Math 140B - Notes

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1 Continuity

The primary goal of this course is to make elementary calculus rigorous. We begin with a review of some basic concepts and conventions.

Sets & Functions We are concerned with functions $f : U \to V$ where both U, V are subsets of the real numbers $\mathbb{R}$:

Domain $\text{dom}(f) = U$; the *inputs* to f. Often implied to be the largest set on which a formula is defined. In calculus examples, the domain is typically a union of intervals of *positive length.*

Codomain $\text{codom}(f) = V$. We often take $V = \mathbb{R}$ by default.

Range $\text{range}(f) = f(U) = \{f(x) : x \in U\}$; the *outputs* of f and a subset of V.

Injectivity f is *injective/one-to-one* if $f(x) = f(y) \implies x = y$.

Surjectivity f is *surjective/onto* if $f(U) = V$.

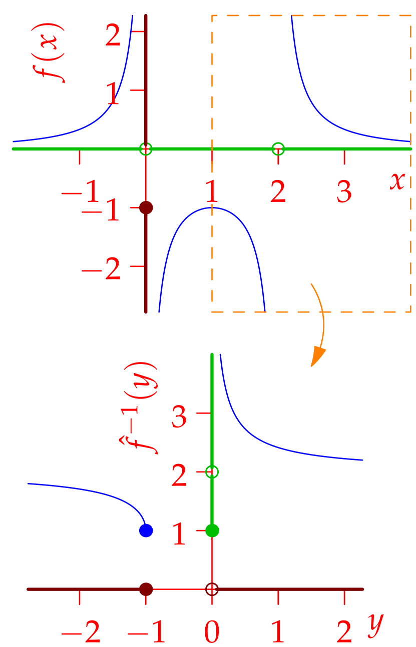
Inverses f is *bijective/invertible* if it is injective and surjective. Equivalently, $\exists f^{-1} : V \to U$ satisfying

$$\forall u \in U,\ f^{-1}\big(f(u)\big) = u \quad \text{and} \quad \forall v \in V,\ f\big(f^{-1}(v)\big) = v$$

Example 1.1. The function defined by $f(x) = \frac{1}{x(x-2)}$ has implied

$$\text{dom}(f) = \mathbb{R} \setminus \{0,2\} = (-\infty, 0) \cup (0,2) \cup (2,\infty)$$

$$\text{range}(f) = (-\infty, -1] \cup (0,\infty)$$



The function is neither injective nor surjective.
By restricting the domain/codomain, we obtain a bijection:

$$\text{dom}(\hat{f}) = [1,2) \cup (2,\infty)$$

$$\text{codom}(\hat{f}) = (-\infty, -1] \cup (0,\infty)$$

with inverse

$$\hat{f}^{-1}(y) = \begin{cases} 1 + y^{-1}\sqrt{y+1} & \text{if } y > 0 \\ 1 - y^{-1}\sqrt{y+1} & \text{if } y \le -1 \end{cases}$$

Now $\text{dom}(\hat{f}^{-1}) = \text{codom}(\hat{f})$ and $\text{codom}(\hat{f}^{-1}) = \text{dom}(\hat{f})$.

Suprema and Infima A set $U \subseteq \mathbb{R}$ is *bounded above* if it has an *upper bound* M:

$$\exists M \in \mathbb{R} \text{ such that } \forall u \in U,\ u \leq M$$

Axiom 1.2 (Completeness). If $U \subseteq \mathbb{R}$ is non-empty and bounded above then it has a *least upper bound*, the *supremum* of U

$$\sup U = \min\{M \in \mathbb{R} : \forall u \in U,\ u \leq M\}$$

By convention, $\sup U = \infty$ if U is unbounded above and $\sup \emptyset = -\infty$; now every subset of $\mathbb{R}$ has a supremum. Similarly, the *infimum* of U is its *greatest lower bound*:

$$\inf U = \begin{cases} \max\{m \in \mathbb{R} : \forall u \in U,\ u \geq m\} & \text{if } U \neq \emptyset \text{ is bounded below} \\ -\infty & \text{if } U \neq \emptyset \text{ is unbounded below} \\ \infty & \text{if } U = \emptyset \end{cases}$$

Examples 1.3. Here are four sets with their suprema and infima stated. You should be able to verify these assertions directly from the definitions.

U	$\{1,2,3,4\}$	$(0,5)$	$(-\infty,\pi]$	$\mathbb{R}$	$\{\frac{1}{n} : n \in \mathbb{N}\}$
$\sup U$	4	5	π	∞	1
$\inf U$	1	0	$-\infty$	$-\infty$	0

Note how the supremum/infimum might or might not lie in the set itself.

Interiors, closures, boundaries and neighborhoods These last concepts might not be review, but they will be used repeatedly.

Definition 1.4. Let $U \subseteq \mathbb{R}$. A value $a \in \mathbb{R}$ is *interior* to U if it lies in some open subinterval of U:

$$\exists \delta > 0 \text{ such that } (a-\delta, a+\delta) \subseteq U$$

A *neighborhood* of a is any set to which a is interior: the interval $(a-\delta, a+\delta)$ is an *open δ-neighborhood* of a. A *punctured neighborhood* of a is a neighborhood with a deleted.

The set of points interior to U is denoted U°.

A *limit point* of U is the limit of some sequence $(x_n) \subseteq U$. The *closure* $\overline{U}$ is the set of limit points.

The *boundary* is the set $\partial U = \overline{U} \setminus U^\circ$.

Examples 1.5. 1. If $U = [1,3)$, then $U^\circ = (1,3)$, $\overline{U} = [1,3]$ and $\partial U = \{1,3\}$.

2. $\mathbb{Q}^\circ = \emptyset$ and $\partial\mathbb{Q} = \overline{\mathbb{Q}} = \mathbb{R}$.

3. $(-3,5) \cup (5,7]$ is a punctured neighborhood of 5.

17 Continuity of Functions

Everything in this section[1] *should* be review.

Definition 1.6. A function $f : U \to \mathbb{R}$ is *continuous at* $u \in U$ if either of the following hold:

1. For all sequences $(x_n) \subseteq U$ converging to u, the sequence $(f(x_n))$ converges to $f(u)$.
2. $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x \in U, |x-u| < \delta \implies |f(x) - f(u)| < \epsilon$.

A function f is *continuous on* U if it is continuous at every point $u \in U$.

Examples 1.7. 1. We prove that $f(x) = x^3$ is continuous at $u = 2$.

(a) (Limit method) Let $x_n \to 2$. By the *limit laws* (i.e. $\lim(x_n^k) = (\lim x_n)^k$),

$$\lim_{x_n \to 2} f(x_n) = \lim_{x_n \to 2} x_n^3 = \left(\lim_{x_n \to 2} x_n\right)^3 = 2^3 = f(2)$$

(b) (ϵ–δ method) Let $\epsilon > 0$ be given and let $\delta = \min\left(1, \frac{\epsilon}{19}\right)$.

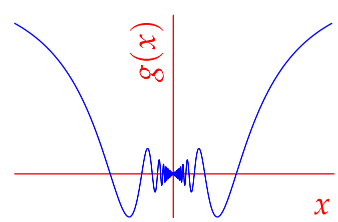
$$|x-2| < \delta \implies |x-2| < 1 \implies 1 < x < 3$$

from which

$$\left|x^3 - 2^3\right| = |x-2| \left|x^2 + 2x + 2^2\right| < 19\,|x-2| \le \epsilon$$

where we used the triangle inequality.

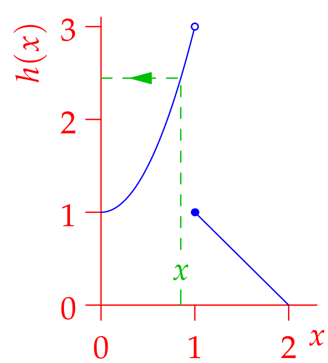
2. Let $g(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$



Then g is continuous at $x = 0$. Again this can be done with limits or an ϵ–δ argument; both are essentially the *squeeze theorem.*

3. The function defined by

$$h(x) = \begin{cases} 1 + 2x^2 & \text{if } x < 1 \\ 2 - x & \text{if } x \ge 1 \end{cases}$$



is discontinuous at $x = 1$.

(a) The sequence with $x_n = 1 - \frac{1}{n}$ converges to 1, yet

$$\lim h(x_n) = 3 \neq 1 = h(1)$$

(b) Choose $\epsilon = 1$ and suppose $\delta > 0$ is given. Now choose $x = \max\{1 - \frac{\delta}{2}, \frac{1}{\sqrt{2}}\}$ to see that

$$|x - 1| < \delta \quad \text{and} \quad |h(x) - h(1)| \ge 1 = \epsilon$$

[1] Section numbers are identical to those in the official textbook.

Theorem 1.8. *The two parts of Definition 1.6 are equivalent.*

Proof. $(1 \Rightarrow 2)$ We prove the contrapositive. Suppose condition 2 is *false*; that is,

$$\exists \epsilon > 0, \text{ such that } \forall \delta > 0, \ \exists x \in U \text{ with } |x-u| < \delta \text{ and } |f(x) - f(u)| \geq \epsilon$$

In particular, for any $n \in \mathbb{N}$ we may let $\delta = \frac{1}{n}$ to obtain

$$\exists \epsilon > 0, \text{ such that } \forall n \in \mathbb{N}, \ \exists x_n \in U \text{ with } |x_n - u| < \frac{1}{n} \text{ and } |f(x_n) - f(u)| \geq \epsilon$$

The sequence (x_n) shows that condition 1 is *false*:

- $\forall n, \ |x_n - u| < \frac{1}{n}$ whence $x_n \to u$.
- $\forall n, \ |f(x_n) - f(u)| \geq \epsilon > 0$, whence $f(x_n)$ *does not converge to* $f(u)$.

$(2 \Rightarrow 1)$ Suppose condition 2 is true, that $(x_n) \subseteq U$ converges to u and that $\epsilon > 0$ is given. Then

$$\exists \delta > 0 \text{ such that } |x-u| < \delta \implies |f(x) - f(u)| < \epsilon$$

However, by the definition of convergence ($x_n \to u$),

$$\exists N \in \mathbb{N} \text{ such that } n > N \implies |x_n - u| < \delta \implies |f(x_n) - f(u)| < \epsilon$$

Otherwise said, $f(x_n) \to f(u)$. ∎

Rather than use these definitions every time, it is helpful to have a working dictionary.

Theorem 1.9 (Common Continuous Functions).

1. *Suppose f and g are continuous at u, that h is continuous at $f(u)$ and that k is constant. Then the following are continuous at u (if defined):*

$$f+g, \quad f-g, \quad fg, \quad \frac{f}{g}, \quad |f|, \quad kf, \quad \max(f,g), \quad \min(f,g), \quad h \circ f$$

2. *Algebraic[2] functions are continuous.*
3. *The common transcendental functions are continuous:* exp, ln, sin, *etc.*

Example 1.10. $f(x) = \sin \frac{\sqrt[3]{x^2+7}}{x-2} + \cos \frac{1}{e^x - 1}$ is continuous on its domain $(-\infty, 0) \cup (0,1) \cup (1, \infty)$.

These claims are tedious to prove using elementary definitions. The first two require many uses of the limit laws, while the transcendental claim is easier to defer until we can define the common functions using power series, after which continuity comeS for free.

[2]Constructed using finitely many addition/subtraction, multiplication/division and n^{th} root operations

Exercises 17 1. Give examples to show that $g \circ f$ being continuous can happen with:

   (a) f continuous and g discontinuous.
   (b) g continuous and f discontinuous.
   (c) Both f, g discontinuous.

   You may use pictures, but make sure they clearly describe the functions f, g.

2. (a) Prove that the function $f(x) = x^3$ is continuous at $x = -2$ using an ϵ–δ argument.
   (b) Prove that $f(x) = x^3$ is continuous at $x = u$ using an ϵ–δ argument.

3. Prove that the following are discontinuous at $x = 0$: use *both* definitions of continuity.

   (a) $f(x) = 1$ for $x < 0$ and $f(x) = 0$ for $x \geq 0$.
   (b) $g(x) = \sin(1/x)$ for $x \neq 0$ and $g(0) = 0$.

4. Suppose f and g are continuous at u. Prove the following using ϵ–δ arguments.

   (a) $f - g$ is continuous at u.
   (b) If h is continuous at $f(u)$, then $h \circ f$ is continuous at u.

5. Contrary to our standing assumption, suppose $f : U \to \mathbb{R}$ is a function whose domain U contains an *isolated point* a: i.e. $\exists r > 0$ such that $(a - r, a + r) \cap U = \{a\}$. Prove that f is continuous at a.

6. Refresh your prerequisites by giving formal proofs of the following:

   (a) (Suprema and sequences) If $M = \sup U$, then $\exists (x_n) \subseteq U$ such that $x_n \to M$. (*Remember that this has to work even if* $M = \infty \ldots$)
   (b) (Limit of a bounded sequence) If $(x_n) \subseteq [a, b]$ and $x_n \to x$, then $x \in [a, b]$.
   (c) (Bolzano–Weierstraß) Every bounded sequence in $\mathbb{R}$ has a convergent subsequence. (*Hint: If* $(x_n) \subseteq [a, b]$*, explain why there exists a family of nested intervals* $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ *such that* infinitely many *of the terms* (x_n) *lie in each interval* I_k*. Hence obtain a subsequence* (x_{n_k}) *and prove that it is* Cauchy.[3])

7. (Hard) Consider the function $f : \mathbb{R} \to \mathbb{R}$ where

$$f(x) = \begin{cases} \frac{1}{q} & \text{whenever } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q > 0 \text{ and } \gcd(p, q) = 1 \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   For example, $f(1) = f(2) = f(-7) = 1$, and $f(\frac{1}{2}) = f(-\frac{1}{2}) = f(\frac{3}{2}) = \cdots = \frac{1}{2}$, etc. Prove that f is continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$ and discontinuous at each point of $\mathbb{Q}$.

[3] This is a good moment to review the notion of a Cauchy sequence

$$\forall \epsilon > 0, \exists N \text{ such that } m, n > N \implies |x_m - x_n| < \epsilon$$

and the discussion of Cauchy completeness: $(x_n) \subseteq \mathbb{R}$ is convergent if and only if it is Cauchy.

18 Properties of Continuous Functions

The goal of this section is to describe the behavior of a continuous function on an interval. We first consider the special case when the domain is a closed bounded interval $[a, b]$.

Theorem 1.11 (Extreme Value Theorem). *A continuous function on a closed, bounded interval is bounded and attains its bounds. Otherwise said, if $f : [a, b] \to \mathbb{R}$ is continuous, then*

$$\exists x, y \in [a, b] \text{ such that } f(x) = \sup \text{range}(f) \text{ and } f(y) = \inf \text{range}(f)$$

In particular, the supremum and infimum are finite.

Proof. Suppose f is continuous with domain $[a, b]$ and let $M = \sup\{f(x) : x \in [a, b]\}$. We invoke the three parts of Exercise 17.6:

- (Part a) There exists a sequence $(x_n) \subseteq [a, b]$ such that $f(x_n) \to M$.
- (Part c) There exists a convergent subsequence (x_{n_k}) with limit x.
- (Part b) $x \in [a, b]$.

Since f is continuous, we now have $f(x) = \lim_{k\to\infty} f(x_{n_k}) = M$. This shows that M is *finite* and that f attains its least upper bound. For the lower bound, apply this to $-f$. ■

It is worth considering how the result can fail when one of the hypotheses is weakened. For example:

f discontinuous $f : [0, 1] \to \mathbb{R} : x \mapsto \begin{cases} x & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$ is bounded but does not attain its bounds.

