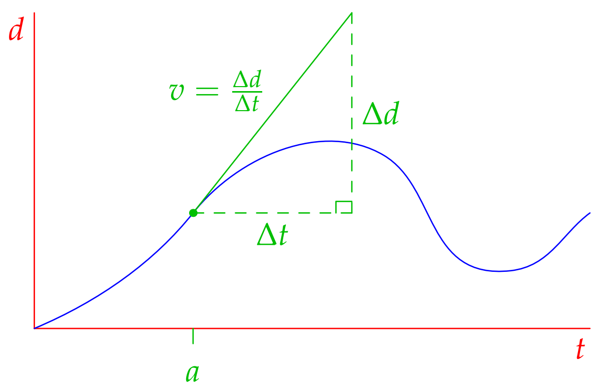
3 Differentiation

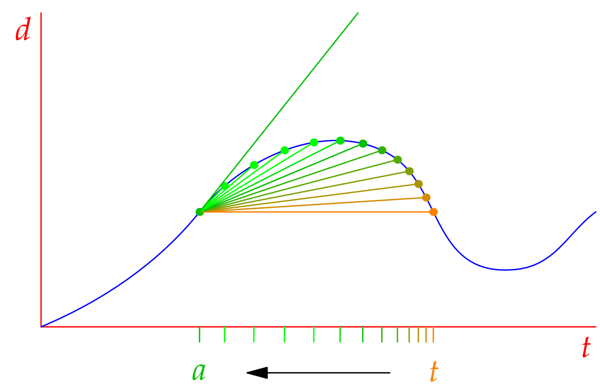
Differentiation grew out of the problem of *instantaneous velocity*. Velocity can only easily be measured as an *average* over a time interval:[1] if an object travels Δd meters in Δt seconds, then its average velocity is $v_{av} = \frac{\Delta d}{\Delta t}\ \mathrm{ms}^{-1}$. An early 'definition' (dating to the 1300's) makes the instantaneous velocity equal to the constant velocity that would be observed if a body were to stop accelerating: while useless for the purposes of measurement, this is essentially Newton's first law regarding inertial motion (1687). We also see the concept of the *tangent line* beginning to appear: if one graphs position against time, then a couple of things should be clear:

- The graph of inertial (constant speed) motion is a straight line whose slope is the velocity.
- The tangent line to a curve at a point has slope equal to the instantaneous velocity at that point.

The problem of finding, defining and computing instantaneous velocity thus morphed into the consideration of tangent lines to curves. With the advent of analytic geometry in the early 1600's, mathematicians such as Fermat and Descartes pioneered versions of the familiar *secant ('cutting') line* method for computing tangents.



Instantaneous velocity equals constant velocity corresponding to tangent line



Secant lines approximate tangent line as $t \to a$

The average velocity of the particle over the time interval $[a, t]$ is the slope of the secant line, namely

$$v_{av}(a,t) = \frac{d(t) - d(a)}{t - a}$$

Since the secant lines approximate the tangent line as t approaches a, it seems reasonable that we should compute the instantaneous velocity in this manner:

$$v(a) = \lim_{t \to a} v_{av}(a,t) = \lim_{t \to a} \frac{d(t) - d(a)}{t - a}$$

This is, of course, the modern definition of the *derivative*.

[1] Even a modern technique such as Doppler-shift compares measurements separated by the extremely small period of a light or soundwave. These are still therefore *average* velocities, albeit taken over very small time intervals.

Definition 3.1. Let $f : U \to \mathbb{R}$ and let $a \in U$. We say that f *is differentiable at a* if the following limit exists (is *finite*!)

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

We call this limit the *derivative of f at a* and denote its value by either $f'(a)$ or $\frac{\mathrm{d}f}{\mathrm{d}x}\Big|_{x=a}$.

If $f'(a)$ exists for all $a \in U$ then f is *differentiable (on U)*; the derivative becomes a function $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$.

Notation The contrasting styles are partly attributable, to the primary founders of calculus, Issac Newton and Gottfried Leibniz. Each has its pros and cons and you should be comfortable with both.

One-sided derivatives Since the defining limit is *two-sided*, differentiability only makes sense at *interior points* of U. *Left-* and *right-derivatives* may be defined via one-sided limits; differentiability is equivalent to these being equal. All results in this section hold for one-sided derivatives with suitable (sometimes tedious) modifications. It is quite common, though strictly incorrect, to say that f is differentiable on an interval $[a, b)$ if it is differentiable on the interior (a, b) and *right*-differentiable at a; however, we will strictly adhere to differentiable meaning *two-sided.*

Examples 3.2. 1. Let $f(x) = x^2 + 4x$. Then, for any $a \in \mathbb{R}$,

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^2+4x-a^2-4a}{x-a} = \lim_{x\to a} \frac{(x-a)(x+a+4)}{x-a}$$
$$= \lim_{x\to a}(x+a+4) = 2a+4$$

Note how the definition of $\lim_{x\to a}$ allows us to cancel the $x-a$ terms from the numerator and denominator. We conclude that f is differentiable (on $\mathbb{R}$) and that $f'(x) = 2x+4$.

2. Let $g(x) = \frac{x+1}{2x-3}$. Then, for any $a \neq \frac{3}{2}$,

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{1}{x-a}\left[\frac{x+1}{2x-3} - \frac{a+1}{2a-3}\right] = \lim_{x\to a} \frac{5a-5x}{(x-a)(2x-3)(2a-3)}$$
$$= \lim_{x\to a} \frac{-5}{(2x-3)(2a-3)} = \frac{-5}{(2a-3)^2}$$

f is therefore differentiable on its domain $\mathbb{R} \setminus \{\frac{3}{2}\}$ with derivative $f'(x) = \frac{-5}{(2x-3)^2}$.

The familiar expressions

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \qquad f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

are equivalent to the original definition (see Exercise 5). While seemingly simpler, they sometimes lead to nastier calculations: see what happens if you try the previous example in this language…

We now turn to possibly the most well-known result of Freshman Calculus.

Theorem 3.3 (Power Law). *Let $r \in \mathbb{R}$. Then $f(x) = x^r$ is differentiable with $f'(x) = rx^{r-1}$.*

The domains of f and f' depend messily on r, but the above certainly holds on the interval $(0, \infty)$. We leave a complete proof to the exercises and instead consider a few generalizable examples.

Examples 3.4. 1. If $n \in \mathbb{N}$ and $a \in \mathbb{R}$, a simple factorization yields

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})}{x - a} \qquad (*)$$
$$= \lim_{x \to a}(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}) = na^{n-1}$$

We conclude that $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$.

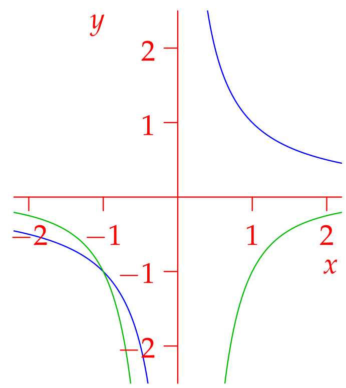
2. If $f(x) = x^{-1}$ and $a \neq 0$, then

$$\lim_{x \to a} \frac{x^{-1} - a^{-1}}{x - a} = \lim_{x \to a} \frac{a - x}{ax(x-a)} = \lim_{x \to a} \frac{-1}{ax} = -\frac{1}{a^2}$$

from which we conclude that $f'(x) = -x^{-2}$.

A similar approach followed by the factorization $(*)$ proves the power law for all negative integer exponents:

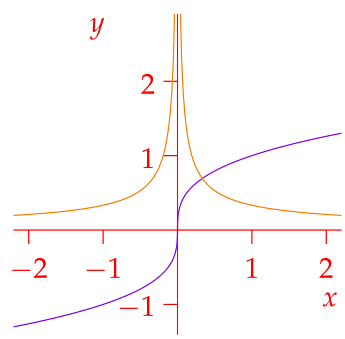
$$\frac{x^{-n} - a^{-n}}{x - a} = \frac{a^n - x^n}{a^n x^n (x - a)} = \cdots$$



3. To differentiate $x^{1/n}$, simply substitute $x = y^n$ and observe case 1. If $g(x) = x^{1/3}$ and $a \neq 0$, then $y = x^{1/3}$ and $b = a^{1/3}$ yield

$$\lim_{x \to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{y \to b} \frac{y - b}{y^3 - b^3} = \frac{1}{3b^2} = \frac{1}{3}a^{-2/3}$$
$$\implies g'(x) = \frac{1}{3}x^{-2/3}$$

Note that g is *not* differentiable at $x = 0$!



We could similarly compute the derivative for all rational exponents, though it is much easier to wait for the chain rule. The power law for irrational exponents is somewhat more ticklish.

