}$, let $f_n(x) = nx^n$ when $x \in [0,1)$ and $f_n(1) = 0$.

   (a) Prove that $f_n \to 0$ pointwise on $[0,1]$.
   (*Hint: recall Exercise 24.4 if you're not sure how to prove this*)

   (b) By considering the integrals $\int_0^1 f_n(x)\,\mathrm{d}x$ show that $f_n \to 0$ is not uniform.

2. Prove that if $f_n \to f$ uniformly, then the sequence (f_n) is uniformly Cauchy.

3. (a) Suppose (f_n) is a sequence of bounded functions on U and suppose that $f_n \to f$ converges uniformly on U. Prove that f is bounded on U.

   (b) Give an example of a sequence of bounded functions (f_n) converging pointwise to f on $[0,\infty)$, but for which f is *unbounded.*

4. The sequence defined by $f_n(x) = \frac{nx}{1+nx^2}$ (Exercise 24.1) converges uniformly on any closed interval $[a,b]$ where $0 < a < b$.

   (a) Check explicitly that $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, where $f = \lim f_n$.

   (b) Is the same thing true for derivatives?

5. Let $f_n(x) = n^{-1}\sin n^2 x$ be defined on $\mathbb{R}$.

   (a) Prove that f_n converges uniformly on $\mathbb{R}$.

   (b) Check that $\int_0^x f_n(t)\,\mathrm{d}t$ converges for any $x \in \mathbb{R}$.

   (c) Does the derived sequence (f_n') converge? Explain.

6. Use the M-test to prove that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ defines a continuous function on $[-1,1]$.

7. Prove that $\sum_{n=1}^{\infty} \frac{x^n \sin x}{(n+1)^3 2^n}$ converges uniformly to a continuous function on the interval $[-2,2]$.

8. Prove that if $\sum g_k$ converges uniformly on a set U and if h is a bounded function on U, then $\sum hg_k$ converges uniformly on U.
   (*Warning: you cannot simply write* $\sum hg_k = h\sum g_k$)

9. Consider Example 2.20.2.

   (a) Check explicitly that the convergence isn't uniform by computing $\sup_{x\in[-1,1]} |f_n(x) - f(x)|$

   (b) Prove that $f_n \to 0$ pointwise on $(0,1]$ using the ϵ–N definition of convergence: that is, given $\epsilon > 0$ and $x \in (0,1]$, find an explicit $N(x,\epsilon)$ such that

   $$n > N \implies |f(x)| < \epsilon$$

   What happens to your choice of $N(x,\epsilon)$ as $x \to 0^+$?

10. Suppose (f_n') converges uniformly on $[a,b]$ and that each f_n' is continuous.

    (a) Use the fact that (f_n') is uniformly Cauchy to prove that (f_n) is uniformly Cauchy and thus converges uniformly to some function f.
    (*Hint:* $|f_n(x) - f_m(x)| = \left|\int_a^x f_n'(t) - f_m'(t)\,\mathrm{d}t\right| \ldots$)

    (b) Explain why we need not have assumed the existence of f in Theorem 2.22.

26 Differentiation and Integration of Power Series

In this section we specialize our recent results to power series. While everything will be stated for series centered at $x = 0$, all are easily translated to arbitrary centers.

Theorem 2.29. *Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$ and let $T \in (0, R)$. Then:*

1. *The series converges uniformly on $[-T, T]$.*
2. *The series is uniformly continuous on $[-T, T]$ and continuous on $(-R, R)$.*

Proof. This is an easy application of the M-test. For each k, define $M_k = |a_k| T^k$,

$$T < R \implies \sum a_n T^n \text{ converges absolutely } \implies \sum M_k \text{ converges}$$

By the M-test and Corollary 2.26, the power series converges uniformly on $[-T, T]$ to a uniformly continuous function.

Finally, every $x \in (-R, R)$ lies in some such interval (take $T = |x|$), whence the power series is continuous on $(-R, R)$.