$\text{dom}(f)$ *not closed* $f : [0, 1) \to \mathbb{R} : x \mapsto x$ is bounded but does not attain its bounds.

$\text{dom}(f)$ *not bounded* $f : [0, \infty) \to \mathbb{R} : x \mapsto x$ is unbounded.

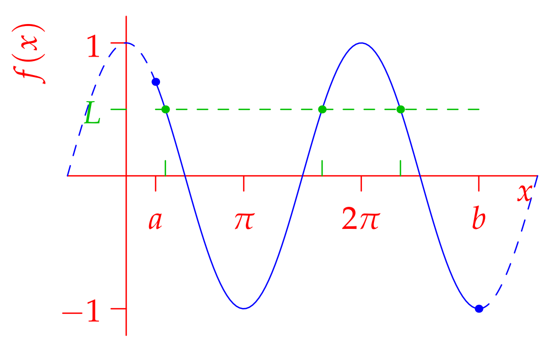
We now generalize to functions on arbitrary intervals. Our next result should be familiar from elementary calculus and is intuitively obvious from the naïve notion of continuity: graph such a function without taking your pen from the page.

Theorem 1.12 (Intermediate Value Theorem). *Let $f : I \to \mathbb{R}$ be continuous on an interval I. Suppose $a, b \in I$ with $a < b$ and that $f(a) \neq f(b)$. If L lies between $f(a)$ and $f(b)$, then $\exists \xi \in (a, b)$ such that $f(\xi) = L$.*

Example 1.13. Let $f(x) = \cos x$ with $a = \frac{\pi}{4}$, $b = 3\pi$ and $L = \frac{1}{2}$; then

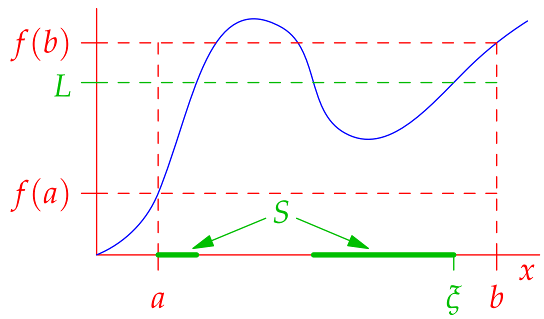
$$f(\xi) = L \iff \xi \in \left\{ \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \right\}$$

There may therefore be several suitable values of ξ. It is even possible (see Exercise 18.2) for there to be *infinitely many*.



Proof. Suppose WLOG that $f(a) < L < f(b)$ and let

$$S = \{x \in [a,b] : f(x) < L\}$$



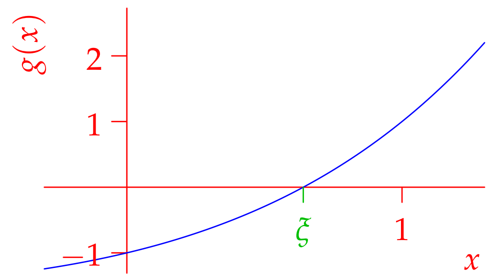
Plainly $S \subseteq [a,b)$ is non-empty, hence $\xi := \sup S$ exists and $\xi \in [a,b]$. It remains to show that ξ satisfies the required properties.

By Exercise 6, $\exists (s_n) \subseteq S$ with $\lim s_n = \xi$. Since f is continuous, $f(\xi) = \lim f(s_n) \leq L$. In particular, $\xi \neq b$.

To finish the proof, we can play a similar game with the sequence defined by $t_n = \min\{b, \xi + \frac{1}{n}\}$; this is left to Exercise 4. ∎

Example 1.14. The intermediate value theorem is particularly useful for demonstrating the existence of solutions to equations. For example, we can use the following steps to show that the equation $x2^x = 1$ has a solution.

- $g(x) = x2^x - 1$ is continuous.
- $g(0) = -1 < 0$.
- $g(1) = 1 > 0$.
- By the intermediate value theorem $\exists \xi \in (0,1)$ such that $g(\xi) = 0$: that is $\xi \cdot 2^\xi = 1$.



It is inefficient, but one can home in on ξ by repeatedly halving the size of the interval: for instance,

$$g(\tfrac{1}{2}) = \tfrac{\sqrt{2}}{2} - 1 < 0, \quad g(\tfrac{3}{4}) = \tfrac{3}{4} \cdot 2^{3/4} - 1 \approx 0.26 > 0 \ldots \implies \tfrac{1}{2} < \xi < \tfrac{3}{4}$$

We finish with a useful corollary.

Corollary 1.15. *Continuous functions map intervals to intervals (or points).*

Proof. An interval I is characterized by the following property

$$\forall x_1, x_2 \in I,\ x \in \mathbb{R},\ x_1 < x < x_2 \implies x \in I$$

Let $f : I \to \mathbb{R}$ be continuous and suppose its range $f(I)$ is not a single point. If $f(a) < L < f(b)$, then $\exists \xi$ between a, b such that $f(\xi) = L$. Otherwise said, $L \in f(I)$ and so $f(I)$ is an interval. ∎

More generally, if $\operatorname{dom}(f) = \bigcup I_n$ is written as a union of disjoint intervals and f is continuous, then

$$\operatorname{range}(f) = \bigcup f(I_n)$$

is also a union of intervals, though these need not be disjoint: a continuous function can bring intervals together, but cannot break an interval apart.

For example, $f(x) = \sqrt{x^2 - 4}$ has domain $(-\infty, -2] \cup [2, \infty)$ and range $[0, \infty)$: both original intervals get mapped to the same interval by f.

A more general statement from topology says that if $f : U \to V$ is continuous between topological spaces and a, b lie in the same *component* of U, then $f(a)$ and $f(b)$ lie in the same component of $f(U)$. In single-variable real analysis each component is an interval.

Exercises 18 1. Give examples of the following:

(a) An unbounded discontinuous function on a closed bounded interval.

(b) An unbounded continuous function on a non-closed bounded interval.

(c) A bounded continuous function on a closed unbounded interval which fails to attain its bounds.

2. Consider the function $f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(a) Explain why f is continuous on any interval I.

(b) Suppose $a < 0 < b$ and that $f(a), f(b)$ have opposite signs. If $L = 0$, show that the intermediate value theorem is satisfied by *infinitely many* distinct values ξ.

3. Use the intermediate value theorem to prove that the equation $8x^3 - 12x^2 - 2x + 1 = 0$ has at least 3 real solutions (and thus, by the fundamental theorem of algebra, exactly 3).

4. Complete the proof of the intermediate value theorem by defining $t_n = \min(b, \xi + \frac{1}{n})$.

5. (a) Suppose $f : U \to \mathbb{R}$ is continuous and that $U = \bigcup_{k=1}^{n} I_k$ is the union of a finite sequence (I_k) of closed bounded intervals. Prove that f is bounded and attains its bounds.

(b) Let $U = \bigcup_{n=1}^{\infty} I_n$, where $I_n = [\frac{1}{2n}, \frac{1}{2n-1}]$ for each $n \in \mathbb{N}$. Give an example of a continuous function $f : U \to \mathbb{R}$ which is either unbounded or does not attain its bounds. Explain.

19 Uniform Continuity

Recall the ϵ–δ definition of continuity: $f : U \to \mathbb{R}$ is continuous at all points[4] $y \in U$, we require

$$\forall y \in U,\ \forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x \in U)\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

Note the order of the quantifiers: δ is permitted to depend on *both* y and ϵ. In the naïve sense of continuity (x close to $y \implies f(x)$ close to $f(y)$), the meaning of *close* is seen to depend on the *location* y. Uniform continuity is a stronger condition where the meaning of 'close' is *independent* of location.

Definition 1.16. $f : U \to \mathbb{R}$ is *uniformly continuous* if

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } (\forall x, y \in U)\ |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

We've included the (typically) hidden quantifiers $(\forall x, y)$ in both definitions to make clear that ϵ and δ are independent of x and y. Note also that the definition is now symmetric in x and y.

Example 1.17. Consider $f(x) = \frac{1}{x}$.

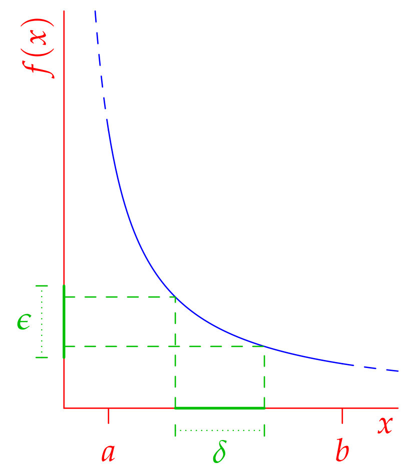
1. If $0 < a < b \leq \infty$, then f is uniformly continuous on $[a, b)$.
   Let $\epsilon > 0$ be given and let $\delta = a^2\epsilon$. Then $\forall x, y \in [a, b)$,

$$|x - y| < \delta \implies \left|\frac{1}{x} - \frac{1}{y}\right| = \left|\frac{y - x}{xy}\right| < \frac{\delta}{xy} \leq \frac{\delta}{a^2} = \epsilon$$

2. If $0 < b \leq \infty$, then f is *not* uniformly continuous on $(0, b)$.

   Let $\epsilon = 1$ and suppose $\delta > 0$ is given.
   Let $x = \min(\delta, 1, \frac{b}{2})$ and $y = \frac{x}{2}$.
   Certainly $x, y \in (0, b)$ and $|x - y| = \frac{x}{2} \leq \frac{\delta}{2} < \delta$. However,

$$|f(x) - f(y)| = \frac{1}{x} \geq 1 = \epsilon$$



Think about how ϵ and δ must relate as one slides the intervals in the picture up/down and left/right.

Some intuition will help make sense of the above examples.

Bounded/unbounded gradient In part 1, $\epsilon = \delta a^2$, where $\frac{1}{a^2} = |f'(a)|$ bounds the gradient of f. By contrast, the slope of f is *unbounded* in part 2.

Extendability In part 1 (if $b \neq \infty$), the domain of f may be extended: $g : [a, b] \to \mathbb{R} : x \mapsto \frac{1}{x}$ is continuous. In part 2, this is impossible: there is no continuous function $g : [0, b) \to \mathbb{R}$ such that $g(x) = \frac{1}{x}$ whenever $x > 0$.

If the gradient of a continuous function is bounded or if you can 'fill in the holes' at the endpoints of its domain, then the function is uniformly continuous. While the utility of uniform continuity is often in proofs when the independence of ϵ and location are critical, it is often one of the above properties that is being invoked. The remainder of this section involves making thse observations watertight.

[4] To promote the symmetry in the coming definition, we use y instead of u for a generic point of $\text{dom}(f)$.

Theorem 1.18. *Let $f : I \to \mathbb{R}$ be differentiable on an interval I. If the derivative f' is bounded on the interior I°, then f is uniformly continuous on I.*

The proof depends on the mean value theorem, which we'll prove later in the term.

Proof. Suppose $|f'(x)| \le M$ on I°. Let $\epsilon > 0$ be given, let $\delta = \frac{\epsilon}{M}$ and suppose $x, y \in I$ with $x > y$. Then

$$|x-y| < \delta \implies \exists \xi \in I^\circ \text{ such that } f'(\xi) = \frac{f(x)-f(y)}{x-y} \qquad \text{(MVT)}$$
$$\implies |f(x)-f(y)| = |f'(\xi)|\,|x-y| < M\delta = \epsilon$$

■

Theorem 1.18 isn't a biconditional: for instance, Exercise 19.5 shows that $f(x) = \sqrt{x}$ on $[0,\infty)$ and $g(x) = x^{1/3}$ on $\mathbb{R}$ are both uniformly continuous even though they have unbounded slope.
We now discuss the idea of extendability and how uniform continuity relates to continuity on closed sets. First we see that for closed bounded sets, uniform continuity is nothing new.

Theorem 1.19. *If $g : [a,b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.*

Proof. Suppose g is continuous but not uniformly so. Then

$$\exists \epsilon > 0 \text{ such that } \forall \delta > 0,\ \exists x, y \in [a,b] \text{ for which } |x-y| < \delta \text{ and } |g(x)-g(y)| \ge \epsilon \qquad (*)$$

For each $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ to see that there exists sequences $(x_n), (y_n) \subseteq [a,b]$ satisfying the above. By Bolzano–Weierstraß, the bounded sequence (x_n) has a convergent subsequence $x_{n_k} \to x \in [a,b]$. Clearly

$$|x_{n_k} - y_{n_k}| < \frac{1}{n_k} \to 0 \implies y_{n_k} \to x$$

But then $|g(x_{n_k}) - g(y_{n_k})| \to 0$ which contradicts $(*)$.

■

Now we build to a partial converse of this.

Lemma 1.20. *If $f : U \to \mathbb{R}$ is uniformly continuous and $(x_n) \subseteq U$ is a Cauchy sequence, then $(f(x_n))$ is also Cauchy.*

Proof. Let $\epsilon > 0$ be given. Then:

- (Uniform Continuity) $\exists \delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < \epsilon$.
- (Cauchy) $\exists N \in \mathbb{N}$ such that $m, n > N \implies |x_m - x_n| < \delta$.

Putting these together, we see that

$$\exists N \in \mathbb{N} \text{ such that } m, n > N \implies |f(x_m) - f(x_n)| < \epsilon$$

Otherwise said, $(f(x_n))$ is Cauchy.

■

We now see that a function $f : I \to \mathbb{R}$ is uniformly continuous on a bounded interval if and only it is has a *continuous extension* $g : \overline{I} \to \mathbb{R}$ defined on the closure of its domain.

Theorem 1.21. *Suppose $f : I \to \mathbb{R}$ is continuous where I is a bounded interval with endpoints $a < b$. Define $g : [a, b] \to \mathbb{R}$ via*

$$g(x) = \begin{cases} f(x) & \text{if } x \in I \\ \lim f(x_n) & \text{whenever } (x_n) \subseteq I \text{ and } x_n \to a \\ \lim f(x_n) & \text{whenever } (x_n) \subseteq I \text{ and } x_n \to b \end{cases}$$

Then f is uniformly continuous if and only g is well-defined (g is continuous, if *well-defined).*

Proof. ($\Rightarrow$) Suppose f is uniformly continuous on I and that $a \notin I$. Let $(x_n), (y_n) \subseteq I$ be sequences converging to a. To show that g is well-defined, we must prove that $(f(x_n))$ and $(f(y_n))$ are convergent, and to the same limit.

Define a sequence

$$(u_n) = (x_1, y_1, x_2, y_2, x_3, y_3, \ldots)$$

Since (x_n) and (y_n) have the same limit a, we see that $u_n \to a$. But then (u_n) is Cauchy; by Lemma 1.20, $(f(u_n))$ is also Cauchy and thus convergent. Since $(f(x_n))$ and $(f(y_n))$ are subsequences of a convergent sequence, they must also converge to the same (finite!) limit.

The argument when $b \notin I$ is identical.

($\Leftarrow$) Certainly if g is well-defined then it is continuous. By Theorem 1.19 it is uniformly so. Since $f = g$ on a subset of $\operatorname{dom}(g)$, the same choice of δ will work for f as for g: f is therefore uniformly continuous. ∎

Examples 1.22.