Corollary 3.5 (Basic Transcendental Functions). *Recalling our development of power series in the previous chapter, the power law (for positive integers!) is all we need to see that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

It is also possible to develop these results independently of power series (see e.g. Exercise 9).

Failure of differentiability

It is instructive to consider when a function can fail to be differentiable. First a simple result shows that functions are not differentiable at discontinuities.

Theorem 3.6. *If f is differentiable at a then f is continuous at a.*

Proof. Simply take the limit (think carefully why this works!):

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)+f(a)\right] = f'(a)(0-0)+f(a) = f(a) \qquad \blacksquare$$

It remains to consider situations when a function is continuous but not differentiable.

Examples 3.7. The following cover all situations where a function is continuous on an interval and differentiable everywhere *except* at a single interior point; similarly to isolated discontinuities, these are classified by considering the three ways in which the derivative limit might not exist.

1. A *vertical tangent line* occurs when the derivative is infinite. For instance, $g(x) = x^{1/3}$ at $x = 0$.

2. *Corners* occur when the one-sided derivatives are unequal (could be infinite). For instance, $f(x) = |x|$ is not differentiable at zero, the one-sided derivatives being

$$\lim_{x\to 0^+}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^+}\frac{x}{x} = 1 \neq \lim_{x\to 0^-}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^-}\frac{-x}{x} = -1$$

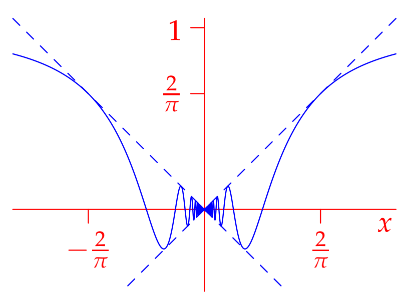
   Indeed f is differentiable everywhere except at zero, with

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

3. A *singularity* is where left- and/or right-derivatives do not exist. The standard example in this case is

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   which is continuous on $\mathbb{R}$ and differentiable everywhere except at zero: the details are in Exercise 8.



Singularities and vertical tangent lines can also prevent one-sided differentiability.

More esoteric examples of non-differentiability are also possible:

- Utilizing series, we can create functions which are continuous on an interval but *nowhere differentiable*! For a classic example, see page 28.

- It is also possible to construct a function which differentiable (and thus continuous) at precisely one point; can you think of an example?

The Basic Rules of Differentiation

Theorem 3.8. *Let f, g be differentiable and k, l be constants.*

1. *(Linearity) The function $kf + lg$ is differentiable with $(kf + lg)' = kf' + lg'$.*
2. *(Product rule) The function fg is differentiable with $(fg)' = f'g + fg'$.*
3. *(Inverse functions) If f is bijective with non-zero derivative, then f^{-1} is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. Parts 1 and 2 follow from the limit laws:

$$\lim_{x\to a} \frac{(kf+lg)(x) - (kf+lg)(a)}{x-a} = \lim_{x\to a} \left[k\frac{f(x)-f(a)}{x-a} + l\frac{g(x)-g(a)}{x-a} \right] = kf'(a) + lg'(a)$$

$$\lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x-a} = \lim_{x\to a} \left[\frac{f(x)-f(a)}{x-a} g(x) + f(a)\frac{g(x)-g(a)}{x-a} \right] = f'(a)g(a) + f(a)g'(a)$$

Note where we used the continuity of g in the second line ($\lim g(x) = g(a)$). Part 3 is an exercise. ■

The inverse function rule is intuitive since the graphs of f and f^{-1} are related by reflection in the line $y = x$; gradients at corresponding points are therefore reciprocal. In Leibniz notation the result reads $\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$.

Examples 3.9.
1. Linearity allows us to differentiate any polynomial: for instance

$$\frac{\mathrm{d}}{\mathrm{d}x}(7x^2 + 13x^4) = 7\frac{\mathrm{d}}{\mathrm{d}x}x^2 + 13\frac{\mathrm{d}}{\mathrm{d}x}x^4 = 14x + 52x^3$$

2. The product rule extends the reach of differentiation somewhat:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^4 \sin x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}x^4\right)\sin x + x^4\frac{\mathrm{d}}{\mathrm{d}x}\sin x = 4x^3 \sin x - x^4 \cos x$$

3. The inverse trigonometric functions can now be differentiated. For instance,

$$y = \sin^{-1} x \implies \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1} x = \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

4. Define natural log to be the inverse of the (bijective!) exponential function $\exp(x)$:

$$y = \ln x \iff x = \exp y$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\exp y} = \frac{1}{x}$$

The full details, and the justification that $\exp x = e^x$, form an optional exercise.

Theorem 3.10 (Chain Rule). *If g is differentiable at a and f is differentiable at $g(a)$ then $f \circ g$ is differentiable at a with derivative*

$$(f \circ g)'(a) = f'\big(g(a)\big)\, g'(a)$$

In Leibniz notation this reads $\frac{\mathrm{d}(f\circ g)}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$ which looks like a simple cancellation of the $\mathrm{d}g$ terms![2]

Proof. Define $\gamma : \mathrm{dom}(f) \to \mathbb{R}$ via

$$\gamma(v) = \begin{cases} \frac{f(v)-f(g(a))}{v-g(a)} & \text{if } v \neq g(a) \\ f'\big(g(a)\big) & \text{if } v = g(a) \end{cases} \tag{$*$}$$

Since f is differentiable at $g(a)$, we see that γ is continuous and $\lim\limits_{v\to g(a)} \gamma(v) = f'\big(g(a)\big)$.
Since g is differentiable at a, there exists an open interval $U \ni a$ for which $x \in U \implies g(x) \in \mathrm{dom}(f)$.
Now compute: for any $x \in U \setminus \{a\}$, let $v = g(x)$ in $(*)$, whence

$$\frac{f\big(g(x)\big) - f\big(g(a)\big)}{x-a} = \gamma\big(g(x)\big)\frac{g(x)-g(a)}{x-a}$$

Take limits as $x \to a$ for the result. ∎

Corollary 3.11 (Quotient Rule). *Suppose f and g are differentiable. Then $\frac{f}{g}$ is differentiable whenever $g(x) \neq 0$. Moreover*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

The proof is an exercise.

Examples 3.12.
1. By the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

2. We can now differentiate highly involved combinations of elementary functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan(e^{4x^2}) - \frac{7x}{\sin x}\right] = 8xe^{4x^2}\sec^2(e^{4x^2}) - \frac{7\sin x - 7x\cos x}{\sin^2 x}$$

[2]This is completely unjustified since $\mathrm{d}g$ does not (for us) mean anything on its own! The same problem appears in the famously faulty one-line 'proof' of the chain rule:

$$\lim_{x\to a}\frac{f(g(x)) - f(g(a))}{x-a} \stackrel{?}{=} \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{g(x)-g(a)}\lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$

The second limit cannot exist unless $g(x) \neq g(a)$ for all x near, but not equal to, a. The faulty argument is repaired by replacing the second difference quotient with $f'(g(a))$ whenever $g(x) = g(a)$, *before* taking the limit. This is precisely what $\gamma(g(x))$ does in the correct proof.

Exercises 28 1. Use Definition 3.1 to calculate the derivatives.

(a) $f(x) = x^3$ at $x = 2$

(b) $g(x) = x + 2$ at $x = a$

(c) $f(x) = x^2 \cos x$ at $x = 0$

(d) $r(x) = \dfrac{3x+4}{2x-1}$ at $x = 1$

2. Differentiate the function $f(x) = \cos\left(e^{x^5 - 3x}\right)$ using the chain and product rules.

3. (a) Prove the quotient rule (Corollary 3.11) by combining the chain and product rules.
   (b) Prove the inverse derivative rule (Theorem 3.8, part 3).
   (*Hint: You can't simply differentiate* $1 = \frac{dx}{dx} = \frac{d}{dx} f(f^{-1}(x))$ *using the chain rule; why not?*)

4. (a) Find the derivatives of secant, cosecant and cotangent using the quotient rule.
   (b) Why did we choose the positive square-root when computing $\frac{d}{dx} \sin^{-1} x$? What is the standard domain of arcsine, and what happens at $x = \pm 1$?
   (c) Find the derivatives of the inverse trigonometric functions using the inverse function rule.

5. Using the definition of the derivative, and supposing that f is differentiable at a, prove that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}$$

6. Prove that the function $f(x) = x\,|x|$ is differentiable everywhere and compute its derivative.

7. Show that following function is differentiable everywhere and compute its derivative:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Moreover, prove that the derivative f' is *discontinuous* at $x = 0$.