■

Example 2.30. On its interval of convergence $(-1, 1)$, the geometric series $\sum\limits_{n=0}^{\infty} x^n$ converges pointwise to $\frac{1}{1-x}$; convergence is uniform on any interval $[-T, T] \subseteq (-1, 1)$.
We needn't use the Theorem for this is simple to verify directly: writing f, f_n for the series and its partial sums,

$$|f_n(x) - f(x)| = \left| \frac{1 - x^{n+1}}{1 - x} - \frac{1}{1 - x} \right| = \left| \frac{x^{n+1}}{1 - x} \right|$$

$$\implies \sup_{x \in [-T, T]} |f_n(x) - f(x)| = \frac{T^{n+1}}{1 - T} \xrightarrow[n \to \infty]{} 0$$

By contrast, the convergence is non-uniform on $(-1, 1)$;

$$\sup_{x \in (-1, 1)} |f_n(x) - f(x)| = \infty$$

Theorem 2.31. *Suppose $\sum\limits_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then the series is integrable and differentiable term-by-term on the interval $(-R, R)$. Indeed for any $x \in (-R, R)$,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int_0^x \sum_{n=0}^{\infty} a_n t^n \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

where both series also have radius of convergence R.

Proof. Let $f(x) = \sum a_n x^n$ have radius of convergence R, and observe that

$$\limsup |na_n|^{1/n} = \lim n^{1/n} \limsup |a_n|^{1/n} = \frac{1}{R}$$

whence $\sum na_n x^n$ also has radius of convergence R. At any given non-zero $x \in (-R, R)$, we may write

$$\sum_{n=1}^{\infty} na_n x^{n-1} = x^{-1} \sum_{n=1}^{\infty} na_n x^n$$

to see that the derived series also has radius of convergence R. On any interval $[-T, T] \subseteq (-R, R)$, the derived series converges uniformly (Theorem 2.29). Since each $a_n x^n$ is continuously differentiable, Corollary 2.26 says that f is differentiable on $[-T, T]$ and that

$$f'(x) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} a_n x^n = \sum_{n=1}^{\infty} na_n x^{n-1}$$

Since any $x \in (-R, R)$ lies in some such interval $[-T, T]$, we are done.

The corresponding result for integrals is Exercise 6. ■

We postpone the canonical examples until after the next result.

Continuity at Endpoints?

There is one small hole in our analysis. If a series has radius of convergence R we know that it converges and is continuous on $(-R, R)$. But what if it additionally converges at $x = \pm R$? Is the series continuous at the endpoints? The answer is an unequivocal yes, though this small benefit requires a lot of work!

Theorem 2.32 (Abel's Theorem). *Power series are continuous on their full interval of convergence.*

Examples 2.33. 1. Apply our results to the geometric series;

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x} \frac{1}{1-x} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\ln(1-x) = -\int_0^x \frac{1}{1-t}\,\mathrm{d}t = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)$$

with both series valid on $(-1, 1)$. In fact the first series has the same interval of convergence, while the second is $[-1, 1)$. By Abel's Theorem and the fact that logarithms are continuous, we have equality at $x = -1$ and the famous identity

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This example shows that while the integrated and differentiated series have the same radius of convergence as the original, convergence at the endpoints need not be the same in all cases.

2. Substitute $x \mapsto -x^2$ in the geometric series and integrate term-by-term: if $|x| < 1$, then

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \implies \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

In fact the arctangent series also converges at $x = \pm 1$; Abel's Theorem says it is continuous on $[-1, 1]$. Since arctangent is continuous (on $\mathbb{R}$!) we recover another famous identity

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

As with the identity for $\ln 2$, this is a very slowly converging alternating series and therefore doesn't provide an efficient method for approximating π.

3. The series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ has radius of convergence ∞. Differentiate to obtain

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$$

This series is also valid for all $x \in \mathbb{R}$. Differentiating again,

$$f''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = -f(x)$$

Recalling that $f(x) = \cos x$ is the unique solution to the initial value problem

$$\begin{cases} f''(x) = -f(x) \\ f(0) = 1,\ f'(0) = 0 \end{cases}$$

We conclude that, $\forall x \in \mathbb{R}$,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \sin x = -f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

These expressions can instead be taken as the *definitions* of sine and cosine. As promised earlier in the course, continuity and differentiability now come for free. One difficulty with this definition is believing that it has anything to do with right-triangles!

We can similarly define other common transcendental functions using power series: for instance

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Example 2.33.1 could be taken as a definition of the logarithm on the interval $(0, 2]$,

$$\ln x = \ln(1 - (1-x)) = -\sum_{n=1}^{\infty} \frac{1}{n}(1-x)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x-1)^n$$

though this is unnecessary since it is more natural to define ln as the inverse of the exponential.

Proof of Abel's Theorem (non-examinable)

This requires a lot of work, so feel free to omit on a first reading!