1. $f : x \mapsto x^2$ is uniformly continuous on $(-3, 10)$ since its derivative $f'(x) = 2x$ is bounded ($|f'(x)| = 2|x| \leq 20$) on its domain. It has the obvious continuous extension $g(x) = x^2$ on $[-3, 10]$.

2. Neither argument works for $f(x) = x^2$ on the domain $(-3, \infty)$: both f' and the domain $(-3, \infty)$ are unbounded, so neither Theorem 1.18 nor 1.21 applies.

   Instead, note that if $\epsilon = 1$, then for any $\delta > 0$, we can choose $x = \frac{1}{\delta}$ and $y = \frac{1}{\delta} + \frac{\delta}{2}$. Clearly

   $$|x - y| = \frac{\delta}{2} < \delta \text{ and } \left|x^2 - y^2\right| = 1 + \frac{\delta^2}{4} > 1 = \epsilon$$

   whence f is not uniformly continuous.

3. $f(x) = x \sin \frac{1}{x}$ is continuous on the interval $(0, \infty)$. Strictly, neither Theorem 1.18 nor 1.21 applies since the derivative

   $$f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$$

   is unbounded as is the domain. However, by breaking the domain into two pieces...

- On $(1,\infty)$, the derivative is bounded: $|f'(x)| \leq 1 + \frac{1}{x^2} \leq 2$ by the triangle inequality. Theorem 1.18 says f is uniformly continuous on $(1,\infty)$.
- f is continuous on $(0,1]$ and, by the squeeze theorem

$$x_n \to 0^+ \implies \lim f(x_n) = 0$$

  Extending f so that $f(0) = 0$ defines a continuous extension. By Theorem 1.21, f is uniformly continuous on $(0,1]$.

Putting this together, f is uniformly continuous on $(0,\infty)$. Indeed the function

$$h(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is uniformly continuous on $\mathbb{R}$.

Exercises 19 1. Which of the following functions are uniformly continuous on the specified set? Justify your answers.

(a) $f(x) = x^4$ on $[-1,1]$.

(b) $f(x) = x^4$ on $(-1,1]$.

(c) $f(x) = x^{-4}$ on $(0,2]$.

(d) $f(x) = x^{-4}$ on $(1,2]$.

(e) $f(x) = x^2 \sin \frac{1}{x}$ on $(0,1]$.

2. Prove that each of the following functions is uniformly continuous on the indicated set by verifying the ϵ–δ property.

(a) $f(x) = 2x - 14$ on $\mathbb{R}$.

(b) $f(x) = x^3$ on $[1,5]$.

(c) $f(x) = x^{-1}$ on $(1,\infty)$.

(d) $f(x) = \frac{x+1}{x+2}$ on $[0,1]$.

3. Prove that $f(x) = x^4$ is not uniformly continuous on $\mathbb{R}$.

4. (a) Suppose that f is uniformly continuous on a bounded interval I. Prove that f is bounded on I.

(b) Use part (a) to write down a bounded interval on which the function $f(x) = \tan x$ is defined, but *not* uniformly continuous.

5. (a) Let $f(x) = \sqrt{x}$ with domain $[0,\infty)$. Show that $f'(x)$ is unbounded, but that f is still uniformly continuous on $[0,\infty)$.
(Hint: try $\delta = \epsilon^2$ and WLOG assume $0 \leq y \leq x$. Now compute $(\sqrt{y} + \epsilon)^2 \ldots$)

(b) Prove that $g(x) = x^{1/3}$ is uniformly continuous on $\mathbb{R}$.
(Hint: try $\delta = (\frac{\epsilon}{2})^3$ and consider the cases $x \geq y \geq 0$, $x \leq y \leq 0$ and $x > 0 > y$ separately)

20 Limits of Functions

You've likely seen many calculations of the following form in elementary calculus:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6$$

Our next goal is to make this notation precise and to tie it to our earlier notion of limit.

Definition 1.23. Suppose $f:U\to\mathbb{R}$, that $S\subseteq U$, and that a is the limit of a sequence[5] in S. We write $\lim\limits_{x\to a^S} f(x)=L$ and say that L is the *limit of $f(x)$ as x tends to a along S*, provided

$$\forall (x_n)\subseteq S,\ \lim x_n=a \implies \lim f(x_n)=L$$

We can now define one-sided and two-sided limits:

Right-hand limit: $\lim\limits_{x\to a^+} f(x)=L$ means $\exists S=(a,b)\subseteq U$ for which $\lim\limits_{x\to a^S} f(x)=L$

Left-hand limit: $\lim\limits_{x\to a^-} f(x)=L$ means $\exists S=(c,a)\subseteq U$ for which $\lim\limits_{x\to a^S} f(x)=L$

Two-sided limit: $\lim\limits_{x\to a} f(x)=L$ means $\exists S=(c,a)\cup(a,b)\subseteq U$ for which $\lim\limits_{x\to a^S} f(x)=L$

- The one-sided definitions apply when $a=\pm\infty$, though we omit the $\pm$ modifiers: for instance,

  $$\lim_{x\to\infty} f(x)=L \iff \lim_{x\to\infty^S} f(x)=L \text{ for some } S=(c,\infty)\subseteq U$$

- The subtlety in the definition is that for $\lim\limits_{x\to a} f(x)$ to be defined, the domain U of f must contain a *punctured neighborhood* S of a: i.e. $a\in U^\circ$. The one-sided limits similarly require a *one-sided punctured neighborhood*. These conditions are always satisfied if U is a disjoint union of intervals of positive length, in which case $\lim\limits_{x\to a^{(\pm)}} f(x)=L$ if and only if

  $$\lim f(x_n)=L,\ \forall (x_n)\subseteq U\setminus\{a\} \text{ tending to } a \text{ (from above/below)}$$

  In this situation, Definition 1.6 recovers the familiar idea from elementary calculus:

  $$f \text{ is continuous at } a\in U \iff f(a)=\begin{cases}\lim\limits_{x\to a} f(x) & \text{when } a\in U^\circ\\ \lim\limits_{x\to a^\pm} f(x) & \text{when } a\in U\setminus\partial U\end{cases} \qquad (*)$$

- By modifying the proof of Theorem 1.8 in the case that $a,L\in\mathbb{R}$ are finite, the above can be written in ϵ-language. For example $\lim\limits_{x\to a} f(x)=L$ means

  $$\forall\epsilon>0,\ \exists\delta>0 \text{ such that } (\forall x\in\mathbb{R})\ 0<|x-a|<\delta \implies |f(x)-L|<\epsilon$$

  If a and/or L is infinite, use the language of unboundedness: e.g. $\lim\limits_{x\to a} f(x)=\infty$ means

  $$\forall M>0,\ \exists\delta>0 \text{ such that } 0<|x-a|<\delta \implies f(x)>M$$

  There are *fifteen* distinct combinations: *three* two-sided and *six* each of the one-sided limits!

[5] I.e. $a\in\overline{S}$ or perhaps $a=\pm\infty$ if S is unbounded.

Examples 1.24. 1. Let $f(x)=\frac{2+x}{x}$ where $\operatorname{dom}(f)=U=\mathbb{R}\setminus\{0\}=(-\infty,0)\cup(0,\infty)$

The following should be clear:

$$\lim_{x\to 3} f(x)=\frac{5}{3} \qquad \lim_{x\to\infty} f(x)=1$$

To compute the first, for instance, we could choose $S=(0,3)\cup(3,\infty)$; if $(x_n)\subseteq S$ and $x_n\to 3$, then the limit laws justify the first claim

$$\lim_{n\to\infty} f(x_n)=\frac{2+3}{3}=\frac{5}{3}$$

as does the fact that f is continuous at $x=3$. The second claim can be checked similarly.

We can take one-sided limits at $x=0$:

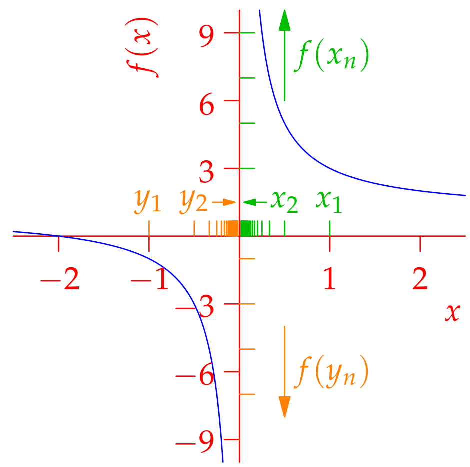
$$\lim_{x\to 0^+} f(x)=\infty \quad \text{and} \quad \lim_{x\to 0^-} f(x)=-\infty$$

For instance, let $(x_n)\subseteq(0,\infty)$ satisfy $x_n\to 0$. Again, the limit laws show that $\lim\limits_{n\to\infty} f(x_n)=\infty$, which is enough to justify the first claim.

Finally, the sequences defined by $x_n=\frac{1}{n}$ and $y_n=-\frac{1}{n}$ both lie in $S=\mathbb{R}\setminus\{0\}$ and converge to zero, yet

$$\lim_{n\to\infty} f(x_n)=\infty\neq-\infty=\lim_{n\to\infty} f(y_n)$$

It follows that the two-sided limit $\lim\limits_{x\to 0} f(x)$ does not exist.



2. Let $f(x)=\frac{1}{x^2}$ whenever $x\neq 0$ and additionally let $f(0)=0$. Here the two-sided limit exists

$$\lim_{x\to 0} f(x)=\infty$$

However the value of the function at $x=0$ does not equal this limit: clearly f is discontinuous at $x=0$.

3. We revisit our motivating example. Let $f(x)=\frac{x^2-9}{x-3}$ have domain $U=\mathbb{R}\setminus\{3\}$. Whenever $x_n\neq 3$, we see that

$$f(x_n)=\frac{(x_n-3)(x_n+3)}{x_n-3}=x_n+3$$

By the limit laws, we conclude that $\lim f(x_n)=3+3=6$ and so

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=6$$

Since we referenced the limit laws so often in the above examples, it is appropriate to update them to this new context. We do so without proof.

Corollary 1.25 (Limit Laws). *Suppose $f, g : U \to \mathbb{R}$ satisfy $L = \lim_{x\to a} f(x)$ and $M = \lim_{x\to a} g(x)$ exist. Then,*

1. $\lim_{x\to a}(f+g)(x) = L + M$.
2. $\lim_{x\to a}(fg)(x) = LM$.
3. $\lim_{x\to a}\left(\frac{f}{g}\right)(x) = \frac{L}{M}$ *(requires $M \neq 0$).*
4. *If $L \in \mathbb{R}$ and h is continuous at L, then $\lim_{x\to a}(h\circ f)(x) = h(L)$.*
5. *(Squeeze Theorem) If $L = M$ and $f(x) \le h(x) \le g(x)$ for all $x \in U$, then $\lim_{x\to a} h(x) = L$.*

The corresponding results for one-sided limits also hold.

As with the original limit laws for sequences, parts 1–3 apply provided the limits are not *indeterminate forms* (e.g. $\infty - \infty$, $0\cdot\infty$, $\frac{0}{0}$, $\frac{\infty}{\infty}$). We'll see later how l'Hôpital's rule may be applied to such cases.

Examples 1.26. 1. Since $f(x) = \frac{x^2+5}{3x^2-2}$ is a rational function (continuous at all points of its domain), we quickly conclude that

$$\lim_{x\to 2}\frac{x^2+5}{3x^2-2} = f(2) = \frac{9}{10}$$

Alternatively, we may tediously invoke the other parts of the theorem:

$$\lim_{x\to 2}\frac{x^2+5}{3x^2-2} \overset{(3)}{=} \frac{\lim(x^2+5)}{\lim(3x^2-2)} \overset{(1)}{=} \frac{\lim x^2 + \lim 5}{\lim 3x^2 - \lim 2} \overset{(2)}{=} \frac{(\lim x)^2+5}{(\lim 3)(\lim x)^2 - 2}$$
$$= \frac{2^2+5}{3\cdot 2^2 - 2} = \frac{9}{10}$$

2. As $x \to \infty$, the simplistic approach results in a nonsense indeterminate form:

$$\lim_{x\to\infty}\frac{x^2+5}{3x^2-2} \overset{?}{=} \frac{\lim(x^2+5)}{\lim(3x^2-2)} \overset{?}{=} \frac{\infty}{\infty}$$

However, a little pre-theorem algebra quickly yields[6]

$$\lim_{x\to\infty}\frac{x^2+5}{3x^2-2} = \lim_{x\to\infty}\frac{1+5x^{-2}}{3-2x^{-2}} = \frac{\lim(1+5x^{-2})}{\lim(3-2x^{-2})} = \frac{1}{3}$$

[6]Be careful! The expressions $\frac{x^2+5}{3x^2-2}$ and $\frac{1+5x^{-2}}{3-2x^{-2}}$ do not describe the same function, yet their *limits* at ∞ are equal. Being able easily to equate these limits is one of the advantages of the '$\exists S$' formulation of Definition 1.23. Think about why; what is a suitable set S in this context?

Classification of Discontinuities

We finish this section by considering the ways in which a function can fail to be continuous.

Definition 1.27. Suppose that a function is continuous on an interval except at finitely many values: we call these *isolated discontinuities.*

Examples 1.28. 1. $f(x) = \frac{1}{x}$ has a discontinuity at $x = 0$ since it is continuous on the interval $\mathbb{R}$, except at one point $x = 0$. Note that a function need not be defined at a discontinuity!

2. $f(x) = \frac{1}{\sin \frac{1}{x}}$ has a *non-isolated discontinuity* at $x = 0$: on any interval containing zero, f has infinitely many discontinuities: $x = \frac{1}{\pi n}$ where $|n| \in \mathbb{N}$.

The next result helps us classify isolated discontinuities.

Theorem 1.29. *Let $f : U \to \mathbb{R}$ and suppose $a \in U^\circ$ is an interior point. Then*

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$$

Proof. ($\Rightarrow$) Let $S = (c, a) \cup (a, b)$ satisfy the definition for $\lim_{x\to a} f(x) = L$. Since any sequence (say) in S^+ is also in S, plainly $S^+ = (a, b)$ and $S^- = (c, a)$ satisfy the one-sided definitions.

($\Leftarrow$) Suppose $S^- = (c, a)$ and $S^+ = (a, b)$ satisfy the one-sided definitions and denote $S = S^- \cup S^+$. Let $(x_n) \subseteq S$ be such that $x_n \to a$. Clearly (x_n) is the disjoint union of two subsequences $(x_n) \cap S^+$ and $(x_n) \cap S^-$, both of which[7] converge to a. There are three cases:

L finite: Let $\epsilon > 0$ be given. Because of the one-sided limits,

- $\exists N_1$ such that $n > N_1$ and $x_n > a \implies |f(x_n) - L| < \epsilon$
- $\exists N_2$ such that $n > N_2$ and $x_n < a \implies |f(x_n) - L| < \epsilon$

Now let $N = \max(N_1, N_2)$ in the definition of limit to see that $\lim f(x_n) = L$. Since this holds for all sequences $(x_n) \subseteq S$ converging to a, we conclude that $\lim_{x\to a} f(x) = L$.

$L = \pm\infty$: This is an exercise.

■

Example 1.30. Recalling elementary calculus, we show that the following is continuous at $x = 1$:

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x \geq 1 \\ 3 - 5x & \text{if } x < 1 \end{cases}$$

Step 1: Compute the left- and right-handed limits and check that these are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 3 - 5x = -2, \qquad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x^2 - 3 = -2$$

Step 2: Check that the value of the limits equals that of the function: $f(1) = 1^2 - 3 = -2$.