8. Show that the following function is differentiable everywhere *except* at zero:

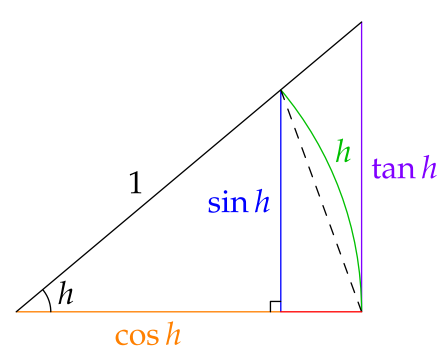
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

9. (a) Suppose $0 < h < \frac{\pi}{2}$. Use the picture to show that

$$0 < \frac{1 - \cos h}{h} < \sin \frac{h}{2} \quad \text{and} \quad \sin h < h < \tan h$$

   Hence conclude that $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \frac{1 - \cos h}{h} = 0$.

   (b) Use part (a) to prove that $\frac{d}{dx} \sin x = \cos x$



10. (Hard) Use induction to prove the Leibniz rule (general product rule):

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

Masochists Corner (non-examinable)

We finish with two *very hard* bonus exercises, though the first is somewhat easier. If you want a challenge, give 'em a go!

The Exponential Function & the General Power Law

Consider the function $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges for all real x.
As we saw when discussing power series, this function satisfies the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x), \qquad \exp(0) = 1$$

Define $e := \exp(1)$. Certainly e^x makes sense whenever $x \in \mathbb{Q}$. When x is irrational, define

$$e^x := \sup\{e^q : q \in \mathbb{Q},\ q < x\}$$

Our primary goal is to *prove* that $\exp(x) = e^x$. As a nice bonus we recover Bernoulli's limit identity $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ and obtain a complete proof of the power law.

(a) For all $x, y \in \mathbb{R}$, prove that $\exp(x + y) = \exp(x)\exp(y)$

(*Hint: use the binomial theorem and change the order of summation*)

(b) Show that $\exp(x)$ is always positive, *even when* $x < 0$.

(c) Prove that $\exp : \mathbb{R} \to (0, \infty)$ is bijective.

(*Hint:* $x \geq 0 \implies \exp(x) \geq 1 + x$*; take limits then apply part (a)*)

(d) Prove that $e^x = \exp(x)$. Do this in three stages:

- If $x \in \mathbb{N}$, use part (a). Now check for $x \in \mathbb{Z}^-$.
- If $x = \frac{m}{n} \in \mathbb{Q}$, first compute $\left[\exp(\frac{m}{n})\right]^n$.
- If x is irrational, start with $\exists (q_n) \subseteq \mathbb{Q}$ such that $q_n < x$ and $e^{q_n} \to e^x \ldots$

(e) Let $\ln : (0, \infty) \to \mathbb{R}$ be the inverse function of exp. Prove the logarithm laws:

$$\ln(xy) = \ln x + \ln y \quad \text{and} \quad \ln x^r = r \ln x$$

(*Just do this when* $r \in \mathbb{N}$*; another argument like part (d) is required in general*)

(f) We've already seen that $\frac{\mathrm{d}}{\mathrm{d}y}\ln y = \frac{1}{y}$. Use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln y = \lim_{h\to 0} \frac{\ln(y+h) - \ln y}{h}$$

to prove that $\exp(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$, thus recovering Bernoulli's definition of e.

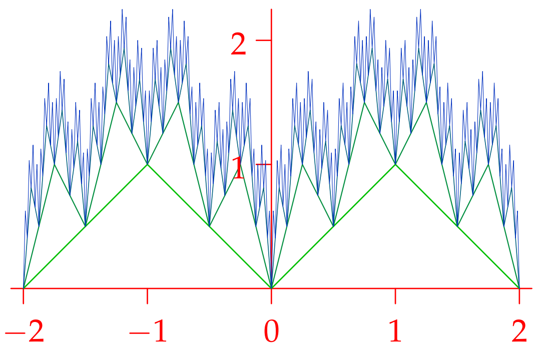
(g) For any $r \in \mathbb{R}$, *define* $x^r := \exp(r \ln x)$. Hence obtain the power law for any exponent.

A Very Strange Function

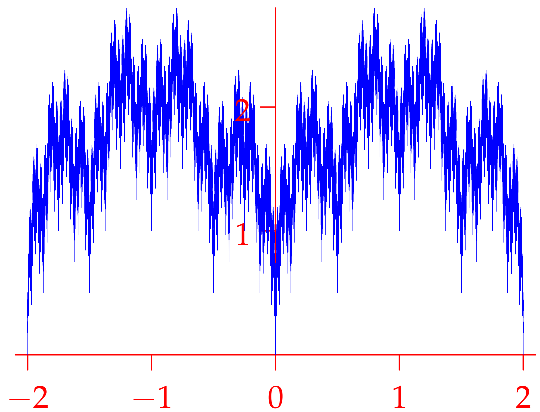
Here is a classic example of a continuous but nowhere-differentiable function!

Let f be the *sawtooth* function defined by $f(x) = |x|$ whenever $x \in [-1, 1]$ and extending periodically to $\mathbb{R}$ so that $f(x+2) = f(x)$. Now define $g : \mathbb{R} \to \mathbb{R}$ via

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n f(4^n x)$$



$f(x)$ and iterations to $n = 3$



$g(x)$ (really $n = 6$, but can you tell?!)

(a) Prove that g is well-defined and continuous on $\mathbb{R}$.

(b) Let $x \in \mathbb{R}$ and $m \in \mathbb{N}$ be fixed. Define $h_m = \pm\frac{1}{2} \cdot 4^{-m}$ where the $\pm$-sign is chosen so that no integers lie strictly between $4^m x$ and $4^m(x + h_m) = 4^m x \pm \frac{1}{2}$.

For each $n \in \mathbb{N}_0$, define

$$k_n = \frac{f\big(4^n(x + h_m)\big) - f(4^n x)}{h_m}$$

Prove the following

i. $|k_n| \leq 4^n$ with equality when $n = m$.

ii. $n > m \implies k_n = 0$.

(*Hint: $|f(y) - f(z)| \leq |y - z|$: when is this an equality?*)

(c) Use part (b) to prove that

$$\left|\frac{g(x + h_m) - g(x)}{h_m}\right| \geq \frac{1}{2}(3^m + 1)$$

Hence conclude that g is *nowhere differentiable.*

29 The Mean Value Theorem

We now turn to one of the central results in calculus.

Theorem 3.13 (Mean Value Theorem/MVT). *Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $\xi \in (a,b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.*

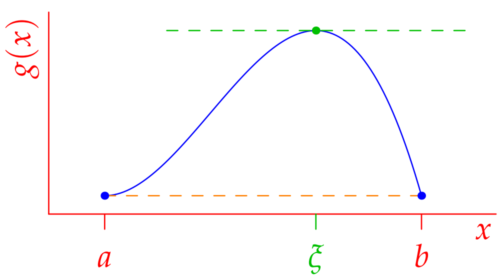
This follows easily from two lemmas.

Lemma 3.14.
1. *(Critical Points) Suppose g is bounded on (a,b) and attains its maximum or minimum at $\xi \in (a,b)$. If g is differentiable at ξ then $g'(\xi) = 0$.*
2. *(Rolle's Theorem) Suppose g is continuous on $[a,b]$, differentiable on (a,b), and that $g(a) = g(b)$. Then there exists $\xi \in (a,b)$ such that $g'(\xi) = 0$.*

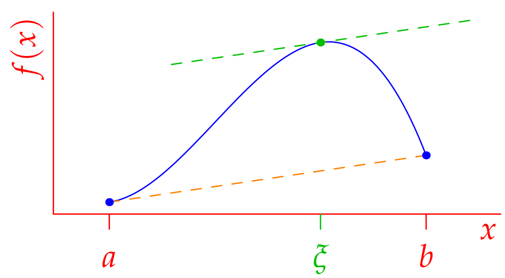
The main result follows by applying Rolle's theorem to

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-b)$$

and observing that $g(a) = f(b) = g(b)$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.



Critical Points/Rolle's Theorem



Mean Value Theorem

In the pictures, the orange and green lines are *parallel*: the average slope over the interval $[a,b]$ equals the gradient/derivative $f'(\xi)$.

Proof of Lemma.
1. Suppose, for a contradiction, that

$$g'(\xi) = \lim_{x \to \xi} \frac{g(x)-g(\xi)}{x-\xi} > 0$$

Let $\epsilon = g'(\xi)$ in the definition of limit: $\exists \delta > 0$ such that

$$0 < |x-\xi| < \delta \implies \left| \frac{g(x)-g(\xi)}{x-\xi} - g'(\xi) \right| < g'(\xi) \implies 0 < \frac{g(x)-g(\xi)}{x-\xi} < 2g'(\xi)$$

In particular, if $x \in (\xi, \xi+\delta)$, then $g(x) > g(\xi)$, contradicting the maximality at ξ.

The argument when $g'(\xi) < 0$ is similar. Finally, apply to $-g$ for the result at a minimum.