First observe that there is nothing to check unless $0 < R < \infty$. By the change of variable $x \mapsto \pm\frac{x}{R}$, it is enough for us to prove the following:

$$\sum_{n=0}^{\infty} a_n \text{ convergent and } f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ on } (-1,1) \implies \lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$$

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ and write $s = \lim s_n = \sum a_n$. It is an easy exercise to check that

$$\sum_{k=0}^{n} a_k x^k = s_n x^n + (1-x)\sum_{k=0}^{n-1} s_k x^k$$

If $|x| < 1$, then (since $s_n \to s$) $\lim s_n x^n = 0$, whence we obtain

$$\forall x \in (-1,1),\ f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$$

Let $\epsilon \in (0,1)$ be given and fix $x \in (0,1)$. Then

$$\exists N \in \mathbb{N} \text{ such that } n > N \implies |s_n - s| < \frac{\epsilon}{2} \qquad (*)$$

Use the geometric series formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and write $h(x) = (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n\right|$ to observe

$$\begin{aligned}
|f(x) - s| &= \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s\right| = \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s(1-x)\sum_{n=0}^{\infty} x^n\right| \\
&= \left|(1-x)\sum_{n=0}^{\infty}(s_n - s)x^n\right| = (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n + \sum_{n=N+1}^{\infty}(s_n - s)x^n\right| \\
&\le (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n\right| + (1-x)\left|\sum_{n=N+1}^{\infty}(s_n - s)x^n\right| && (\triangle\text{-inequality}) \\
&< h(x) + \frac{\epsilon}{2}(1-x)\left|\sum_{n=N+1}^{\infty} x^n\right| && (\text{by } (*)) \\
&\le h(x) + \frac{\epsilon}{2}
\end{aligned}$$

Since $h > 0$ is continuous and $h(1) = 0$, $\exists \delta > 0$ such that $x \in (1-\delta, 1) \implies h(x) < \frac{\epsilon}{2}$ (the computation of a suitable δ is another exercise).

We conclude that $\lim_{x\to 1^-} f(x) = s$. ∎

Exercises 26 1. (a) Prove that $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

(b) Evaluate $\sum_{n=1}^{\infty} \frac{n}{2^n}$, $\sum_{n=1}^{\infty} \frac{n}{4^n}$ and $\sum_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$

2. (a) Starting with a power series centered at $x = 0$, evaluate the integral $\int_0^{1/2} \frac{1}{1+x^4}\,\mathrm{d}x$ as an infinite series.

   (b) (Harder) Repeat part (a) but for $\int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x$. What extra ingredients do you need?

3. The probability that a standard normally distributed random variable X lies in the interval $[a, b]$ is given by the integral

$$\mathbb{P}(a \le X \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{x^2}{2}\right)\,\mathrm{d}x$$

Find $\mathbb{P}(-1 \le X \le 1)$ as an infinite series.

4. Define $c(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $s(x) = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

   (a) Prove that $c'(x) = s(x)$ and that $s'(x) = c(x)$.

   (b) Prove that $c(x)^2 - s(x)^2 = 1$ for all $x \in \mathbb{R}$.

(*These functions are the hyperbolic sine and cosine:* $s(x) = \sinh x$ *and* $c(x) = \cosh x$)

5. Let $a, b \in (-1, 1)$. Extending Example 2.30, show that the convergence $\sum x^n = \frac{1}{1-x}$ is non-uniform on any interval of the form $(-1, a)$ or $(b, 1)$.

6. Prove the integration part of Theorem 2.31.

7. Prove or disprove: If a series converges absolutely at the *endpoints* of its interval of convergence then its convergence is uniform on the entire interval.

8. Complete the proof of Abel's Theorem:

   (a) Let $s_n = \sum_{k=0}^{n} a_k$ be the partial sum of the series $\sum a_n$. For each n, prove that,

$$\sum_{k=0}^{n} a_k x^k = s_n x^n + (1-x) \sum_{k=0}^{n-1} s_k x^k$$

   (b) Suppose $x > 0$. Let $S = \max\{|s_n - s| : n \le N\}$ and prove that $h(x) \le S(1 - x^{N+1})$. Hence find an explicit δ that completes the final step.

27 The Weierstraß Approximation Theorem

A major theme of analysis is *approximation*; for instance power series are an example of (uniform) approximation by polynomials. It is reasonable to ask whether any function can be so approximated. In 1885, Weierstraß answered a specific case in the affirmative.