[7] It is possible for *one* of these subsequences to be finite; say if $x_n > a$ for all large n. This is of no concern; one of the ϵ-N conditions would be empty and thus vacuously true.

Recalling $(*)$ on page 13, we describe the different types of isolated discontinuity at some point a.

Removable discontinuity The two-sided limit $\lim_{x\to a} f(x) = L$ is finite, and either:

$$f(a) \neq L \text{ or } f(a) \text{ is undefined.}$$

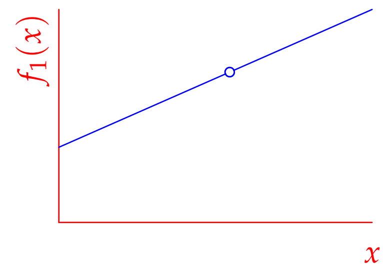
The term comes from the fact that we can remove the discontinuity by changing the behavior of f only at $x = a$:

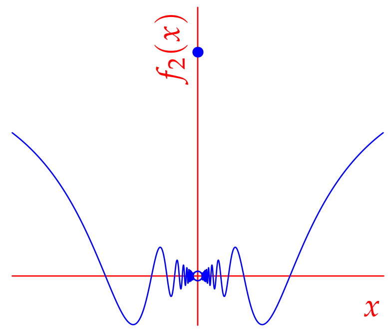
$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \neq a \\ \lim_{x\to a} f(x) & \text{if } x = a \end{cases}$$

is now continuous at $x = a$. In the pictures,

$$f_1(x) = \frac{x^2 - 9}{x - 3} \quad \text{and} \quad f_2(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

have removable discontinuities at $x = 3$ and 0 respectively.

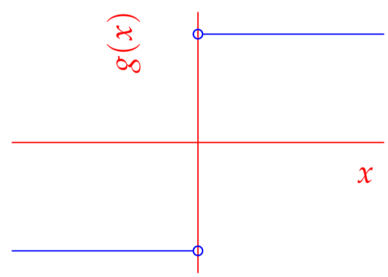




Jump Discontinuity The one-sided limits are finite but *not equal*. A jump discontinuity cannot be removed by changing or inserting a value at $x = a$. The picture shows

$$g(x) = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

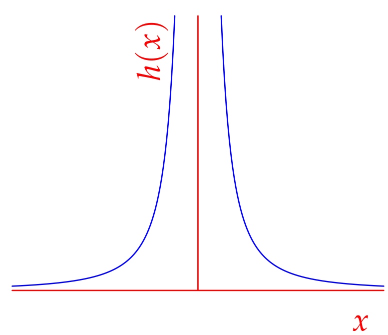
with a jump discontinuity at $x = 0$.



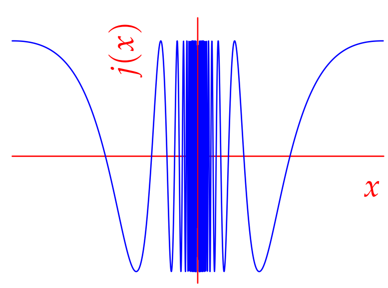
Infinite discontinuity The one-sided limits exist but at least one is infinite. We call the line $x = a$ a *vertical asymptote*. The picture shows

$$h(x) = \frac{1}{x^2}$$

with an infinite discontinuity $x = 0$. The fact that the one-sided limits of h are equal (and infinite) is irrelevant.



Essential discontinuity At least one of the one-sided limits does not exist. The picture shows $j(x) = \sin \frac{1}{x}$ for which neither of the limits $\lim_{x\to 0^\pm} j(x)$ exist.



It is also reasonable to refer to removable, infinite or essential discontinuities at interval endpoints.

Exercises 20 1. For the function $f(x) = \frac{x^3}{|x|}$, determine the limits $\lim_{x\to\infty} f(x)$, $\lim_{x\to-\infty} f(x)$, $\lim_{x\to 0^-} f(x)$, $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0} f(x)$, if they exist.

2. Evaluate the following limits *using the methods of this section*

   (a) $\lim_{x\to a} \dfrac{\sqrt{x}-\sqrt{a}}{x-a}$ (b) $\lim_{x\to a} \dfrac{x^{-3/2}-a^{-3/2}}{x-a}$

   (c) $\lim_{x\to 0} \dfrac{\sqrt{1+3x^2}-1}{x^2}$ (d) $\lim_{x\to -\infty} \dfrac{\sqrt{4+3x^2}-2}{x}$

3. Suppose that the limits $L = \lim_{x\to a^+} f(x)$ and $M = \lim_{x\to a^+} g(x)$ exist.

   (a) Suppose $f(x) \le g(x)$ for all x in some interval (a,b). Prove that $L \le M$.

   (b) Do we have the same conclusion if we have $f(x) < g(x)$ on (a,b), or can we conclude that $L < M$? Prove your assertion, or give a counter-example.

4. Suppose that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$. Using *only* this information, which of the following can you evaluate? Prove your assertions in each case.

   (a) $\lim_{x\to\infty} (f+g)(x)$ (b) $\lim_{x\to\infty} (f-g)(x)$

   (c) $\lim_{x\to\infty} (fg)(x)$ (d) $\lim_{x\to\infty} (f/g)(x)$

5. Complete the proof of Theorem 1.29 by considering the $L = \pm\infty$ cases.

6. Graph $f : \mathbb{R} \to \mathbb{R}$, find and identify the types of its discontinuities.

$$f(x) = \begin{cases} 0 & x = 0, \pm 1 \\ \frac{x}{|x|} & 0 < |x| < 1 \\ x^2 & |x| > 1 \end{cases}$$

7. Find the discontinuities and identify their types for the following function

$$f(x) = \begin{cases} \frac{1}{x}\sin\frac{1}{x} & \text{if } x < 0 \text{ or } x > 1 \\ \frac{1}{x} & \text{if } 0 < x \le 1 \end{cases}$$

8. Let $a \in U^\circ$. Verify the claim following Definition 1.23: $\lim_{x\to a} f(x) = L$ if and only if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x) - L| < \epsilon$$

9. Recall Exercise 17.5, where we saw that a function $f : U \to \mathbb{R}$ is continuous at any isolated point $a \in U$.

   (a) Any function with domain $\text{dom}(f) = \mathbb{Z}$ is continuous everywhere! Explain why we cannot define any limits $\lim_{x\to a^{(\pm)}} f(x)$ for such a function.

   (*Hint: Being unable to define a limit is different from saying* $\lim f(x) = DNE$*: see page 13.*)

   (b) Suppose $g(x) = x^2 h(x)$ has $\text{dom}(g) = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}\}$, where h is any function taking values in the interval $[-1,1]$. Explain why g is continuous at every point of its domain.

(*These awkward examples of continuity can be avoided if we follow our usual approach where a domain is a union of intervals of positive length. This restriction is essentially baked in to the Definition 1.23.*)

2 Sequences and Series of Functions

If (f_n) is a sequence of functions, what should we mean by $\lim f_n$? This question is of great relevance to the history of calculus; Issac Newton's work in the late 1600's made great use of *power series*, which are naturally constructed as limits of sequences of polynomials.

For instance, for each $n \in \mathbb{N}_0$, we might consider the polynomial function $f_n : \mathbb{R} \to \mathbb{R}$ defined by

$$f_n(x) = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n$$

This is easy to work with, to differentiate and integrate using the power law. What, however, are we to make of the following *series*?

$$f(x) := \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

Does this make sense? What is its domain? Does it equal the limit of the sequence (f_n) in a meaningful way? Is it continuous, differentiable or integrable? Can we compute its limit/derivative/integral term-by-term in the obvious way; for instance, is it legitimate to write

$$f'(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \cdots ?$$

To many in Newton's time, these questions were of diminished importance when compared to the burgeoning applications of calculus to the natural sciences. However, for the 18$^{\text{th}}$ and 19$^{\text{th}}$ century mathematicians who followed, the widespread application of calculus only increased the imperative to rigorously address these issues.

23 Power Series

First we recall some of the important definitions, examples and results regarding infinite series.

Definition 2.1. Let $(b_n)_{n=m}^{\infty}$ be a sequence of real numbers. The *(infinite) series* $\sum b_n$ is the limit of the sequence (s_n) of *partial sums*,

$$s_n = \sum_{k=m}^{n} b_n = b_m + b_{m+1} + \cdots + b_n, \qquad \sum_{n=m}^{\infty} b_n = \lim_{n\to\infty} s_n$$

The series $\sum b_n$ converges, diverges to infinity or diverges by oscillation[8] if the sequence (s_n) does so. $\sum b_n$ is *absolutely convergent* if $\sum |b_n|$ converges. A convergent series that is not absolutely convergent is *conditionally convergent*.

[8] Recall that every sequence (s_n) has subsequences tending to each of

$$\limsup s_n = \lim_{N\to\infty} \sup\{x_n : n > N\} \quad \text{and} \quad \liminf s_n = \lim_{N\to\infty} \inf\{x_n : n > N\}$$

If (s_n) converges, or diverges to $\pm\infty$, then $\lim s_n = \limsup s_n = \liminf s_n$. The remaining case, divergence by oscillation, is when $\liminf s_n \neq \limsup s_n$: there exist (at least) two subsequences tending to different limits.

Examples 2.2. These examples form the standard reference dictionary for analysis of more complex series. Make sure you are familiar with them![9]

1. (Geometric series) Let r be a constant, then $s_n = \sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$. It follows that

$$\sum_{n=0}^{\infty} r^n \begin{cases} \text{converges (absolutely) to } \frac{1}{1-r} & \text{if } -1 < r < 1 \\ \text{diverges to } \infty & \text{if } r \geq 1 \\ \text{diverges by oscillation} & \text{if } r \leq -1 \end{cases}$$

2. (Telescoping series) If $b_n = \frac{1}{n(n+1)}$, then $s_n = \sum_{k=1}^{n} b_n = 1 - \frac{1}{n+1} \implies \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is (absolutely) convergent. In fact $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, though checking this explicitly is tricky.
4. (Harmonic series) $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent to ∞.
5. (Alternating harmonic series) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is conditionally convergent.

Theorem 2.3 (Root Test). *Given a series $\sum b_n$, let $\beta = \limsup |b_n|^{1/n}$,*

- *If $\beta < 1$ then the series converges absolutely.*
- *If $\beta > 1$ then the series diverges.*

[9] We give sketch proofs, and/or refer you to a standard 'test.' Review these if you are unfamiliar.

1. $s_n - rs_n = 1 + r + \cdots + r^n - (r + \cdots + r^n + r^{n+1}) = 1 - r^{n+1} \implies s_n = \frac{1-r^{n+1}}{1-r}$.
2. $b_n = \frac{1}{n} - \frac{1}{n+1} \implies s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1}$.
3. Use the comparison or integral tests. Alternatively: For each $n \geq 2$, we have $\frac{1}{n^2} < \frac{1}{n(n-1)}$. By part 2,

$$s_n = \sum_{k=1}^{n} \frac{1}{k^2} < 1 + \sum_{k=1}^{n} \frac{1}{k(k-1)} \leq 2$$

   Since (s_n) is a monotone up sequence, bounded above by 2, we conclude that $\sum \frac{1}{n^2}$ is convergent.
4. Use the integral test. Alternatively, observe that

$$s_{2^{n+1}} - s_{2^n} = \sum_{k=2^n-1}^{2^{n+1}} \frac{1}{k} \geq \frac{2^n}{2^{n+1}} = \frac{1}{2} \implies s_{2^n} \geq \frac{n}{2} \xrightarrow[n\to\infty]{} \infty$$

   Since $s_n = \sum_{k=1}^{n} \frac{1}{k}$ defines an increasing sequence we conclude that $s_n \to \infty$.
5. Use the alternating series test, or explicitly check that both the even and odd partial sums (s_{2n}) and (s_{2n+1}) are convergent (monotone and bounded) to the same limit.

Root Test: $\beta < 1 \implies \exists \epsilon > 0$ such that $|b_n|^{1/n} \leq 1 - \epsilon$ (for large n) $\implies \sum |b_n|$ converges by comparison with the convergent geometric series $\sum (1-\epsilon)^n$.

$\beta > 1 \implies$ a subsequence of $(|b_n|^{1/n})$ converges to $\beta > 1$, whence $b_n \nrightarrow 0 \implies \sum b_n$ diverges (n^{th}-term test).

The root test is inconclusive if $\beta = 1$. Some simple inequalities[10] yield a simpler test.

Corollary 2.4 (Ratio Test). *Given a series $\sum b_n$,*

- *If* $\limsup \left|\frac{b_{n+1}}{b_n}\right| < 1$ *then* $\sum b_n$ *converges absolutely.*
- *If* $\liminf \left|\frac{b_{n+1}}{b_n}\right| > 1$ *then* $\sum b_n$ *diverges.*

We can now properly define and analyze our main objects of interest.

Definition 2.5. A *power series centered at* $c \in \mathbb{R}$ is a formal expression

$$\sum_{n=m}^{\infty} a_n (x-c)^n$$

where $(a_n)_{n=m}^{\infty}$ is a sequence of real numbers and x is considered a variable.

It is common to refer simply to a *series*, and modify by infinite/power when clarity requires. We almost always have $m = 0$ or 1, and it is common for examples to be centered at $c = 0$.

Example 2.6. Using the geometric series formula, we see that

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-4)^n = \frac{1}{1-\frac{-(x-4)}{2}} = \frac{2}{x-2} \quad \text{whenever} \quad \left|-\frac{x-4}{2}\right| < 1 \iff 2 < x < 6$$

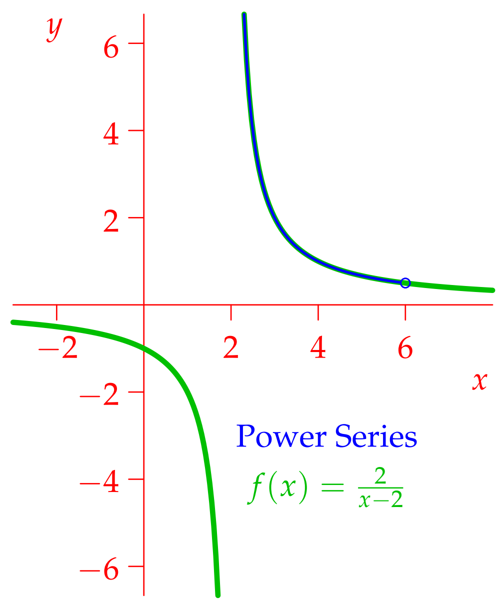
The series is valid (converges) only on a small subinterval of the implied domain of the function $x \mapsto \frac{2}{x-2}$. The behavior of both as $x \to 2^+$ should not be a surprise; evaluating the power series results in the divergent infinite series

$$\sum 1 = +\infty$$

By contrast, as $x \to 6^-$, we see that limits and infinite series do not interact the way we might expect,

$$\lim_{x\to 6^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}(x-4)^n = \lim_{x\to 6^-} \frac{2}{x-2} = \frac{1}{2}$$
$$\sum_{n=0}^{\infty} \lim_{x\to 6^-} \frac{(-1)^n}{2^n}(x-4)^n = \sum (-1)^n = \text{DNE}$$

with the last divergent by oscillation.



As the example shows, we cannot take limits inside an infinite sum; understanding when we can do this is one of our primary goals.