2. By the extreme value theorem, g is bounded and attains its bounds. If the maximum and minimum *both* occur at the endpoints a, b, then g is constant: any $\xi \in (a,b)$ satisfies the result. Otherwise, at least one extreme value occurs at some $\xi \in (a,b)$: part 1 says that $g'(\xi) = 0$. ■

Examples 3.15. 1. Let $f(x) = (x-1)^2(4-x) + x$ on $[a,b] = [1,4]$: this is roughly the above picture illustrating the mean value theorem. We compute the average slope and the derivative:

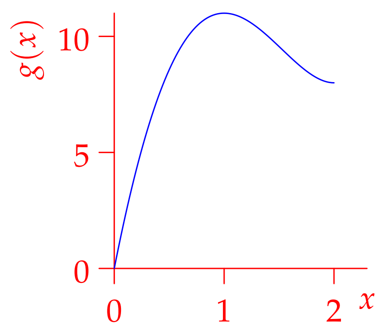
$$\frac{f(b)-f(a)}{b-a} = 1, \qquad f'(x) = 2(x-1)(4-x) - (x-1)^2 + 1 = -3x^2 + 12x - 8$$

and observe that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a} \iff 3\xi^2 - 12\xi + 9 = 0 \iff \xi = 1 \text{ or } 3$$

Since only 3 lies in the interval $(1,4)$, this is the value ξ satisfying the mean value theorem.

2. We find the maximum and minimum values of $g(x) = x^4 - 14x^2 + 24x$ on the interval $[0,2]$. The function is differentiable, with



$$g'(x) = 4x^3 - 28x + 24 = 4(x-2)(x-1)(x+3)$$

By the Lemma, the locations of the extrema are either the end-points $x = 0, 2$ or locations with zero derivative ($x = 1$). Since

$$f(0) = 0, \quad f(1) = 11, \quad f(2) = 8$$

we conclude that $\max(f) = f(1) = 11$ and $\min(f) = f(0) = 0$.

Consequences of the Mean Value Theorem Several simple corollaries relate to monotonicity.

Definition 3.16. Suppose $f : I \to \mathbb{R}$ is defined on an interval I. We say that f is:

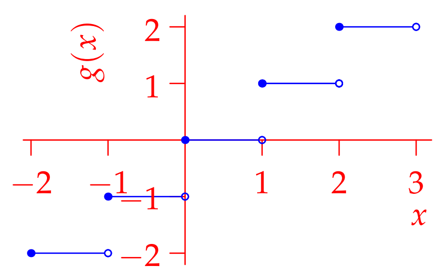
Increasing (monotone-up) on I if $x < y \implies f(x) \le f(y)$

Decreasing (monotone-down) on I if $x < y \implies f(x) \ge f(y)$

We say *strictly* increasing/decreasing if the inequalities are strict.

Examples 3.17. 1. $f : x \mapsto x^2$ is strictly increasing on $[0,\infty)$ and strictly decreasing on $(-\infty, 0]$.

2. The floor function $f : x \mapsto \lfloor x \rfloor$ (the greatest integer less than or equal to x) is increasing, but not strictly, on $\mathbb{R}$.



Corollary 3.18. *Suppose f is differentiable on an interval I, then*

1. $f' \ge 0$ *on* $I \iff f$ *is increasing on* I
2. $f' \le 0$ *on* $I \iff f$ *is decreasing on* I
3. $f' = 0$ *on* $I \iff f$ *is constant on* I

Proof. ($\Rightarrow$) Let $x < y$ where $x, y \in I$. By the mean value theorem, $\exists \xi \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(\xi) \quad \text{whence} \quad f'(\xi) \geq 0 \implies f(y) \geq f(x)$$

($\Leftarrow$) For the converse, use the definition of derivative: $f'(\xi) = \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi}$. If f is increasing, then

$$x > \xi \implies f(x) \geq f(\xi) \implies f'(\xi) \geq 0$$

Parts 2 and 3 are similar. ∎

Corollary 3.18 yields a couple of flashbacks to elementary calculus.

Corollary 3.19. *Let I be an interval.*

1. *(Anti-derivatives on an interval) If $f'(x) = g'(x)$ on I, then $\exists c$ such that $g(x) = f(x) + c$ on I.*
2. *(First derivative test) Suppose f is continuous on I and differentiable except perhaps at ξ. If*

$$\begin{cases} f'(x) < 0 & \text{whenever } x < \xi, \text{ and} \\ f'(x) > 0 & \text{whenever } x > \xi \end{cases} \qquad \text{then } f \text{ has its minimum value at } x = \xi$$

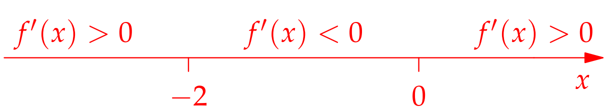
The statement for a maximum is similar.

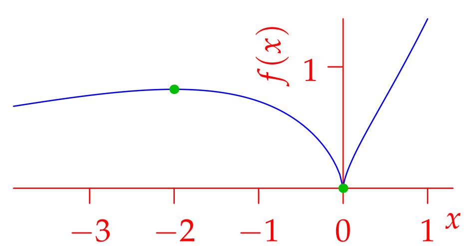
Examples 3.20. 1. Since $\frac{\mathrm{d}}{\mathrm{d}x} \sin(3x^2 + x) = (6x + 1)\cos(3x^2 + x)$ on (the interval) $\mathbb{R}$, whence all anti-derivatives of $f(x) = (6x + 1)\cos(3x^2 + x)$ are given by

$$\int f(x)\,\mathrm{d}x = \int (6x + 1)\cos(3x^2 + x)\,\mathrm{d}x = \sin(3x^2 + x) + c$$

As is typical, we use the *indefinite integral* notation $\int f(x)\,\mathrm{d}x$ for anti-derivatives.

2. If $f(x) = x^{2/3}e^{x/3}$, then $f'(x) = \frac{1}{3}x^{-1/3}(2 + x)e^{x/3}$.
By Lemma 3.14, the only possible critical points are at $x = 0$ or -2. The sign of the derivative is also clear:





By the 1st derivative test, f has a maximum at $x = -2$ and a minimum at $x = 0$.

We finish this section by tying together the mean and intermediate value theorems.

Theorem 3.21 (IVT for Derivatives). *Suppose f is differentiable on an interval I containing $a < b$, and that L lies between $f'(a)$ and $f'(b)$. Then $\exists \xi \in (a, b)$ such that $f'(\xi) = L$.*

If $f'(x)$ is *continuous*, this is just the intermediate value theorem applied to f'. A full proof is left to the exercises; surprisingly, continuity is *not* required...

Exercises 29 1. Determine whether the conclusion of the mean value theorem holds for each function on the given interval. If so, find a suitable point ξ. If not, state which hypothesis fails.

(a) x^2 on $[-1, 2]$ (b) $\sin x$ on $[0, \pi]$ (c) $|x|$ on $[-1, 2]$
(d) $1/x$ on $[-1, 1]$ (e) $1/x$ on $[1, 3]$

2. Suppose f and g are differentiable on an open interval I, that $a < b$ and $f(a) = f(b) = 0$. By considering $h(x) = f(x)e^{g(x)}$, prove that $f'(\xi) + f(\xi)g'(\xi) = 0$ for some $\xi \in (a, b)$.

3. Use the Mean Value Theorem to prove the following:
   (a) $x < \tan x$ for all $x \in (0, \pi/2)$.
   (b) $\frac{x}{\sin x}$ is a strictly increasing function on $(0, \pi/2)$.
   (c) $x \leq \frac{\pi}{2} \sin x$ for all $x \in [0, \pi/2]$.

4. Suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all $x, y \in \mathbb{R}$. Prove that f is a constant function.

5. (a) Prove that $f' > 0$ on an interval $I \implies f$ is *strictly* increasing on I.
   (b) Show that the converse of part (a) is *false.*
   (c) Carefully prove the first derivative test (Corollary 3.19).

6. If f is differentiable on an interval I such that $f'(x) \neq 0$ for all $x \in I$, use the intermediate value theorem for derivatives to prove that f is either strictly increasing or strictly decreasing.

7. We prove the intermediate value theorem for derivatives. Let f, a, b and L be as in the Theorem, define $g : I \to \mathbb{R}$ by $g(x) = f(x) - Lx$, and let $\xi \in [a, b]$ be such that

$$g(\xi) = \min\{g(x) : x \in [a, b]\}$$

   (a) Why can we be sure that ξ exists? If $\xi \in (a, b)$, explain why $f'(\xi) = L$.
   (b) Now assume WLOG that $f'(a) < f'(b)$. Prove that $g'(a) < 0 < g'(b)$. By considering $\lim_{x \to a^+} \frac{g(x) - g(a)}{x - a}$, show that $\exists x > a$ for which $g(x) < g(a)$. Hence complete the proof.