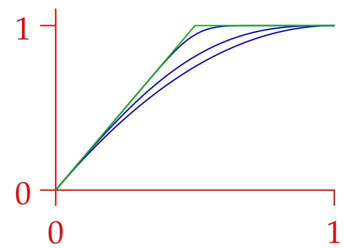
Theorem 2.34 (Weierstraß). *If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists a sequence of polynomials converging uniformly to f on $[a, b]$.*

Suitable polynomials can be defined in various ways. By scaling the domain, it is enough to do this on $[a, b] = [0, 1]$ where perhaps the simplest approach is via the *Bernstein Polynomials*,

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \qquad \left(\binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ is the binomial coefficient}\right)$$

We omit the proof due to length; Weierstraß' original argument was completely different. Instead we compute a couple of examples and give an important interpretation/application.

Examples 2.35. 1. Suppose $f(x) = 2x$ if $x < \frac{1}{2}$ and $f(x) = 1$ otherwise.



$$B_1 f(x) = f(0)(1-x)f(0) + f(1)x = x$$

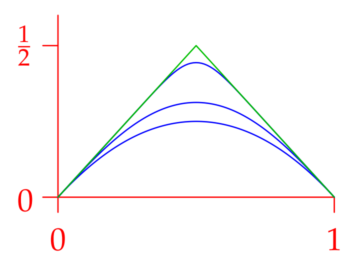
$$\begin{aligned} B_2 f(x) &= f(0)(1-x)^2 + 2f(\tfrac{1}{2})x(1-x) + f(1)x^2 \\ &= 2x(1-x) + x^2 \\ &= x(2-x) \end{aligned}$$

$$\begin{aligned} B_3 f(x) &= f(0)(1-x)^3 + 3f(\tfrac{1}{3})x(1-x)^2 + 3f(\tfrac{2}{3})x^2(1-x) + f(1)x^3 \\ &= 0(1-x)^3 + 2x(1-x)^2 + 3x^2(1-x) + x^3 \\ &= x(2-x) = B_2 f(x) \end{aligned}$$

$$\begin{aligned} B_4 f(x) &= 0(1-x)^4 + 2x(1-x)^3 + 6x^2(1-x)^2 + 4x^3(1-x) + x^4 \\ &= x(x^3 - 2x^2 + 2) \end{aligned}$$

The Bernstein polynomials $B_2 f(x)$, $B_4 f(x)$ and $B_{50} f(x)$ are drawn.

2. Now assume $f(x) = x$ if $x < \frac{1}{2}$ and $1 - x$ otherwise.



$$B_1 f(x) = f(0)(1-x) + f(1)x = 0$$

$$B_2 f(x) = x(1-x)$$

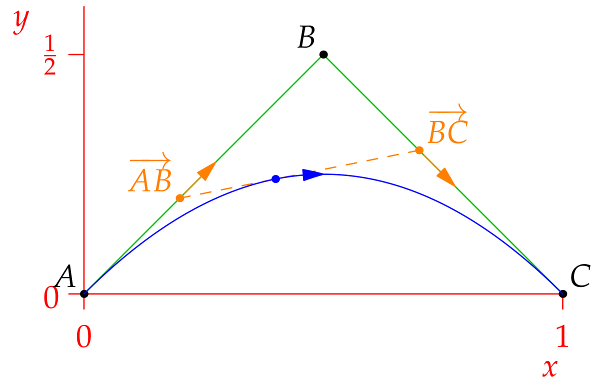
$$\begin{aligned} B_3 f(x) &= 0(1-x)^3 + x(1-x)^2 + x^2(1-x) + 0x^3 \\ &= x(1-x) = B_2 f(x) \end{aligned}$$

$$\begin{aligned} B_4 f(x) &= f(0)(1-x)^4 + f(\tfrac{1}{4}) \cdot 4x(1-x)^3 + f(\tfrac{1}{2}) \cdot 6x^2(1-x)^2 + f(\tfrac{3}{4}) \cdot 4x^3(1-x) + f(1)x^4 \\ &= x(1-x)^3 + 3x^2(1-x)^2 + x^3(1-x) \\ &= x(1-x)(1 + x - x^2) \end{aligned}$$

Bézier curves (just for fun!)