[10] $\liminf \left|\frac{b_{n+1}}{b_n}\right| \le \liminf |b_n|^{1/n} \le \limsup |b_n|^{1/n} \le \limsup \left|\frac{b_{n+1}}{b_n}\right|$

Radius and Interval of Convergence

At any real number x, a series may converge absolutely, converge conditionally, diverge to $\pm\infty$, or diverge by oscillation. A series defines a *function* whose implied domain is the set on which the series converges. In the previous example, the domain was an *interval* $(2,6)$. By applying the root test (Theorem 2.3), we can show that this holds for *every* series.

Theorem 2.7 (Root Test for Power Series). *Given a series* $\sum_{n=0}^{\infty} a_n(x-c)^n$, *define*[11]

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

Exactly one of the following is true:

$R=\infty$	*the series converges absolutely for all* $x \in \mathbb{R}$
$R=0$	*the series converges only when* $x=c$
$R \in \mathbb{R}^+$	*the series converges absolutely when* $\lvert x-c\rvert < R$ *and diverges when* $\lvert x-c\rvert > R$

Proof. For each fixed $x \in \mathbb{R}$, let $b_n = a_n(x-c)^n$ and apply the root test to $\sum b_n$, noting that

$$\limsup |b_n|^{1/n} = \begin{cases} 0 & \text{if } x=c \text{ or } R=\infty \\ \infty & \text{if } x\neq c \text{ and } R=0 \\ \limsup |a_n|^{1/n}|x-c| = \frac{1}{R}|x-c| & \text{otherwise} \end{cases}$$

In the final situation, $\limsup |b_n|^{1/n} < 1 \iff |x-c| < R$, etc. ∎

Definition 2.8. The *radius of convergence* is the value R defined in Theorem 2.7. The *interval of convergence* is the set of values x for which the series converges; the implied domain.

Radius of convergence	Interval of convergence
∞	$\mathbb{R} = (-\infty,\infty)$
0	$\{c\}$
R	$(c-R,c+R),\ (c-R,c+R],\ [c-R,c+R),$ or $[c-R,c+R]$

In the third case convergence/divergence at the endpoints of the interval of convergence must be tested separately.

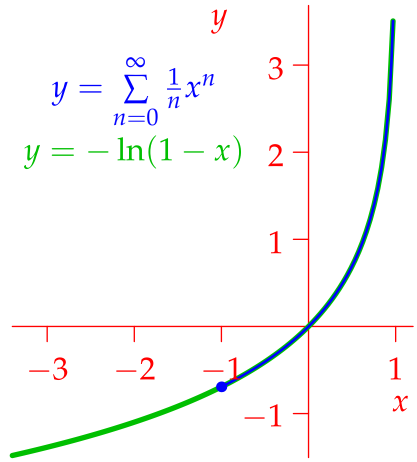
By applying Corollary 2.4, we obtain a more user-friendly result.

Corollary 2.9 (Ratio Test for Power Series). *If the limit exists,* $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.

[11] Since $|a_n| \geq 0$, we here adopt the convention that $\frac{1}{0} = \infty$ and $\frac{1}{\infty} = 0$. With similar caveats, it is also reasonable to write $R = \liminf |a_n|^{-1/n}$.

Examples 2.10. 1. The series $\sum_{n=1}^{\infty} \frac{1}{n}x^n$ is centered at 0. The ratio test tells us that

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{1/n}{1/(n+1)} = \lim_{n\to\infty}\frac{n+1}{n} = 1$$



Test the endpoints of the interval of convergence separately:

$x = 1 \quad \sum \frac{1}{n} = \infty$ diverges

$x = -1 \quad \sum \frac{(-1)^n}{n}$ converges (conditionally)

We conclude that the interval of convergence is $[-1, 1)$.

It can be seen that the series converges to $-\ln(1-x)$ on its interval of convergence. As in Example 2.6, this function has a larger domain $(-\infty, -1)$, than that of the series.

2. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}x^n$ similarly has

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)^2}{n^2} = 1$$

Since $\sum \frac{1}{n^2}$ is absolutely convergent, we conclude that the power series also converges absolutely at $x = \pm 1$; the interval of convergence is $[-1, 1]$.

3. The series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges absolutely for all $x \in \mathbb{R}$, since

$$R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1) = \infty$$

You should recall from elementary calculus that this series converges to the natural exponential function $\exp(x) = e^x$ everywhere on $\mathbb{R}$; indeed this is one of the common *definitions* of the exponential!

4. The series $\sum_{n=0}^{\infty} n!x^n$ has $R = 0$, and thus only converges at its center $x = 0$.

5. Let $a_n = \left(\frac{2}{3}\right)^n$ if n is even and $\left(\frac{3}{2}\right)^n$ if n is odd. If we try to apply the ratio test to the series $\sum_{n=0}^{\infty} a_n x^n$, we see that

$$\left|\frac{a_n}{a_{n+1}}\right| = \begin{cases} \left(\frac{2}{3}\right)^{2n+1} & \text{if } n \text{ even} \\ \left(\frac{3}{2}\right)^{2n+1} & \text{if } n \text{ odd} \end{cases} \implies \limsup\left|\frac{a_n}{a_{n+1}}\right| = \infty \neq 0 = \liminf\left|\frac{a_n}{a_{n+1}}\right|$$

The ratio test therefore fails. However, by the root test,

$$|a_n|^{1/n} = \begin{cases} \frac{2}{3} & \text{if } n \text{ even} \\ \frac{3}{2} & \text{if } n \text{ odd} \end{cases} \implies R = \frac{1}{\limsup |a_n|^{1/n}} = \frac{1}{3/2} = \frac{2}{3}$$

It is easy to check that the series diverges at $x = \pm\frac{2}{3}$; the interval of convergence is $\left(-\frac{2}{3}, \frac{2}{3}\right)$.

With the help of the root test, we can understand the domain of a power series. The issues of limits, continuity, differentiability and integrability are more delicate. We will return to these once we've developed some of the ideas around *convergence* for *sequences of functions.*

Exercises 23 1. For each power series, find the radius and interval of convergence:

(a) $\sum \frac{(-1)^n}{n^2 4^n} x^n$ (b) $\sum \frac{(n+1)^2}{n^3}(x-3)^n$ (c) $\sum \sqrt{n} x^n$

(d) $\sum \frac{1}{n\sqrt{n}}(x+7)^n$ (e) $\sum (x-\pi)^{n!}$ (f) $\sum \frac{3^n}{\sqrt{n}} x^{2n+1}$

2. For each $n \in \mathbb{N}$ let $a_n = \left(\frac{4+2(-1)^n}{5}\right)^n$

   (a) Find $\limsup |a_n|^{1/n}$, $\liminf |a_n|^{1/n}$, $\limsup \left|\frac{a_{n+1}}{a_n}\right|$ and $\liminf \left|\frac{a_{n+1}}{a_n}\right|$.

   (b) Do the series $\sum a_n$ and $\sum (-1)^n a_n$ converge? Why?

   (c) Find the interval of convergence of the power series $\sum a_n x^n$.

3. Suppose that $\sum a_n x^n$ has radius of convergence R. If $\limsup |a_n| > 0$, prove that $R \le 1$.

4. On the interval $(-\frac{2}{3}, \frac{2}{3})$, express the series in Example 2.10.5 as a simple function.

   (*Hints: Use geometric series formulæ and the fact that the value of an absolutely convergent series is independent of rearrangements*)

5. Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{3^n n}(x-7)^{5n+1} = \frac{1}{3}(x-7) + \frac{1}{18}(x-7)^6 + \frac{1}{81}(x-7)^{11} + \cdots$$

   Since only one in five of the terms are non-zero, it is a little tricky to analyze using a naïve application of our standard tests.

   (a) Explain why the ratio test for power series (Corollary 2.9) does not apply.

   (b) Writing the series as $\sum a_m (x-7)^m$, observe that

$$a_m = \begin{cases} \frac{5}{3^{\frac{m-1}{5}}(m-1)} & \text{if } m \equiv 1 \mod 5 \\ 0 & \text{otherwise} \end{cases}$$

   Use the root test (Theorem 2.7) and your understanding of elementary limits to directly compute the radius of convergence.

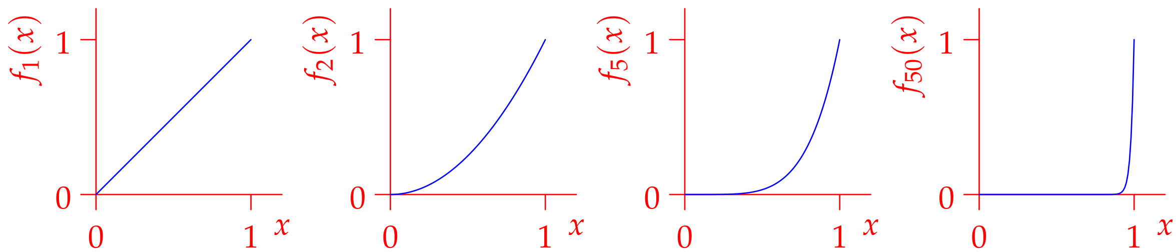
   (c) Alternatively, write $\sum \frac{1}{3^n n}(x-7)^{5n+1} = \sum b_n$. Apply the ratio test for *infinite* series (Corollary 2.4): what do you observe? Use your observation to compute the radius of convergence of the original series in a simpler manner than part (a).

   (d) Finally, check the endpoints to determine the interval of convergence.

24 Uniform Convergence

In this section we consider sequences (f_n) of *functions* $f_n : U \to \mathbb{R}$.

Example 2.11. For each $n \in \mathbb{N}$, consider $f_n : (0,1) \to \mathbb{R} : x \mapsto x^n$.



There turn out to be several good notions of convergence for sequences of functions; the simplest it where, for each x, $(f_n(x))$ is treated as a separate sequence of real numbers.

Definition 2.12. Suppose a function f and a sequence of functions (f_n) are given. We say that (f_n) *converges pointwise to f on U* if,

$$\forall x \in U, \lim_{n\to\infty} f_n(x) = f(x)$$

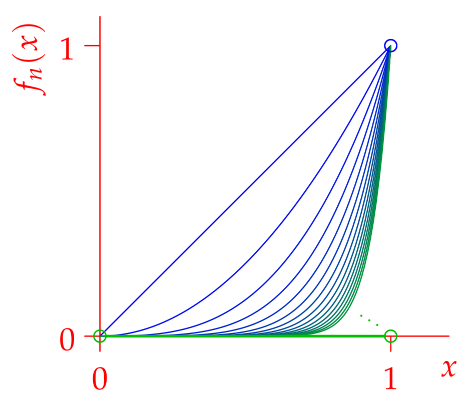
It is common to write '$f_n \to f$ pointwise.' For reference, we state two equivalent rephrasings:

1. $\forall x \in U, \ |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$;
2. $\forall x \in U, \forall \epsilon > 0, \exists N$ such that $n > N \implies |f_n(x) - f(x)| < \epsilon$.

As we'll see shortly, the relative positions of the quantifiers ($\forall x$ and $\exists N$) is crucial: in this definition, the value of N is permitted to depend on x as well as ϵ.

Example (2.11, mk. II). The sequence (f_n) converges pointwise on the domain $(0,1)$ to

$$f : (0,1) \to \mathbb{R} : x \mapsto 0$$



We prove this explicitly as a sanity check. First observe that

$$|f_n(x) - f(x)| = x^n$$

Suppose $x \in (0,1)$, that $\epsilon > 0$ is given, and let $N = \frac{\ln \epsilon}{\ln x}$. Then

$$n > N \implies n \ln x < \ln \epsilon \implies x^n < \epsilon$$

where the inequality switches sign since $\ln x < 0$.

The example is nice in that a sequence of continuous functions converges pointwise to a continuous function. Unfortunately, this desirable situation is not universal.

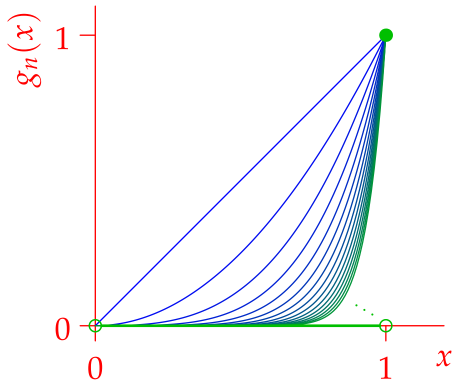
Example (2.11, mk. III). Extend the domain to include $x = 1$; define

$$g_n : (0, 1] \to \mathbb{R} : x \mapsto x^n$$

Each g_n is a continuous function, however its pointwise limit

$$g(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}$$

has a *jump discontinuity* at $x = 1$.

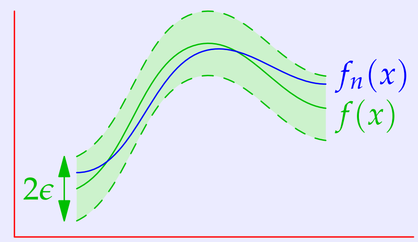


With the goal of having convergence of functions preserve continuity, we make a tighter definition.

Definition 2.13. (f_n) *converges uniformly* to f on U if either

1. $\sup_{x \in U} |f_n(x) - f(x)| \xrightarrow[n \to \infty]{} 0$, or,
2. $\forall \epsilon > 0$, $\exists N$ such that $\forall x \in U, n > N \implies |f_n(x) - f(x)| < \epsilon$

A common notation is $f_n \rightrightarrows f$, though we won't use it.



Whenever $n > N$, the graph of $f_n(x)$ must lie between those of $f(x) \pm \epsilon$.

We'll show that statements 1 and 2 are equivalent momentarily. For the present, compare with the corresponding statements for pointwise convergence:

- As with *continuity* versus *uniform continuity*, the distinction comes in the *order of the quantifiers*: in uniform convergence, x is quantified *after* N and so *the same N works for all x*.
- Uniform convergence implies pointwise convergence.

For the last time, we revisit our main example.

Example (2.11, mk. IV). If $f_n : (0, 1) \to \mathbb{R} : x \mapsto x^n$ and f are defined as before, then the pointwise convergence $f_n \to f$ is *non-uniform*. We show this using both criteria.

1. For *every* n,

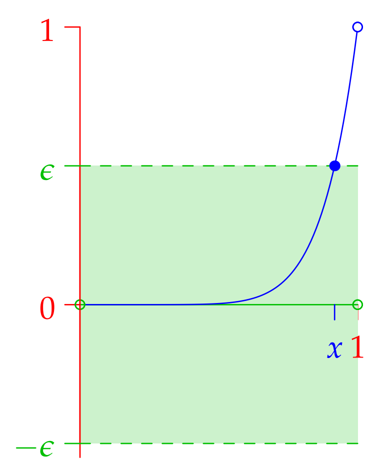
$$\sup_{x \in (0,1)} |f_n(x) - f(x)| = \sup\{x^n : 0 < x < 1\} = 1 \not\to 0$$

2. Suppose the convergence were uniform and let $\epsilon = \frac{1}{2}$. Then

$$\exists N \in \mathbb{N} \text{ such that } \forall x \in (0, 1),\ n > N \implies x^n < \frac{1}{2}$$

Since $N \in \mathbb{N}$, a simple choice results in a contradiction;

$$x = \left(\frac{1}{2}\right)^{\frac{1}{N+1}} \in (0, 1) \implies x^{N+1} = \frac{1}{2}$$



Theorem 2.14. *The criteria for uniform convergence in Definition 2.13 are equivalent.*

Proof. (1 ⇒ 2) This follows from the fact that

$$\forall x \in U, \ |f_n(x) - f(x)| \le \sup_{x \in U} |f_n(x) - f(x)|$$

(2 ⇒ 1) Suppose $\epsilon > 0$ is given. Then

$$\exists N \in \mathbb{R} \text{ such that } \forall x \in U, \ n > N \implies |f_n(x) - f(x)| < \frac{\epsilon}{2}$$

But then

$$n > N \implies \sup_{x \in U} |f_n(x) - f(x)| \le \frac{\epsilon}{2} < \epsilon$$

■

Somewhat amazingly, the subtle change of definition results in the preservation of continuity.