8. Suppose f' exists on (a, b), and is continuous except for a discontinuity at $c \in (a, b)$.
   (a) Obtain a contradiction if $\lim_{x \to c^+} f'(x) = L < f'(c)$. Hence argue that f' cannot have a *removable* or a *jump* discontinuity at $x = c$.
   (*Hint: let* $\epsilon = \frac{f'(c) - L}{2}$ *in the definition of limit then apply IVT for derivatives*)
   (b) Similarly, obtain a contradiction if $\lim_{x \to c^+} f'(x) = \infty$ and conclude that f' cannot have an *infinite* discontinuity at $x = c$.
   (c) It remains to see that f' can have an essential discontinuity. Recall (Exercise 28.7) that

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   is differentiable on $\mathbb{R}$, but has discontinuous derivative at $x = 0$.
   i. By considering $x_n = \frac{1}{2n\pi}$ and $y_n = \frac{1}{(2n+1)\pi}$, show that f' has an essential discontinuity at $x = 0$.
   ii. Prove that if $s_n \to 0$ and $f'(s_n)$ converges to some M, then $M \in [-1, 1]$.
   iii. Use IVT for derivatives to show that for any $L \in [-1, 1]$, $\exists (t_n) \subseteq \mathbb{R} \setminus \{0\}$ such that $\lim_{n \to \infty} f'(t_n) = L$.

30 L'Hôpital's Rule

We are often forced to consider limits known as *indeterminate forms*, which do not yield easily to the standard limits laws. For example, it is tempting to try to write

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-1}=\frac{\lim\limits_{x\to 0}\sin 2x}{\lim\limits_{x\to 0}e^{3x}-1}=\frac{0}{0} \qquad (*)$$

This is an incorrect application of the limit laws since the resulting quotient has no meaning.

Definition 3.22. An *indeterminate form* is a limit where a naïve application of the limit laws results in a meaningless expression: the primary types are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $0\cdot\infty$, 0^0, 0^∞, and 1^∞.

Examples 3.23. 1. $\lim\limits_{x\to 7^+}(x-7)^{\frac{1}{x-7}}$ is an indeterminate form of type 0^∞.

2. The above indeterminate form $(*)$ may be evaluated using the definition of the derivative

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-1}=\lim_{x\to 0}\frac{\sin 2x-0}{x-0}\frac{x-0}{e^{3x}-1}=\left(\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=0}\sin 2x\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=0}e^{3x}\right)^{-1}=\frac{2}{3}$$

By considering $\lim\limits_{x\to 0}\frac{3a\sin 2x}{2(e^{3x}-1)}$, we see that an indeterminate form of type $\frac{0}{0}$ can take *any value* a!

This approach generalizes: if $f(a)=0=g(a)$, we obtain the simplest version of l'Hôpital's rule;

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\frac{x-a}{g(x)-g(a)}=\frac{f'(a)}{g'(a)}$$

This obviously isn't rigorous. Our goal is to make it so and to extend to the following situations:

- Limits where $a=\pm\infty$.
- When the RHS cannot be cleanly evaluated: for instance $g'(a)=0$ or if the original limit is $\pm\infty$.

Covering all cases makes the proof an absolute behemoth! Because of this, and because such limits can often be evaluated more instructively using elementary methods, the rule is often discouraged in Freshman calculus. To prepare for the upcoming monster, we first generalize the MVT.

Lemma 3.24 (Extended Mean Value Theorem). *Fix $a<b$, suppose f,g are continuous on $[a,b]$ and differentiable on (a,b). Then there exists $\xi\in(a,b)$ such that*

$$(f(b)-f(a))g'(\xi)=(g(b)-g(a))f'(\xi)$$

Proof. Simply apply the standard mean value theorem (really Rolle's Theorem) to

$$h(t)=(f(b)-f(a))g(t)-(g(b)-g(a))f(t)$$

which satisfies $h(a)=h(b)$. ∎

Theorem 3.25 (l'Hôpital's rule). *Let $a \in \mathbb{R} \cup \{\pm\infty\}$ and suppose functions f and g satisfy:*

1. $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$ *for some* $L \in \mathbb{R} \cup \{\pm\infty\}$
2. *(a)* $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, *or* *(b)* $\lim\limits_{x\to a} g(x) = \infty$ *(no condition on f)*

Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L$. *The same result holds for one-sided limits.*

Examples 3.26.

1. If $f(x) = e^{4x}$ and $g(x) = 21x - 17$, then
$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{4e^{4x}}{21} = \infty \implies \lim_{x\to\infty} \frac{e^{4x}}{21x-17} = \infty$$
This is an example of type $\frac{\infty}{\infty}$.
2. For an example of type $\frac{0}{0}$, consider $f(x) = x^2 - 9$ and $g(x) = \ln(4-x)$:
$$\lim_{x\to 3^-} \frac{f'(x)}{g'(x)} = \lim_{x\to 3^-} \frac{2x}{-1/(4-x)} = \lim_{x\to 3^-} 2x(x-4) = -6 \implies \lim_{x\to 3^-} \frac{x^2-9}{\ln(4-x)} = -6$$
3. One can apply the rule repeatedly: for example
$$\lim_{x\to 0} \frac{e^{4x} - 1 - 4x}{x^2} = \lim_{x\to 0} \frac{4e^{4x} - 4}{2x} = \lim_{x\to 0} \frac{16e^{4x}}{2} = 8$$
There is an abuse of protocol here, since the existence of the first limit is dependent on the last. The approach is acceptable, though you should understand why it is an abuse. Indeed...
4. It is important that the limit $\lim \frac{f'}{g'}$ be seen to exist *before* applying l'Hôpital's rule! Consider $f(x) = x + \cos x$ and $g(x) = x$: certainly $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)}$ has type $\frac{\infty}{\infty}$, however
$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} 1 - \sin x$$
does not exist! In this case the rule is unnecessary, since
$$\frac{f(x)}{g(x)} = 1 + \frac{\cos x}{x} \xrightarrow[x\to\infty]{} 1$$
by the squeeze theorem.
5. Finally, a short example to explain why l'Hôpital's rule is often prohibited in Freshman calculus. Consider the calculation:
$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\cos x}{1} = 1$$
This appears to be a legitimate application of the rule. However, recall (Exercise 28.9) that one purpose of this limit is to demonstrate that $\frac{d}{dx} \sin x = \cos x$; to use this fact to calculate the limit on which it depends is the very definition of circular logic!

Other Indeterminate Forms

The remaining indeterminate forms listed in Definition 3.22 may be modified so that l'Hôpital's rule applies. Since you've likely seen several such examples in elementary calculus, we give just a couple.

Examples 3.27. 1. An indeterminate form of type $\infty - \infty$ is transformed to one of type $\frac{0}{0}$ before applying the rule (twice):

$$\begin{aligned}\lim_{x\to 0^+} \frac{1}{e^x-1} - \frac{1}{x} &= \lim_{x\to 0^+} \frac{x+1-e^x}{x(e^x-1)} && (\text{type } \tfrac{0}{0})\\ &= \lim_{x\to 0^+} \frac{1-e^x}{e^x-1+xe^x} && (\text{still type } \tfrac{0}{0})\\ &= \lim_{x\to 0^+} \frac{-e^x}{2e^x+xe^x} = -\frac{1}{2}\end{aligned}$$

2. For an indeterminate form of type 1^∞, we use the log laws & the continuity of the exponential:

$$\begin{aligned}\lim_{x\to 0^+} (1+\sin x)^{1/x} &= \exp\left(\lim_{x\to 0^+} \frac{1}{x}\ln(1+\sin x)\right) && (\text{type } \tfrac{0}{0})\\ &= \exp\left(\lim_{x\to 0^+} \frac{\cos x}{1+\sin x}\right)\\ &= e^1 = e\end{aligned}$$

Proving l'Hôpital's Rule

The complete argument is very long; if you do nothing else, read the following proof of the simplest case. Everything else is a modification.

Proof (type $\frac{0}{0}$ with right limits). We prove first for *right-limits* $x \to a^+$. First observe that condition 1. forces the existence of an interval (a,b) on which f, g are differentiable and $g'(x) \neq 0$.