The Bernstein polynomials arise naturally when considering *Bézier curves.* These have many applications, particularly in computer graphics. Given three points A, B, C, define points on the line segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$ for each $t \in [0, 1]$, via

$$\overrightarrow{AB}(t) = (1-t)A + tB \qquad \overrightarrow{BC}(t) = (1-t)B + tC$$



These points move at a constant speed along the corresponding segments. Now consider a point on the *moving* segment between the points defined above:

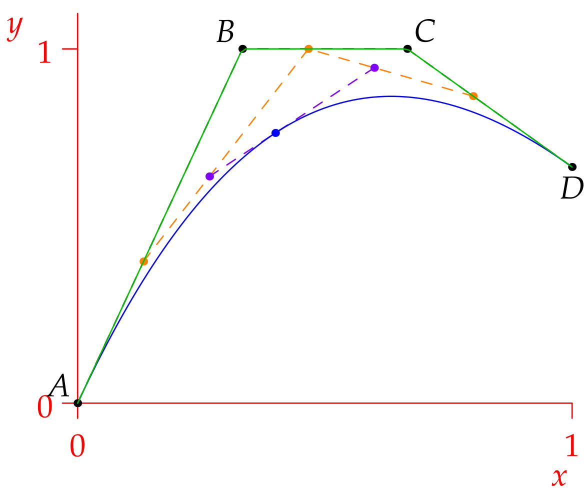
$$R(t) := (1-t)\overrightarrow{AB}(t) + t\overrightarrow{BC}(t) = (1-t)^2 A + 2t(1-t)B + t^2 C$$

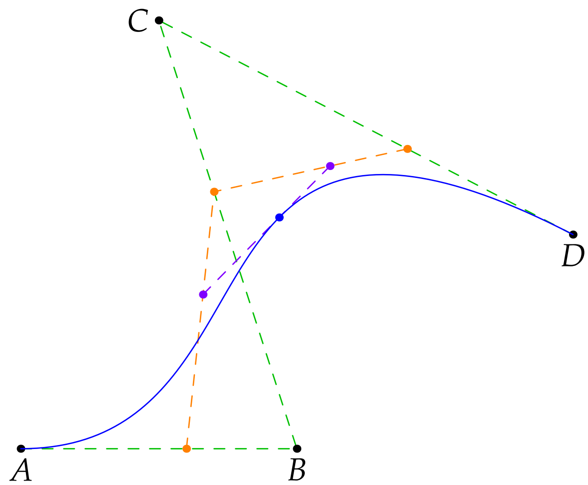
This is the *quadratic Bézier curve with control points A, B, C.* The 2^{nd} Bernstein polynomial for a function f is simply the quadratic Bézier curve with control points $(0, f(0))$, $\left(\frac{1}{2}, f(\frac{1}{2})\right)$ and $(1, f(1))$. The picture[7] above shows $B_2 f(x)$ for the above example.

We can repeat the construction with more control points: with four points A, B, C, D, one constructs $\overrightarrow{AB}(t)$, $\overrightarrow{BC}(t)$, $\overrightarrow{CD}(t)$, then the second-order points between these, and finally the cubic Bézier curve

$$\begin{aligned} R(t) :&= (1-t)\left((1-t)\overrightarrow{AB}(t) + t\overrightarrow{BC}(t)\right) + t\left((1-t)\overrightarrow{BC}(t) + t\overrightarrow{CD}(t)\right) \\ &= (1-t)^3 A + 3t(1-t)^2 B + 3t^2(1-t)C + t^3 D \end{aligned}$$

where we now recognize the relationship to the 3^{rd} Bernstein polynomial.





The pictures show cubic Bézier curves: the first is the graph of the Bernstein polynomial

$$B_3 f(x) = 0(1-x)^3 + 3x(1-x)^2 + 3x^2(1-x) + \tfrac{2}{3}x^3$$

while the second is for the four given control points A, B, C, D.

[7] To see these pictures move, visit `https://www.math.uci.edu/~ndonalds/math140b/bezier.html`

Exercises 27 1. Show that the closed bounded interval assumption in the approximation theorem is required by giving an example of a continuous function $f : (-1, 1) \to \mathbb{R}$ which is *not* the uniform limit of a sequence of polynomials.

2. If $g : [a, b] \to \mathbb{R}$ is continuous, then $f(x) := g\big((b-a)x + a\big)$ is continuous on $[0, 1]$. If $P_n \to f$ uniformly on $[0, 1]$, prove that $Q_n \to g$ uniformly on $[a, b]$, where

$$Q_n(x) = P_n\left(\frac{x-a}{b-a}\right)$$

3. Use the binomial theorem to check that every Bernstein polynomial for $f(x) = x$ is $B_n f(x) = x$ itself!

4. Find a parametrization of the cubic Bézier curve with control points $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. Now sketch the curve.
(*Use a computer algebra package if you like!*)

5. (Hard) Show that the Bernstein polynomials for $f(x) = x^2$ are given by

$$B_n f(x) = \frac{1}{n}x + \frac{n-1}{n}x^2$$

and thus verify explicitly that $B_n f \to f$ uniformly.