Theorem 2.15. *Suppose that (f_n) is a sequence of continuous functions. If $f_n \to f$ uniformly, then f is continuous.*

Proof. We demonstrate the continuity of f at $a \in U$. Let $\epsilon > 0$ be given.

- Since $f_n \to f$ uniformly,

$$\exists N \text{ such that } \forall x \in U, \ n > N \implies |f(x) - f_n(x)| < \frac{\epsilon}{3}$$

- Choose any $n > N$. Since f_n is continuous at a,

$$\exists \delta > 0 \text{ such that } |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3} \qquad (\dagger)$$

Simply put these together with the triangle inequality to see that

$$\begin{aligned} |x - a| < \delta \implies |f(x) - f(a)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \end{aligned}$$

■

We need not have fixed a at the start of the proof. Rewriting $(\dagger)$ to become

$$\exists \delta > 0 \text{ such that } \forall x, a \in U, \ |x - a| < \delta \implies |f_n(x) - f_n(a)| < \frac{\epsilon}{3}$$

proves a related result.

Corollary 2.16. *If $f_n \to f$ uniformly where each f_n is* uniformly *continuous, then f is uniformly continuous.*

Examples 2.17. 1. Let $f_n(x) = x + \frac{1}{n}x^2$. This is continuous on $\mathbb{R}$ for all x, and converges pointwise to the continuous function $f : x \mapsto x$.

(a) On any bounded interval $[-M, M]$ the convergence $f_n \to f$ is uniform,

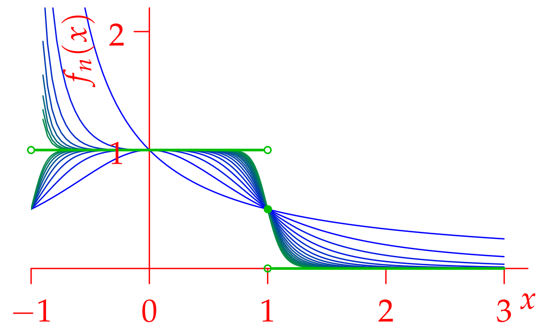
$$\sup_{x\in[-M,M]} |f_n(x) - f(x)| = \sup\left\{\frac{1}{n}x^2 : x \in [-M, M]\right\} = \frac{M^2}{n} \xrightarrow[n\to\infty]{} 0$$

(b) On any unbounded interval, $\mathbb{R}$ say, the convergence is non-uniform,

$$\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup\left\{\frac{1}{n}x^2 : x \in \mathbb{R}\right\} = \infty$$

2. Consider $f_n(x) = \frac{1}{1+x^n}$; this is continuous on $(-1, \infty)$ and converges pointwise to

$$f(x) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \\ 1 & \text{if } -1 < x < 1 \end{cases}$$



We consider the convergence $f_n \to f$ on several intervals.

(a) On $[2, \infty)$, the pointwise limit is continuous. Moreover, $f_n(x)$ is decreasing, whence

$$\sup_{x\in[2,\infty)} |f_n(x) - 0| = \frac{1}{1+2^n} \xrightarrow[n\to\infty]{} 0$$

and the convergence is uniform. Alternatively; if $\epsilon \in (0, 1)$, let $N = \log_2(\epsilon^{-1} - 1)$, then

$$\forall x \geq 2,\ n > N \implies |f_n(x) - 0| = \frac{1}{1+x^n} \leq \frac{1}{1+2^n} < \frac{1}{1+2^N} = \epsilon$$

The same argument shows that $f_n \to f$ uniformly on any interval $[a, \infty)$ where $a > 1$.

(b) On $[1, \infty)$ the convergence is not uniform, since the pointwise limit is discontinuous,

$$f(x) = \begin{cases} 0 & \text{if } x > 1 \\ \frac{1}{2} & \text{if } x = 1 \end{cases}$$

(c) The convergence is not even uniform on the open interval $(1, \infty)$,

$$\sup_{x\in[1,\infty)} |f_n(x) - f(x)| = \sup\left\{\frac{1}{1+x^n} : x > 1\right\} = \frac{1}{2} \not\xrightarrow[n\to\infty]{} 0$$

(d) Similarly, for any $a \in (0, 1)$, the convergence $f_n \to f$ is uniform on $[0, a]$, this time to the (continuous) constant function $f(x) = 1$,

$$\sup_{x\in[0,a]} |f_n(x) - 1| = \left|1 - \frac{1}{1+a^n}\right| = \frac{a^n}{1+a^n} \xrightarrow[n\to\infty]{} 0$$

(e) Finally, on $(-1, 1)$ the convergence is not uniform,

$$\sup_{x\in[0,1)} |f_n(x) - f(x)| = \sup\left\{\frac{x^n}{1+x^n} : x \in [0, 1)\right\} = \frac{1}{2} \not\xrightarrow[n\to\infty]{} 0$$

Exercises 24 1. For each sequence of functions defined on $[0, \infty)$:

(i) Find the pointwise limit $f(x)$ as $n \to \infty$.

(ii) Determine whether $f_n \to f$ uniformly on $[0,1]$.

(iii) Determine whether $f_n \to f$ uniformly on $[1, \infty)$.

(a) $f_n(x) = \dfrac{x}{n}$ (b) $f_n(x) = \dfrac{x^n}{1+x^n}$ (c) $f_n(x) = \dfrac{x^n}{n+x^n}$

(d) $f_n(x) = \dfrac{x}{1+nx^2}$ (e) $f_n(x) = \dfrac{nx}{1+nx^2}$

2. Let $f_n(x) = \left(x - \frac{1}{n}\right)^2$. If $f(x) = x^2$, we clearly have $f_n \to f$ pointwise on any domain.

   (a) Prove that the convergence is uniform on $[-1, 1]$.

   (b) Prove that the convergence is non-uniform on $\mathbb{R}$.

3. For each sequence, find the pointwise limit and decide if the convergence is uniform.

   (a) $f_n(x) = \frac{1+2\cos^2(nx)}{\sqrt{n}}$ for $x \in \mathbb{R}$.

   (b) $f_n(x) = \cos^n(x)$ on $[-\pi/2, \pi/2]$.

4. For each $n \in \mathbb{N}$, consider the continuous function

   $$f_n : [0,1] \to \mathbb{R} : x \mapsto nx^n(1-x)$$

   (a) Given $0 \le x < 1$, let $a \in (x, 1)$. Explain why $\exists N$ such that

   $$n > N \implies |f_{n+1}(x)| \le a\,|f_n(x)|$$

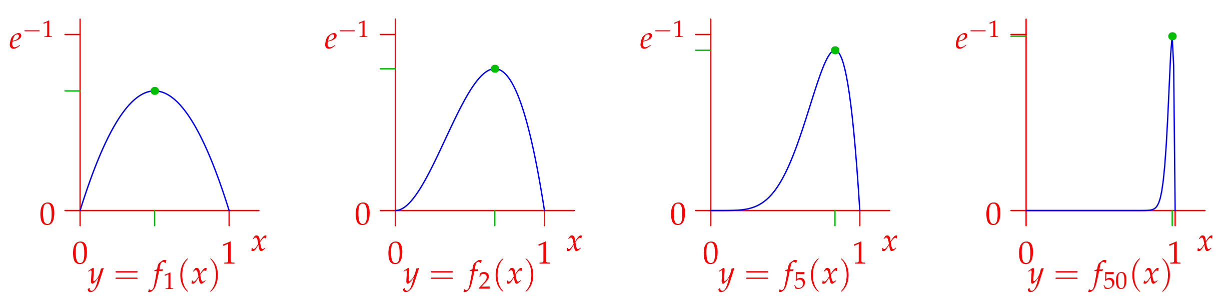
   Hence conclude that the pointwise limit of (f_n) is the zero function.

   (b) Use elementary calculus ($f_n'(x) = 0 \iff \ldots$) to prove that the maximum value of f_n is located at $x_n = \frac{n}{1+n}$. Hence compute

   $$\sup_{x \in [0,1]} |f_n(x) - f(x)|$$

   and use it to show that the convergence $f_n \to 0$ is non-uniform.

This shows that the converse to Theorem 2.15 is false, even on a bounded interval: the continuous sequence (f_n) converges non-uniformly *to a continuous function. Sketches of several f_n are below.*



5. Explain where the proof of Theorem 2.15 fails if $f_n \to f$ *non-uniformly*.

25 More on Uniform Convergence

While we haven't yet developed calculus, our familiarity with basic differentiation and integration makes it natural to pause to consider the interaction of these operations with sequences of functions.

We also consider a Cauchy-criterion for uniform convergence, which leads to the useful Weierstraß M-test.

Example 2.18. Recall that $f_n(x) = x^n$ converges uniformly to $f(x) = 0$ on any interval $[0, a]$ where $a < 1$. We easily check that

$$\int_0^a f_n(x)\,\mathrm{d}x = \frac{1}{n+1}a^{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^a f(x)\,\mathrm{d}x$$

In fact the sequence of derivatives converge here also

$$\frac{\mathrm{d}}{\mathrm{d}x} f_n(x) = nx^{n-1} \xrightarrow[n\to\infty]{} 0 = f'(x)$$

It is perhaps surprising that integration interacts more nicely with uniform limits than does differentiation. We therefore consider integration first.

Theorem 2.19. *Let $f_n \to f$ uniformly on $[a, b]$ where the functions f_n are integrable. Then f is integrable on $[a, b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n(x)\mathrm{d}x = \int_a^b f(x)\mathrm{d}x$$

Proof. Given $\epsilon > 0$, note that $\int_a^b \frac{\epsilon}{2(b-a)}\,\mathrm{d}x = \frac{\epsilon}{2}$. Since $f_n \to f$ uniformly, $\exists N$ such that[12]

$$\begin{aligned}
\forall x \in [a, b],\ n > N &\implies |f_n(x) - f(x)| < \frac{\epsilon}{2(b-a)} \\
&\implies f_n(x) - \frac{\epsilon}{2(b-a)} < f(x) < f_n(x) + \frac{\epsilon}{2(b-a)} \\
&\implies \int_a^b f_n(x)\,\mathrm{d}x - \frac{\epsilon}{2} \le \int_a^b f(x)\,\mathrm{d}x \le \int_a^b f_n(x)\,\mathrm{d}x + \frac{\epsilon}{2} \\
&\implies \left| \int_a^b f_n(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x \right| \le \frac{\epsilon}{2} < \epsilon
\end{aligned}$$

■

The appearance of uniform convergence in the proof is subtle. If $N = N(\epsilon)$ were allowed to depend on x, then the integral $\int_a^b f_n(x)\,\mathrm{d}x$ would be meaningless: Which n would we consider? Larger than $N(x, \epsilon)$ for *which* x? Taking n 'larger' than *all* the $N(x, \epsilon)$ might produce the absurdity $n = \infty$!

[12] This assumes f is already integrable. Once we've properly defined (Riemann) integrability at the end of the course, we can insert the following

$$\int_a^b f_n(x)\,\mathrm{d}x - \frac{\epsilon}{2} \le L(f) \le U(f) \le \int_a^b f_n(x)\,\mathrm{d}x + \frac{\epsilon}{2} \implies 0 \le U(f) - L(f) \le \epsilon \implies U(f) = L(f)$$

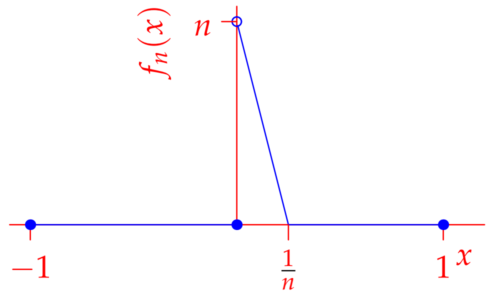
where $U(f)$ and $L(f)$ are the upper and lower Darboux integrals of f; their equality shows that f is integrable on $[a, b]$.

Examples 2.20. 1. Uniform convergence is not required for the integrals to converge as we'd like. For instance, recall that extending the previous example to the domain $[0,1]$ results in non-uniform convergence; however, we still have

$$\int_0^1 f_n(x)\,dx = \frac{1}{n+1} \xrightarrow[n\to\infty]{} 0 = \int_0^1 f(x)\,dx$$

2. To obtain a sequence of functions $f_n \to f$ for which $\int f_n \not\to \int f$ requires a bit of creativity. Consider the sequence

$$f_n : [-1,1] \to \mathbb{R} : x \mapsto \begin{cases} n - n^2 x & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$



If $0 < x < 1$, then for large $n \in \mathbb{N}$ we have

$$x \geq \frac{1}{n} \implies f_n(x) = 0$$

We conclude that $f_n \to 0$ pointwise. Since the area under f_n is a triangle with base $\frac{1}{n}$ and height n, the integral is constant and *non-zero*;

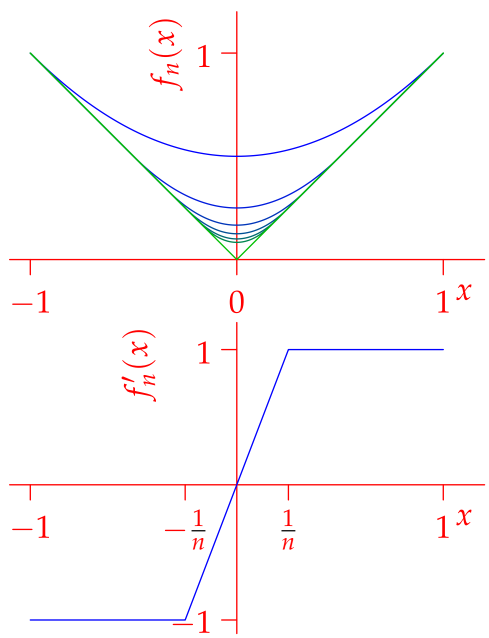
$$\int_{-1}^1 f_n(x)\,dx = \frac{1}{2} \neq 0 = \int_{-1}^1 f(x)\,dx$$

It should be obvious why the convergence $f_n \to 0$ is non-uniform; why?

Derivatives and Uniform Limits We've already seen that a uniform limit of differentiable functions *might* be differentiable (Example 2.18), but this shouldn't be expected in general since even uniform limits of differentiable functions can have corners!

Example 2.21. For each $n \in \mathbb{N}$, consider the function

$$f_n : [-1,1] \to \mathbb{R} : x \mapsto \begin{cases} |x| & \text{if } |x| \geq \frac{1}{n} \\ \frac{n}{2}x^2 + \frac{1}{2n} & \text{if } |x| < \frac{1}{n} \end{cases}$$



- f_n converges pointwise to $f(x) = |x|$.
- $f_n \to f$ uniformly since

$$\sup_{x\in[-1,1]} |f_n(x) - f(x)| = \frac{1}{2n} \to 0$$

- Each f_n is differentiable: $f_n'(x) = \begin{cases} 1 & \text{if } x \geq \frac{1}{n} \\ nx & \text{if } |x| < \frac{1}{n} \\ -1 & \text{if } x \leq -\frac{1}{n} \end{cases}$
- The uniform limit f is *not differentiable* at $x = 0$.