Assume we have a form of type $\frac{0}{0}$ (case 2. (a)) and assume additionally that a and L are finite. Everything follows from the definition of limit (condition 1.) and Lemma 3.24:

$$\text{Given } \epsilon > 0,\ \exists \delta \in (0, b-a) \text{ such that } a < \xi < a+\delta \implies \left|\frac{f'(\xi)}{g'(\xi)} - L\right| < \frac{\epsilon}{2} \qquad (*)$$

$$a < y < x < a+\delta \implies \exists \xi \in (y,x) \text{ such that } \frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(\xi)}{g'(\xi)} \qquad (\dagger)$$

Since $g' \neq 0$, the usual mean value theorem says we never divide by zero in $(\dagger)$:

$$\exists c \in (y,x) \text{ such that } g(x) - g(y) = g'(c)(x-y) \neq 0$$

Observe that $\left|\frac{f(x)-f(y)}{g(x)-g(y)} - L\right| = \left|\frac{f'(\xi)}{g'(\xi)} - L\right| < \frac{\epsilon}{2}$, let $y \to a^+$ and use 2. (a) to see that

$$\forall x \in (a, a+\delta), \quad \left|\frac{f(x)}{g(x)} - L\right| \leq \frac{\epsilon}{2} < \epsilon$$

which is the required result. ∎

We now describe some modifications.

If $a = -\infty$: Replace the blue part of $(*)$ as follows:

$$\text{Given } \epsilon > 0, \exists m \le b \text{ such that } \xi < m \implies \left|\frac{f'(\xi)}{g'(\xi)} - L\right| < \frac{\epsilon}{2}$$

The rest of the proof goes through after replacing a with $-\infty$ and $a+\delta$ with m.

If $L = \infty$: Replace the green parts of $(*)$ with Given $M > 0$ and $\frac{f'(\xi)}{g'(\xi)} > 2M$. Fixing the rest of the proof is again straightforward.

If $L = -\infty$: Replace the green parts of $(*)$ with Given $M > 0$ and $\frac{f'(\xi)}{g'(\xi)} < -2M$.

Left-limits: If f, g are differentiable on (c, a), then the blue part may be replaced with either:

- (a finite) $\exists \delta \in (0, a-c)$ such that $a - \delta < \xi < a$
- ($a = \infty$) $\exists m \ge c$ such that $\xi > m$

The blue and green parts of $(*)$ can be replaced independently. This completes the proof for all indeterminate forms of type $\frac{0}{0}$.

Proof (case 2. (b) when $\lim g(x) = \infty$*).* This requires a little more modification.[3] Since $g' \neq 0$, and $\lim_{x\to a^+} g(x) = \infty$, Exercise 29.6 says that g is *strictly decreasing* on (a, b). By replacing b by some $\tilde{b} \in (a, b)$, if necessary, we may assume that

$$a < y < x < b \implies 0 < g(x) < g(y) \tag{‡}$$

Assume a and L are finite, and obtain $(*)$ and $(\dagger)$ as before. Let $x \in (a, a+\delta)$ be fixed and multiply $(\dagger)$ by $\frac{g(y)-g(x)}{g(y)}$ (this is *positive* by (‡)): a little algebra and the triangle inequality tell us that

$$\begin{aligned} a < y < x &\implies \frac{f(y)}{g(y)} = \frac{f'(\xi)}{g'(\xi)} + \frac{f(x)}{g(y)} - \frac{g(x)}{g(y)} \cdot \frac{f'(\xi)}{g'(\xi)} \\ &\implies \left|\frac{f(y)}{g(y)} - L\right| \le \left|\frac{f'(\xi)}{g'(\xi)} - L\right| + \frac{1}{g(y)}\left(|f(x)| + |g(x)|\left(L + \frac{\epsilon}{2}\right)\right) \end{aligned}$$

Since $\lim\limits_{y\to a^+} g(y) = \infty$ and x is fixed, we see that there exists $\eta \le x - a < \delta$ such that

$$y \in (a, a+\eta) \implies \frac{1}{g(y)}\left(|f(x)| + |g(x)|\left(L + \frac{\epsilon}{2}\right)\right) < \frac{\epsilon}{2}$$

Finally combine with $(*)$: given $\epsilon > 0, \exists \eta > 0$ such that $y \in (a, a+\eta) \implies \left|\frac{f(y)}{g(y)} - L\right| < \epsilon$.
The same modifications listed previously complete the proof. ∎

[3] *Forms of type* $\frac{\infty}{\infty}$*?* Instead of assumption 2. (b), why not simply assume $\lim f = \lim g = \infty$ and write $\frac{f}{g} = \frac{1/g}{1/f}$ to obtain a form of type $\frac{0}{0}$? The problem is that the derivative of the 'new' denominator $\frac{d}{dx}\frac{1}{f} = \frac{-f'}{f^2}$ need not be non-zero on any interval (a, b) and so condition 1. need not hold. We could modify this, but it would make for a weaker theorem. Example 3.26.4 illustrates this: $f'(x) = 1 + \sin x$ has zeros on any unbounded interval.

After the 2. (b) case is proved and we know that $\lim \frac{f}{g} = L$, it is then clear that $\lim f$ must also be infinite (unless $L = 0$ in which case $\lim f$ could be anything and need not exist). This situation therefore really does deal with forms of type $\frac{\infty}{\infty}$.

Exercises 30 1. Evaluate the following limits, if they exist:

(a) $\displaystyle\lim_{x\to 0} \frac{x^3}{\sin x - x}$ (b) $\displaystyle\lim_{x\to \frac{\pi}{2}^-} \tan x - \frac{2}{\pi - 2x}$

(c) $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$ (d) $\displaystyle\lim_{x\to 0} (1+2x)^{1/x}$

(e) $\displaystyle\lim_{x\to \infty} (e^x + x)^{1/x}$

2. Let f be differentiable on (c, ∞) and suppose that $\lim_{x\to\infty} [f(x) + f'(x)] = L$ is finite.

   (a) Prove that $\lim_{x\to\infty} f(x) = L$ and that $\lim_{x\to\infty} f'(x) = 0$.
   (*Hint: write* $f(x) = \frac{f(x)e^x}{e^x}$)

   (b) Does anything change if L exists and is *infinite*?

3. If $p_n(x)$ is a polynomial of degree n, use induction to prove that $\lim_{x\to\infty} p_n(x)e^{-x} = 0$

4. Let $f(x) = x + \sin x \cos x$, $g(x) = e^{\sin x} f(x)$ and $h(x) = \dfrac{2\cos x}{e^{\sin x}(f(x) + 2\cos x)}$

   (a) Prove that $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$ but that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

   (b) If $\cos x \neq 0$, and x is large, show that $\frac{f'(x)}{g'(x)} = h(x)$.

   (c) Prove that $\lim_{x\to\infty} h(x) = 0$. Explain why this does not contradict part (a)!

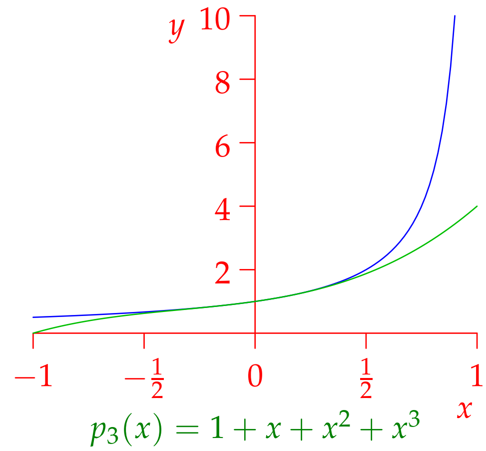
31 Taylor's Theorem

A primary goal of power series is the approximation of functions. As such, there are two natural questions to ask of a given function f:

1. Given $c \in \text{dom}(f)$, is there a series $\sum a_n(x-c)^n$ which equals $f(x)$ on an interval containing c?
2. If we take the first n terms of such a series, how accurate is this polynomial approximation?

Example 3.28. Recall the geometric series

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ whenever } -1 < x < 1$$



The polynomial approximation

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

has error

$$R_n(x) = f(x) - p_n(x) = \frac{x^{n+1}}{1-x}$$

If x is close to 0, this is likely very small; for instance if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then

$$|R_n(x)| \le \frac{1}{1-\frac{1}{2}}\left(\frac{1}{2}\right)^{n+1} = 2^{-n}$$

However, when x is close to 1, the error is unbounded!

The behavior in the Example occurs in general: the truncated polynomial approximations are better near the center of the series. To see this, we first need to consider higher-order derivatives.

Definition 3.29. We write f'' for the *second derivative* of f, namely the derivative of its derivative

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x-a}$$

The existence of $f''(a)$ presupposes that f' exists on an (open) interval containing a. We can similarly consider third, fourth, and higher-order derivatives. As a function, the n^{th} derivative is written

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}$$

By convention, the *zeroth derivative* is the function itself $f^{(0)}(x) = f(x)$. We say that f is *n times differentiable* at a if $f^{(n)}(a)$ exists, and *infinitely differentiable* (or *smooth*) if derivatives of all orders exist.

Example 3.30. $f(x) = x^2|x|$ is twice differentiable, with $f''(x) = 6|x|$. It is smooth everywhere except at $x = 0$, where third (and higher-order) derivatives do not exist.

