1. (a) Give the $\varepsilon$–$\delta$ definition of what it means for a real-valued function $f$ to be uniformly continuous on a set $U$. 
(b) How does the condition on $\delta$ change if $f$ is merely continuous on $U$? 
(c) Prove, using $\varepsilon$–$\delta$ arguments, that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ for any $a > 0$. 
(d) Prove, either directly or by quoting a theorem, that $f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$.

2. Suppose that $f$ is real-valued and continuous on $\mathbb{R}$, and that $f(a)f(b) < 0$ for some $a, b \in \mathbb{R}$. Prove the following:
   (a) $a \neq b$. 
   (b) There exists $x$ between $a, b$ such that $f(x) = 0$.

3. Prove that the series of functions $\sum_{n=1}^{\infty} \frac{x^3}{n^2 + x^2}$ converges uniformly on $|x| \leq 1$.

4. Let $(a_n)$ be the sequence such that $a_{2k} = 1$ for all $k$, and every other term is zero. Calculate the exact interval of convergence of the series $\sum_{n=0}^{\infty} a_nx^n$.

5. Consider $g_n(x) = \frac{nx}{1+nx^2}$ defined on $\mathbb{R}$.
   (a) Find the pointwise limit $g$ of $g_n$. 
   (b) Is the convergence $g_n \to g$ uniform on $\mathbb{R}$? Explain your answer. 
   (c) Find the maximum of the function $|g_n(x) - g(x)|$ on $[a, \infty)$ where $a > 0$ and thus prove that $g_n \to g$ uniformly on any such interval.