Transferring differentiability to the limit of a sequence of functions is a bit messy.

Theorem 2.22. *Suppose (f_n) is a sequence and g is a function on $[a,b]$ for which:*

- *$f_n \to f$ pointwise;*
- *Each f_n is differentiable with continuous derivative;*[13]
- *$f_n' \to g$ uniformly.*

Then $f_n \to f$ uniformly on $[a,b]$ and f is differentiable with derivative g.

The issue in the previous example is that the *pointwise limit* of the derived sequence (f_n') is discontinuous at $x = 0$ and therefore $f_n' \to g$ isn't uniform!

Proof. For any $x \in [a,b]$, the fundamental theorem of calculus tells us that

$$\int_a^x f_n'(t)\,\mathrm{d}t = f_n(x) - f_n(a)$$

By Theorem 2.19, the left side converges to $\int_a^x g(t)\,\mathrm{d}t$, while the right converges to $f(x) - f(a)$. Since $f_n' \to g$ uniformly, we see that g is continuous and we can apply the fundamental theorem again: $\int_a^x g(t)\,\mathrm{d}t = f(x) - f(a)$ is differentiable with derivative g.
The uniformity of the convergence $f_n \to f$ follows from Exercise 10. ∎

Uniformly Cauchy Sequences and the Weierstraß M-Test

Recall that one may use Cauchy sequences to demonstrate convergence *without knowing the limit in advance.* An analogous discussion is available for sequences of functions.

Definition 2.23. A sequence of functions (f_n) is *uniformly Cauchy* on U if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall x \in U,\ m, n > N \implies |f_n(x) - f_m(x)| < \epsilon$$

Example 2.24. Let $f_n(x) = \sum\limits_{k=1}^{n} \frac{1}{k^2} \sin k^2 x$ be defined on $\mathbb{R}$. Given $\epsilon > 0$, let $N = \frac{1}{\epsilon}$, then

$$\begin{aligned} m > n > N \implies |f_m(x) - f_n(x)| &= \left| \sum_{k=n+1}^{m} \frac{1}{k^2} \sin k^2 x \right| \le \sum_{k=n+1}^{m} \frac{1}{k^2} \le \sum_{k=n+1}^{m} \frac{1}{k(k-1)} \\ &= \sum_{k=n+1}^{m} \frac{1}{k-1} - \frac{1}{k} = \frac{1}{n} - \frac{1}{m} < \frac{1}{N} = \epsilon \end{aligned}$$

whence (f_n) is uniformly Cauchy.

[13]Without the continuity assumption, the fundamental theorem of calculus doesn't apply and the proof requires an alternative approach. One can also weaken the hypotheses: if $f_n' \to g$ uniformly and that $(f_n(x))$ converges for *at least one* $x \in [a,b]$, then there exists f such that $f_n \to f$ is uniform and $f' = g$.

As with sequences of real numbers, uniformly Cauchy sequences converge; in fact uniformly!

Theorem 2.25. *A sequence (f_n) is uniformly Cauchy on U if and only if it converges uniformly to some $f : U \to \mathbb{R}$.*

Proof. ($\Rightarrow$) Let (f_n) be uniformly Cauchy on U. For each $x \in U$, the sequence $(f_n(x)) \subseteq \mathbb{R}$ is Cauchy and thus convergent. Define $f : U \to \mathbb{R}$ via

$$f(x) := \lim_{n\to\infty} f_n(x)$$

We claim that $f_n \to f$ uniformly. Let $\epsilon > 0$ be given, then

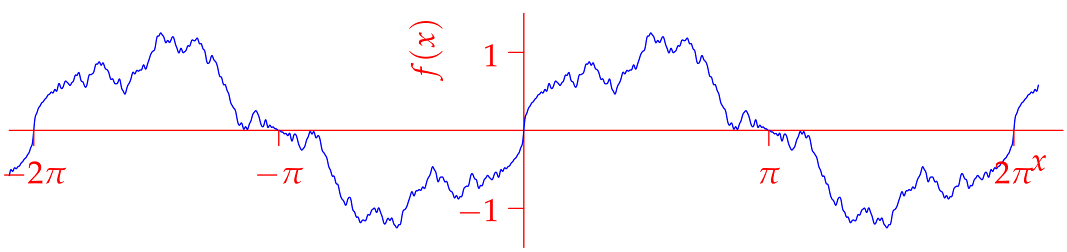
$$\begin{aligned}\exists N \in \mathbb{N} \text{ such that } m > n > N &\implies |f_n(x) - f_m(x)| < \frac{\epsilon}{2} \\ &\implies f_n(x) - \frac{\epsilon}{2} < f_m(x) < f_n(x) + \frac{\epsilon}{2} \\ &\implies f_n(x) - \frac{\epsilon}{2} \le f(x) \le f_n(x) + \frac{\epsilon}{2} \qquad \text{(take limits as } m \to \infty) \\ &\implies |f_n(x) - f(x)| \le \frac{\epsilon}{2} < \epsilon\end{aligned}$$

($\Leftarrow$) This is Exercise 2. ∎

Example (2.24, mk. II). Since (f_n) is uniformly Cauchy on $\mathbb{R}$, it converges uniformly to some $f : \mathbb{R} \to \mathbb{R}$. It seems reasonable to write

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin n^2 x$$

The graph of this function looks somewhat bizarre:



Since each f_n is (uniformly) continuous, Theorem 2.15 says that f is also (uniformly) continuous. By Theorem 2.19, $f(x)$ is integrable, indeed

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{k=1}^{n} -k^{-4} \cos k^2 x \Big|_a^b = \sum_{n=1}^{\infty} \frac{1}{n^4}(\cos n^2 a - \cos n^2 b)$$

which converges (comparison test) for all a, b. By contrast, the derived sequence

$$f_n'(x) = \sum_{k=1}^{n} \cos k^2 x$$

does not converge for *any* x since $\cos n^2 x \underset{k\to\infty}{\not\longrightarrow} 0$. We should thus expect (though we offer no proof) that f is nowhere differentiable.

The example generalizes. Suppose (g_k) is a sequence of functions on U and define the *series* $\sum g_k(x)$ as the pointwise limit of the sequence (f_n) of partial sums

$$\sum_{k=k_0}^{\infty} g_k(x) := \lim_{n\to\infty} f_n(x) \quad \text{where} \quad f_n(x) = \sum_{k=k_0}^{n} g_k(x)$$

whenever the limit exists. The series is said to converge uniformly whenever (f_n) does so. Theorems 2.15, 2.19 and 2.22 immediately translate.

Corollary 2.26. *Let $\sum g_k$ be a series of functions converging uniformly on U. Then:*

1. *If each g_k is (uniformly) continuous then $\sum g_k$ is (uniformly) continuous.*
2. *If each g_k is integrable, then $\int \sum g_k(x)\,\mathrm{d}x = \sum \int g_k(x)\,\mathrm{d}x$.*
3. *If each g_k is continuously differentiable, and the sequence of derived partial sums f_n' converges uniformly, then $\sum g_k$ is differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}\sum g_k(x) = \sum g_k'(x)$.*

As an application of the uniform Cauchy criterion, we obtain an easy test for uniform convergence.

Theorem 2.27 (Weierstraß M-test). *Suppose (g_k) is a sequence of functions on U. Moreover assume:*

1. *(M_k) is a non-negative sequence such that $\sum M_k$ converges.*
2. *Each g_k is bounded by M_k; that is $|g_k(x)| \le M_k$.*

Then $\sum g_k(x)$ converges uniformly on U.

Proof. Let $f_n(x) = \sum_{k=k_0}^{n} g_k(x)$ define the sequence of partial sums. Since $\sum M_k$ converges, its sequence of partial sums is Cauchy (the *Cauchy criterion* for infinite series); given $\epsilon > 0$,

$$\exists N \text{ such that } m > n > N \implies \sum_{k=n+1}^{m} M_k < \epsilon$$

However, by assumption,

$$m > n > N \implies |f_m(x) - f_n(x)| = \left|\sum_{k=n+1}^{m} g_k(x)\right| \le \sum_{k=n+1}^{m} |g_k(x)| \le \sum_{k=n+1}^{m} M_k < \epsilon$$ ■

The sequence of partial sums is uniformly Cauchy and thus uniformly convergent.

Example 2.28. Given the series $\sum_{n=1}^{\infty} \frac{1+\cos^2(nx)}{n^2}\sin(nx)$, we clearly have

$$\left|\frac{1+\cos^2(nx)}{n^2}\sin(nx)\right| \le \frac{2}{n^2} \text{ for all } x \in \mathbb{R}$$

Since $\sum \frac{2}{n^2}$ converges, the M-test shows that the original series converges uniformly on $\mathbb{R}$.

Exercises 25 1. For each $n \in \mathbb{N}$, let $f_n(x) = nx^n$ when $x \in [0,1)$ and $f_n(1) = 0$.

(a) Prove that $f_n \to 0$ pointwise on $[0,1]$.
(*Hint: recall Exercise 24.4 if you're not sure how to prove this*)

(b) By considering the integrals $\int_0^1 f_n(x)\,\mathrm{d}x$ show that $f_n \to 0$ is not uniform.

2. Prove that if $f_n \to f$ uniformly, then the sequence (f_n) is uniformly Cauchy.

3. (a) Suppose (f_n) is a sequence of bounded functions on U and suppose that $f_n \to f$ converges uniformly on U. Prove that f is bounded on U.

(b) Give an example of a sequence of bounded functions (f_n) converging pointwise to f on $[0,\infty)$, but for which f is *unbounded.*

4. The sequence defined by $f_n(x) = \frac{nx}{1+nx^2}$ (Exercise 24.1) converges uniformly on any closed interval $[a,b]$ where $0 < a < b$.

(a) Check explicitly that $\int_a^b f_n(x)\,\mathrm{d}x \to \int_a^b f(x)\,\mathrm{d}x$, where $f = \lim f_n$.

(b) Is the same thing true for derivatives?

5. Let $f_n(x) = n^{-1}\sin n^2 x$ be defined on $\mathbb{R}$.

(a) Prove that f_n converges uniformly on $\mathbb{R}$.

(b) Check that $\int_0^x f_n(t)\,\mathrm{d}t$ converges for any $x \in \mathbb{R}$.

(c) Does the derived sequence (f_n') converge? Explain.

6. Use the M-test to prove that $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ defines a continuous function on $[-1,1]$.

7. Prove that $\sum_{n=1}^{\infty} \frac{x^n \sin x}{(n+1)^3 2^n}$ converges uniformly to a continuous function on the interval $[-2,2]$.

8. Prove that if $\sum g_k$ converges uniformly on a set U and if h is a bounded function on U, then $\sum h g_k$ converges uniformly on U.
(*Warning: you cannot simply write* $\sum h g_k = h \sum g_k$)

9. Consider Example 2.20.2.

(a) Check explicitly that the convergence isn't uniform by computing $\sup_{x\in[-1,1]} |f_n(x) - f(x)|$

(b) Prove that $f_n \to 0$ pointwise on $(0,1]$ using the ϵ–N definition of convergence: that is, given $\epsilon > 0$ and $x \in (0,1]$, find an explicit $N(x,\epsilon)$ such that

$$n > N \implies |f(x)| < \epsilon$$

What happens to your choice of $N(x,\epsilon)$ as $x \to 0^+$?

10. Suppose (f_n') converges uniformly on $[a,b]$ and that each f_n' is continuous.

(a) Use the fact that (f_n') is uniformly Cauchy to prove that (f_n) is uniformly Cauchy and thus converges uniformly to some function f.
(*Hint:* $|f_n(x) - f_m(x)| = \left|\int_a^x f_n'(t) - f_m'(t)\,\mathrm{d}t\right| \ldots$)

(b) Explain why we need not have assumed the existence of f in Theorem 2.22.

26 Differentiation and Integration of Power Series

In this section we specialize our recent results to power series. While everything will be stated for series centered at $x = 0$, all are easily translated to arbitrary centers.

Theorem 2.29. *Let $\sum a_n x^n$ be a power series with radius of convergence $R > 0$ and let $T \in (0, R)$. Then:*

1. *The series converges uniformly on $[-T, T]$.*
2. *The series is uniformly continuous on $[-T, T]$ and continuous on $(-R, R)$.*

Proof. This is an easy application of the M-test. For each k, define $M_k = |a_k| T^k$,

$$T < R \implies \sum a_n T^n \text{ converges absolutely } \implies \sum M_k \text{ converges}$$

By the M-test and Corollary 2.26, the power series converges uniformly on $[-T, T]$ to a uniformly continuous function.

Finally, every $x \in (-R, R)$ lies in some such interval (take $T = |x|$), whence the power series is continuous on $(-R, R)$.

■

Example 2.30. On its interval of convergence $(-1, 1)$, the geometric series $\sum_{n=0}^{\infty} x^n$ converges pointwise to $\frac{1}{1-x}$; convergence is uniform on any interval $[-T, T] \subseteq (-1, 1)$.
We needn't use the Theorem for this is simple to verify directly: writing f, f_n for the series and its partial sums,

$$|f_n(x) - f(x)| = \left| \frac{1 - x^{n+1}}{1 - x} - \frac{1}{1 - x} \right| = \left| \frac{x^{n+1}}{1 - x} \right|$$

$$\implies \sup_{x \in [-T, T]} |f_n(x) - f(x)| = \frac{T^{n+1}}{1 - T} \xrightarrow[n \to \infty]{} 0$$

By contrast, the convergence is non-uniform on $(-1, 1)$;

$$\sup_{x \in (-1, 1)} |f_n(x) - f(x)| = \infty$$

Theorem 2.31. *Suppose $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$. Then the series is integrable and differentiable term-by-term on the interval $(-R, R)$. Indeed for any $x \in (-R, R)$,*

$$\frac{\mathrm{d}}{\mathrm{d}x} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad \int_0^x \sum_{n=0}^{\infty} a_n t^n \, \mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$$

where both series also have radius of convergence R.

Proof. Let $f(x) = \sum a_n x^n$ have radius of convergence R, and observe that

$$\limsup |na_n|^{1/n} = \lim n^{1/n} \limsup |a_n|^{1/n} = \frac{1}{R}$$

whence $\sum na_n x^n$ also has radius of convergence R. At any given non-zero $x \in (-R, R)$, we may write

$$\sum_{n=1}^{\infty} na_n x^{n-1} = x^{-1} \sum_{n=1}^{\infty} na_n x^n$$

to see that the derived series also has radius of convergence R. On any interval $[-T, T] \subseteq (-R, R)$, the derived series converges uniformly (Theorem 2.29). Since each $a_n x^n$ is continuously differentiable, Corollary 2.26 says that f is differentiable on $[-T, T]$ and that

$$f'(x) = \sum_{n=0}^{\infty} \frac{\mathrm{d}}{\mathrm{d}x} a_n x^n = \sum_{n=1}^{\infty} na_n x^{n-1}$$

Since any $x \in (-R, R)$ lies in some such interval $[-T, T]$, we are done.

The corresponding result for integrals is Exercise 6. ∎

We postpone the canonical examples until after the next result.

Continuity at Endpoints?

There is one small hole in our analysis. If a series has radius of convergence R we know that it converges and is continuous on $(-R, R)$. But what if it additionally converges at $x = \pm R$? Is the series continuous at the endpoints? The answer is an unequivocal yes, though this small benefit requires a lot of work!