Definition 3.31. Suppose f is n times differentiable at $x = c$. The n^{th} *Taylor polynomial* p_n of f centered at c is

$$p_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The *remainder* $R_n(x)$ is the error in the polynomial approximation

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{j=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

If f is infinitely differentiable at $x = c$, then its *Taylor series* centered at $x = c$ is the power series

$$\mathrm{T}f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

When $c = 0$ this is known as a *Maclaurin series.*[4]

For simplicity we'll most often work with Maclaurin series, with general cases hopefully being clear.

Examples 3.32. 1. If $f(x) = e^{3x}$, then $f^{(n)}(x) = 3^n e^x$, from which the Maclaurin series is

$$\mathrm{T}f(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$$

2. If $g(x) = \sin 7x$, then the sequence of derivatives is

$$7\cos 7x, \quad -7^2 \sin 7x, \quad -7^3 \cos 7x, \quad 7^4 \sin 7x, \quad 7^5 \cos 7x, \quad -7^6 \sin 7x, \ldots$$

At $x = 0$, every even derivative is zero, while the odd derivatives alternate in sign; the Maclaurin series is easily seen to be

$$\mathrm{T}g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 7^{2n+1}}{(2n+1)!}x^{2n+1}$$

3. If $h(x) = \sqrt{x}$, then $h'(x) = \frac{1}{2}x^{-1/2}$, $h''(x) = \frac{-1}{2^2}x^{-3/2}$, and $h'''(x) = \frac{3}{2^3}x^{-5/2}$, from which the third Taylor polynomial centered at $c = 1$ is

$$\begin{aligned} p_2(x) &= h(1) + h'(1)(x-1) + \frac{h''(1)}{2}(x-1)^2 + \frac{h'''(1)}{6}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \end{aligned}$$

Rather than compute more examples, we develop a little theory that makes verifying Taylor series much easier.

[4]Named for Englishman Brook Taylor (1685–1731) and Scotsman Colin Maclaurin (1698–1746). Taylor's general method expanded on examples discovered by James Gregory and Issac Newton in the mid-to-late 1600's.

Differentiation of Taylor Polynomials and Series

Suppose $P(x) = \sum a_j x^j$ is a power series with radius of convergence $R > 0$. As we discovered previously, this is differentiable term-by-term on $(-R, R)$. Indeed

$$P'(x) = \sum_{j=1}^{\infty} a_j j x^{j-1} \implies P'(0) = a_1$$

$$P''(x) = \sum_{j=2}^{\infty} a_j j(j-1)x^{j-2} \implies P''(0) = 2a_2$$

$$P'''(x) = \sum_{j=3}^{\infty} a_j j(j-1)(j-2)x^{j-3} \implies P'''(0) = 3!a_3$$

$$\vdots$$

$$P^{(k)}(x) = \sum_{j=k}^{\infty} a_j j(j-1)\cdots(j-k+1)x^{j-k} = \sum_{j=k}^{\infty} \frac{j!a_j}{(j-k)!}x^{j-k} \implies P^{(k)}(0) = k!a_k$$

Otherwise said, P is its own Maclaurin series! The same discussion holds for polynomials: indeed if $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial, then for all $k \le n$,

$$P^{(k)}(0) = f^{(k)}(0) \iff a_k = \frac{f^{(k)}(0)}{k!}$$

If this holds for *all* $k \le n$, then P must be the Taylor polynomial of f! With a little modification, we've proved the following:

Theorem 3.33.

1. *If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ on a neighborhood of c, then $\sum_{n=0}^{\infty} a_n (x-c)^n$ is the Taylor series of f.*
2. *The n^{th} Taylor polynomial of f centered at $x = c$ is the unique polynomial p_n of degree $\le n$ whose value and first n derivatives agree with those of f at $x = c$: that is*

$$\forall k \le n, \;\; p_n^{(k)}(c) = f^{(k)}(c)$$

This answers our first motivating question: a function can equal at most one power series with a given center. The second question requires a careful study of the *remainder*: we'll do this shortly.

Examples 3.34 (Common Maclaurin Series). These should be familiar from elementary calculus. Each of these functions equals the given series by our previous discussion of power series: by the Theorem, each series is therefore the Maclaurin series of the given function with no requirement to calculate directly!

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad x \in \mathbb{R} \qquad\qquad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad x \in (-1,1)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} \qquad x \in \mathbb{R} \qquad\qquad \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n \qquad x \in (-1,1]$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} \qquad x \in \mathbb{R} \qquad\qquad \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} \qquad x \in [-1,1]$$

Examples 3.35 (Modifying Maclaurin Series). By substituting for x in a common series, we quickly obtain new ones.

1. Substitute $x \mapsto 7x$ in the Maclaurin series for $\sin x$, to recover our earlier example

$$\sin 7x = \sum_{n=0}^{\infty} \frac{(-1)^n 7^{2n+1}}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R}$$

   Note how this requires almost no calculation: since the function equals a series, the Theorem says we have the Maclaurin series for $\sin 7x$!

2. Substitute $x \mapsto x^2$ in the Maclaurin series for e^x to obtain

$$e^{x^2} = \exp(x^2) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}, \quad x \in \mathbb{R}$$

   This would be disgusting to verify directly, given the difficulty of repeatedly differentiating e^{x^2}.

3. We find the Taylor series for $f(x) = \frac{1}{5-x}$ centered at $x = 2$:

$$f(x) = \frac{1}{3+2-x} = \frac{1}{3(1-\frac{2-x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2-x}{3}\right)^n$$

   which is valid whenever $-1 < \frac{2-x}{3} < 1 \iff -1 < x < 5$.

4. Fix $c \in \mathbb{R}$ and observe that, for all $x \in \mathbb{R}$,

$$e^x = e^{c+x-c} = e^c e^{x-c} = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n$$

   We conclude that the series is the Taylor series of e^x centered at $x = c$. Of course this is easily verified using the definition, since $\frac{d^n}{dx^n}\Big|_{x=c} e^x = e^c$.

5. Combining the Theorem with the multiple-angle formula, we obtain the Taylor series for $\sin x$ centered at $x = c$:

$$\begin{aligned}\sin x &= \sin(c + x - c) = \sin c \cos(x-c) + \cos x \sin(x-c) \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n \sin c}{(2n)!}(x-c)^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n \cos c}{(2n+1)!}(x-c)^{2n+1}\end{aligned}$$

Definition 3.36. A function is *analytic* on a domain if for each c there exists a neighborhood of c on which the function equals its Taylor series centered at c.

All the examples we've so far seen are analytic on their domains; indeed the last two of Examples 3.35 *prove* this for the exponential and sine functions. Every analytic function is automatically smooth (infinitely differentiable), however the converse is *false* in that not every smooth function is analytic (see Exercise 10). Analyticity is of greater importance in complex analysis where it is seen to be equivalent to complex-differentiability.

Accuracy of Taylor Approximations

Our final goal is to estimate the accuracy of a Taylor polynomial as an approximation to its generating function. Otherwise said, we want to estimate the size of the remainder $R_n(x) = f(x) - p_n(x)$.

Theorem 3.37 (Taylor's Theorem: Lagrange's form). *Suppose f is $n+1$ times differentiable on an open interval I containing c and let $x \in I \setminus \{c\}$. Then there exists some ξ between c and x for which the remainder centered at c satisfies*

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Proof. For simplicity let $c = 0$. Fix $x \neq 0$, define a constant M_x and a function $g : I \to \mathbb{R}$ by

$$R_n(x) = \frac{M_x}{(n+1)!}x^{n+1} \quad \text{and} \quad g(t) = \frac{M_x}{(n+1)!}t^{n+1} + p_n(t) - f(t) = \frac{M_x}{(n+1)!}t^{n+1} - R_n(t)$$

Observe that

$$\begin{aligned} k \leq n+1 \implies & g^{(k)}(x) = \frac{M_x}{(n+1-k)!}t^{n+1-k} + p_n^{(k)}(t) - f^{(k)}(t) \qquad (*) \\ \implies & g^{(k)}(0) = p_n^{(k)}(0) - f^{(k)}(0) = 0 \quad \text{if } k \leq n \end{aligned}$$

where we invoked Theorem 3.33.
Apply Rolle's Theorem repeatedly (WLOG assume $x > 0$):

- $\exists \xi_1$ between 0 and x such that $g'(\xi_1) = 0$.
- $\exists \xi_2$ between 0 and ξ_1 such that $g''(\xi_2) = 0$, etc.
- Iterate to obtain a sequence (ξ_k) such that

$$0 < \xi_{n+1} < \xi_n < \cdots < \xi_1 < x \quad \text{and} \quad g^{(k)}(\xi_k) = 0$$

Take $\xi = \xi_{n+1}$ and consider $(*)$: since $\deg p_n \leq n$, we see that

$$0 = g^{(n+1)}(\xi) = M_x - f^{(n+1)}(\xi) \implies R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

∎

Corollary 3.38. *Suppose f is smooth on an open interval I containing c and that all derivatives $f^{(n)}$ of all orders are bounded on I. Then f equals its Taylor series (centered at c) on I.*

Proof. For simplicity, let $c = 0$. Suppose $\left|f^{(n+1)}(\xi)\right| \leq K$ for all $\xi \in I$. Choose any $N > |x|$ and observe that

$$n > N \implies |R_n(x)| \leq \frac{K|x|^{n+1}}{(n+1)!} = \frac{K|x|^{n+1}}{N!(N+1)\cdots(n+1)} \leq \frac{K|x|^N}{N!}\left(\frac{|x|}{N}\right)^{n+1-N} \xrightarrow[n\to\infty]{} 0$$

∎

Examples 3.39. 1. The functions sine and cosine have derivatives bounded by 1 on $\mathbb{R}$, and thus both functions equal their Maclaurin series on $\mathbb{R}$. This removes the need to have previously justified these facts using the theory of differential equations.

2. The exponential function does not have bounded derivatives, however we can still apply Taylor's Theorem. For any fixed x, $\exists \xi$ between 0 and x such that

$$|R_n(x)| = \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \xrightarrow[n\to\infty]{} 0$$

by the same argument in the Corollary. Thus e^x equals its Maclaurin series on the real line.

3. Extending Example 3.32.3, we see that the function $h(x) = \sqrt{x}$ has the following linear approximation (1st Taylor polynomial) centered at $c = 9$

$$p_1(x) = h(9) + h'(9)(x-9) = 3 + \frac{1}{6}(x-9)$$

This yields the simple approximation

$$\sqrt{10} \approx p_1(10) = 3 + \frac{1}{6} = \frac{19}{6}$$

Taylor's Theorem can be used to estimate its accuracy (remember to shift the center to 9!):

$$R_1(10) = \frac{h''(\xi)}{2!}(10-9)^2 = -\frac{1}{2^2 \cdot 2!}\xi^{-3/2} = -\frac{1}{8\xi^{3/2}} \quad \text{for some} \quad \xi \in (9,10)$$

Certainly $\xi^{-3/2} < 9^{-3/2} = \frac{1}{27}$, whence

$$-\frac{1}{216} < R_1(10) < 0 \implies \frac{19}{6} - \frac{1}{216} = \frac{683}{216} < \sqrt{10} < \frac{684}{216} = \frac{19}{6}$$

$\frac{19}{6}$ is therefore an overestimate for $\sqrt{10}$, but is accurate to within $\frac{1}{216} < 0.005$.

Alternative Versions of Taylor's Theorem

The two other common expressions for the remainder are typically less easy to use than Lagrange's form, but can sometimes provide sharper estimates for the remainder, particularly when x is far from the center of the series.

Corollary 3.40. *Suppose $f^{(n+1)}$ is continuous on an open interval I containing c, let $x \in I \setminus \{c\}$, and let $R_n(x) = f(x) - p_n(x)$ be the remainder for the Taylor polynomial centered at c. Then:*

1. *(Integral Remainder)* $R_n(x) = \int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt$

2. *(Cauchy's Form)* $\exists \xi$ *between c and x such that* $R_n(x) = \frac{(x-\xi)^n}{n!}(x-c)f^{(n+1)}(\xi)$

Using these expressions it is possible to explicitly prove Newton's binomial series formula:

Theorem 3.41. *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n$$
$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 + \cdots$$

If $\alpha \in \mathbb{N}_0$, this is the usual binomial theorem. Otherwise it is more interesting, for instance,

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$
$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots$$

Of course this last could easily be obtained from $\frac{1}{1+x} = \sum(-1)^n x^n$ by differentiating twice!

Exercises 31 1. Compute the Maclaurin series for $\cos x$ directly from the definition and use Taylor's Theorem to indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

2. Repeat the previous exercise for $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

3. Find the Maclaurin series for the function $\sin(3x^2)$. How do you know you are correct?

4. Find the Taylor series of $f(x) = x^4 - 3x^2 + 2x - 5$ centered at $x = 2$ and show that $\mathrm{T}f(x) = f(x)$.

5. Find a rational approximation to $\sqrt[3]{9}$ using the first Taylor polynomial for $f(x) = \sqrt[3]{x}$. Now use Taylor's Theorem to estimate its accuracy.

6. If $c \neq 1$, use the fact that $1 - x = (1-c)\left(1 - \frac{x-c}{1-c}\right)$ to obtain the Taylor series of $\frac{1}{1-x}$ centered at c. Hence conclude that $\frac{1}{1-x}$ is analytic on its domain $\mathbb{R} \setminus \{1\}$.

7. We use Taylor's Theorem to prove that the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$ converges to $\ln(1+x)$ whenever $0 < x \leq 1$.

   (a) Explicitly compute $\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} \ln(1+x)$.

   (b) Suppose $0 < x \leq 1$. Using Taylor's Theorem, prove that $\lim_{n\to\infty} R_n(x) = 0$.

   (*If* $-1 < x < 0$, *the argument is tougher, being similar to Exercise 11*)

8. Why can't we use Taylor's Theorem to approximate the error in $\frac{1}{1-x} = 1 + x + R_1(x)$ when $x \geq 1$? Try it when $x = 2$, what happens? What about when $x = -2$?

9. Prove Taylor's Theorem with integral remainder when $c = 0$ by using the following as an induction step: for each $n \in \mathbb{N}$, define

$$A_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t$$

   and use integration by parts to prove that $A_{n+1} = A_n - \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(0)$.

   (*The Cauchy form follows from the intermediate value theorem for integrals which we'll see later*)

10. Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove by induction that there exists a degree $2n$ polynomial q_n for which

$$f^{(n)}(x) = q_n\left(\tfrac{1}{x}\right) e^{-1/x} \text{ whenever } x > 0$$

(b) Prove that f is infinitely differentiable at $x = 0$ with $f^{(n)}(0) = 0$ (*use Exercise 30.3*).

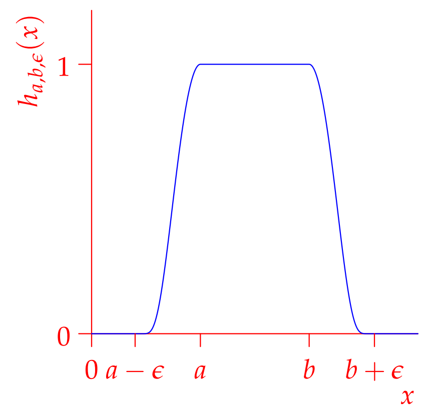
The Maclaurin series of f is identically zero! Moreover, f is smooth (infinitely differentiable) on $\mathbb{R}$ but non-analytic at zero since it does not equal its Taylor series on any open interval containing zero.
A modification allows us to create bump functions, *which find wide use in analysis. If $a < b$, define*

$$g_{a,b} : x \mapsto f(x-a)f(b-x)$$

This is smooth on $\mathbb{R}$ but non-zero only on the interval (a,b). A further modification involving two such functions $g_{a,b}$ creates a smooth function on $\mathbb{R}$ which satisfies

$$h_{a,b,\epsilon}(x) = \begin{cases} 0 & \text{if } x \le a-\epsilon \text{ or } x \ge b+\epsilon \\ 1 & \text{if } a \le x \le b \end{cases}$$

This 'switches on' rapidly from 0 to 1 near a and switches off similarly near b. By letting ϵ be small, we smoothly (but not uniformly) approximate the indicator function on $[a,b]$.



11. (Hard) We prove the binomial series formula. Let $f(x) = (1+x)^\alpha$ and $g(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ where $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. Our goal is to prove that $f = g$ on the interval $(-1,1)$.

(a) Check that $f^{(n)}(0) = n!a_n$ so that g really is the Maclaurin series of f.

(b) i. Prove that the radius of convergence of g is 1.
ii. Prove that $\lim_{n\to\infty} na_n x^n = 0$ whenever $|x| < 1$.
iii. If $|x| < 1$ and ξ lies between 0 and x, prove that $\left|\frac{x-\xi}{1+\xi}\right| \le |x|$.
(*Hint: $\xi = tx$ for some $t \in (0,1)\ldots$*)

(c) Use Taylor's Theorem with Cauchy remainder to prove that

$$|R_n(x)| < (n+1)\,|a_{n+1}|\,|x|^{n+1}\,(1+\xi)^{\alpha-1}$$

Hence conclude that $g = f$ whenever $|x| < 1$.

(d) Here is an alternative argument:
i. Show that $(n+1)a_{n+1} + na_n = \alpha a_n$.
ii. Differentiate term-by-term to prove directly that g satisfies the differential equation $(1+x)g'(x) = \alpha g(x)$. Solve this to show that $g = f$ whenever $|x| < 1$.