Theorem 2.32 (Abel's Theorem). *Power series are continuous on their full interval of convergence.*

Examples 2.33. 1. Apply our results to the geometric series;

$$\frac{1}{(1-x)^2} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-x} = \sum_{n=1}^{\infty} nx^{n-1} = \sum_{n=0}^{\infty}(n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

$$\ln(1-x) = -\int_0^x \frac{1}{1-t}\,\mathrm{d}t = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1} = -\sum_{n=1}^{\infty} \frac{1}{n} x^n = -\left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\right)$$

with both series valid on $(-1, 1)$. In fact the first series has the same interval of convergence, while the second is $[-1, 1)$. By Abel's Theorem and the fact that logarithms are continuous, we have equality at $x = -1$ and the famous identity

$$\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

This example shows that while the integrated and differentiated series have the same radius of convergence as the original, convergence at the endpoints need not be the same in all cases.

2. Substitute $x \mapsto -x^2$ in the geometric series and integrate term-by-term: if $|x| < 1$, then

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \implies \arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

In fact the arctangent series also converges at $x = \pm 1$; Abel's Theorem says it is continuous on $[-1, 1]$. Since arctangent is continuous (on $\mathbb{R}$!) we recover another famous identity

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

As with the identity for $\ln 2$, this is a very slowly converging alternating series and therefore doesn't provide an efficient method for approximating π.

3. The series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ has radius of convergence ∞. Differentiate to obtain

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+1)!} x^{2n+1}$$

This series is also valid for all $x \in \mathbb{R}$. Differentiating again,

$$f''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n)!} x^{2n} = -\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = -f(x)$$

Recalling that $f(x) = \cos x$ is the unique solution to the initial value problem

$$\begin{cases} f''(x) = -f(x) \\ f(0) = 1, \ f'(0) = 0 \end{cases}$$

We conclude that, $\forall x \in \mathbb{R}$,

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \qquad \sin x = -f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

These expressions can instead be taken as the *definitions* of sine and cosine. As promised earlier in the course, continuity and differentiability now come for free. One difficulty with this definition is believing that it has anything to do with right-triangles!

We can similarly define other common transcendental functions using power series: for instance

$$\exp(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Example 2.33.1 could be taken as a definition of the logarithm on the interval $(0, 2]$,

$$\ln x = \ln(1 - (1-x)) = -\sum_{n=1}^{\infty} \frac{1}{n} (1-x)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (x-1)^n$$

though this is unnecessary since it is more natural to define ln as the inverse of the exponential.

Proof of Abel's Theorem (non-examinable)

This requires a lot of work, so feel free to omit on a first reading!

First observe that there is nothing to check unless $0 < R < \infty$. By the change of variable $x \mapsto \pm\frac{x}{R}$, it is enough for us to prove the following:

$$\sum_{n=0}^{\infty} a_n \text{ convergent and } f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ on } (-1,1) \implies \lim_{x\to 1^-} f(x) = \sum_{n=0}^{\infty} a_n$$

Proof. Let $s_n = \sum_{k=0}^{n} a_k$ and write $s = \lim s_n = \sum a_n$. It is an easy exercise to check that

$$\sum_{k=0}^{n} a_k x^k = s_n x^n + (1-x)\sum_{k=0}^{n-1} s_k x^k$$

If $|x| < 1$, then (since $s_n \to s$) $\lim s_n x^n = 0$, whence we obtain

$$\forall x \in (-1,1),\ f(x) = (1-x)\sum_{n=0}^{\infty} s_n x^n$$

Let $\epsilon \in (0,1)$ be given and fix $x \in (0,1)$. Then

$$\exists N \in \mathbb{N} \text{ such that } n > N \implies |s_n - s| < \frac{\epsilon}{2} \qquad (*)$$

Use the geometric series formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ and write $h(x) = (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n\right|$ to observe

$$\begin{aligned}
|f(x) - s| &= \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s\right| = \left|(1-x)\sum_{n=0}^{\infty} s_n x^n - s(1-x)\sum_{n=0}^{\infty} x^n\right| \\
&= \left|(1-x)\sum_{n=0}^{\infty}(s_n - s)x^n\right| = (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n + \sum_{n=N+1}^{\infty}(s_n - s)x^n\right| \\
&\le (1-x)\left|\sum_{n=0}^{N}(s_n - s)x^n\right| + (1-x)\left|\sum_{n=N+1}^{\infty}(s_n - s)x^n\right| && (\triangle\text{-inequality}) \\
&< h(x) + \frac{\epsilon}{2}(1-x)\left|\sum_{n=N+1}^{\infty} x^n\right| && (\text{by } (*)) \\
&\le h(x) + \frac{\epsilon}{2}
\end{aligned}$$

Since $h > 0$ is continuous and $h(1) = 0$, $\exists \delta > 0$ such that $x \in (1-\delta, 1) \implies h(x) < \frac{\epsilon}{2}$ (the computation of a suitable δ is another exercise).

We conclude that $\lim_{x\to 1^-} f(x) = s$. ∎

Exercises 26 1. (a) Prove that $\sum\limits_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ for $|x| < 1$.

   (b) Evaluate $\sum\limits_{n=1}^{\infty} \frac{n}{2^n}$, $\sum\limits_{n=1}^{\infty} \frac{n}{4^n}$ and $\sum\limits_{n=1}^{\infty} \frac{(-1)^n n}{4^n}$

2. (a) Starting with a power series centered at $x = 0$, evaluate the integral $\int_0^{1/2} \frac{1}{1+x^4}\,\mathrm{d}x$ as an infinite series.

   (b) (Harder) Repeat part (a) but for $\int_0^1 \frac{1}{1+x^4}\,\mathrm{d}x$. What extra ingredients do you need?

3. The probability that a standard normally distributed random variable X lies in the interval $[a, b]$ is given by the integral

$$\mathbb{P}(a \le X \le b) = \frac{1}{\sqrt{2\pi}} \int_a^b \exp\left(-\frac{x^2}{2}\right)\,\mathrm{d}x$$

   Find $\mathbb{P}(-1 \le X \le 1)$ as an infinite series.

4. Define $c(x) = \sum\limits_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$ and $s(x) = \sum\limits_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

   (a) Prove that $c'(x) = s(x)$ and that $s'(x) = c(x)$.

   (b) Prove that $c(x)^2 - s(x)^2 = 1$ for all $x \in \mathbb{R}$.

   (*These functions are the hyperbolic sine and cosine:* $s(x) = \sinh x$ *and* $c(x) = \cosh x$)

5. Let $a, b \in (-1, 1)$. Extending Example 2.30, show that the convergence $\sum x^n = \frac{1}{1-x}$ is non-uniform on any interval of the form $(-1, a)$ or $(b, 1)$.

6. Prove the integration part of Theorem 2.31.

7. Prove or disprove: If a series converges absolutely at the *endpoints* of its interval of convergence then its convergence is uniform on the entire interval.

8. Complete the proof of Abel's Theorem:

   (a) Let $s_n = \sum\limits_{k=0}^{n} a_k$ be the partial sum of the series $\sum a_n$. For each n, prove that,

$$\sum_{k=0}^{n} a_k x^k = s_n x^n + (1-x) \sum_{k=0}^{n-1} s_k x^k$$

   (b) Suppose $x > 0$. Let $S = \max\{|s_n - s| : n \le N\}$ and prove that $h(x) \le S(1 - x^{N+1})$. Hence find an explicit δ that completes the final step.

27 The Weierstraß Approximation Theorem

A major theme of analysis is *approximation*; for instance power series are an example of (uniform) approximation by polynomials. It is reasonable to ask whether any function can be so approximated. In 1885, Weierstraß answered a specific case in the affirmative.

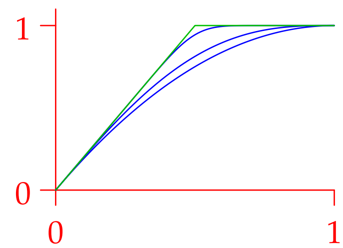
Theorem 2.34 (Weierstraß). *If $f : [a,b] \to \mathbb{R}$ is continuous, then there exists a sequence of polynomials converging uniformly to f on $[a,b]$.*

Suitable polynomials can be defined in various ways. By scaling the domain, it is enough to do this on $[a,b] = [0,1]$ where perhaps the simplest approach is via the *Bernstein Polynomials*,

$$B_n f(x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k} \qquad \left(\binom{n}{k} = \tfrac{n!}{k!(n-k)!} \text{ is the binomial coefficient}\right)$$

We omit the proof due to length; Weierstraß' original argument was completely different. Instead we compute a couple of examples and give an important interpretation/application.

Examples 2.35. 1. Suppose $f(x) = 2x$ if $x < \frac{1}{2}$ and $f(x) = 1$ otherwise.



$$B_1 f(x) = f(0)(1-x)f(0) + f(1)x = x$$

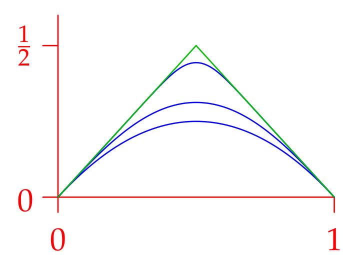
$$\begin{aligned} B_2 f(x) &= f(0)(1-x)^2 + 2f(\tfrac{1}{2})x(1-x) + f(1)x^2 \\ &= 2x(1-x) + x^2 \\ &= x(2-x) \end{aligned}$$

$$\begin{aligned} B_3 f(x) &= f(0)(1-x)^3 + 3f(\tfrac{1}{3})x(1-x)^2 + 3f(\tfrac{2}{3})x^2(1-x) + f(1)x^3 \\ &= 0(1-x)^3 + 2x(1-x)^2 + 3x^2(1-x) + x^3 \\ &= x(2-x) = B_2 f(x) \end{aligned}$$

$$\begin{aligned} B_4 f(x) &= 0(1-x)^4 + 2x(1-x)^3 + 6x^2(1-x)^2 + 4x^3(1-x) + x^4 \\ &= x(x^3 - 2x^2 + 2) \end{aligned}$$

The Bernstein polynomials $B_2 f(x), B_4 f(x)$ and $B_{50} f(x)$ are drawn.

2. Now assume $f(x) = x$ if $x < \frac{1}{2}$ and $1 - x$ otherwise.



$$B_1 f(x) = f(0)(1-x) + f(1)x = 0$$

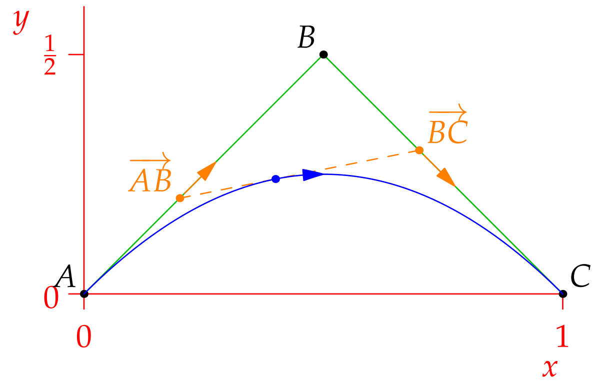
$$B_2 f(x) = x(1-x)$$

$$\begin{aligned} B_3 f(x) &= 0(1-x)^3 + x(1-x)^2 + x^2(1-x) + 0x^3 \\ &= x(1-x) = B_2 f(x) \end{aligned}$$

$$\begin{aligned} B_4 f(x) &= f(0)(1-x)^4 + f(\tfrac{1}{4})\cdot 4x(1-x)^3 + f(\tfrac{1}{2})\cdot 6x^2(1-x)^2 + f(\tfrac{3}{4})\cdot 4x^3(1-x) + f(1)x^4 \\ &= x(1-x)^3 + 3x^2(1-x)^2 + x^3(1-x) \\ &= x(1-x)(1+x-x^2) \end{aligned}$$

Bézier curves (just for fun!)

The Bernstein polynomials arise naturally when considering *Bézier curves*. These have many applications, particularly in computer graphics. Given three points A, B, C, define points on the line segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$ for each $t \in [0, 1]$, via



$$\overrightarrow{AB}(t) = (1-t)A + tB \qquad \overrightarrow{BC}(t) = (1-t)B + tC$$

These points move at a constant speed along the corresponding segments. Now consider a point on the *moving* segment between the points defined above:

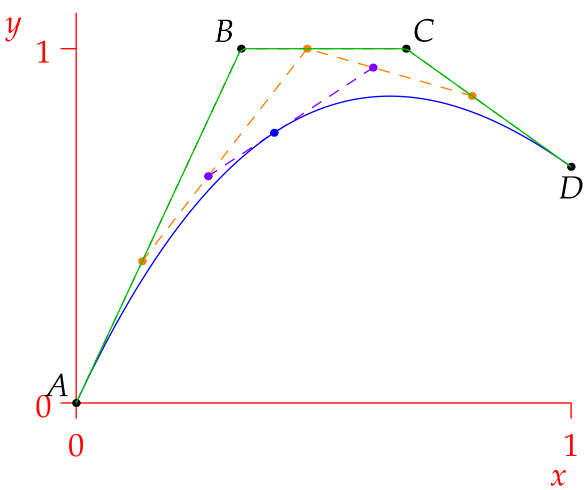
$$R(t) := (1-t)\overrightarrow{AB}(t) + t\overrightarrow{BC}(t) = (1-t)^2 A + 2t(1-t)B + t^2 C$$

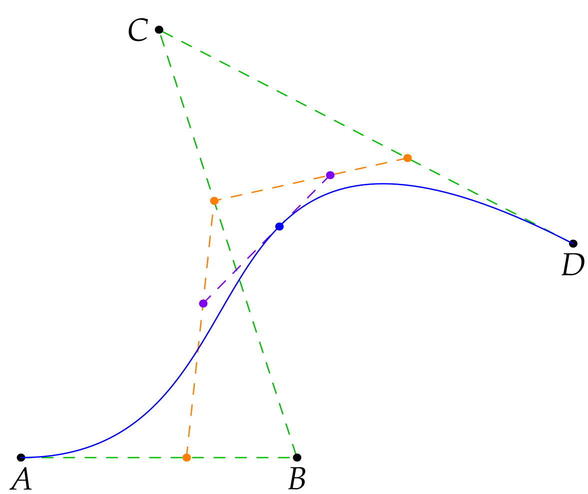
This is the *quadratic Bézier curve with control points* A, B, C. The 2^{nd} Bernstein polynomial for a function f is simply the quadratic Bézier curve with control points $(0, f(0))$, $\left(\frac{1}{2}, f(\frac{1}{2})\right)$ and $(1, f(1))$. The picture[14] above shows $B_2 f(x)$ for the above example.

We can repeat the construction with more control points: with four points A, B, C, D, one constructs $\overrightarrow{AB}(t)$, $\overrightarrow{BC}(t)$, $\overrightarrow{CD}(t)$, then the second-order points between these, and finally the cubic Bézier curve

$$\begin{aligned} R(t) :&= (1-t)\left((1-t)\overrightarrow{AB}(t) + t\overrightarrow{BC}(t)\right) + t\left((1-t)\overrightarrow{BC}(t) + t\overrightarrow{CD}(t)\right) \\ &= (1-t)^3 A + 3t(1-t)^2 B + 3t^2(1-t)C + t^3 D \end{aligned}$$

where we now recognize the relationship to the 3^{rd} Bernstein polynomial.





The pictures show cubic Bézier curves: the first is the graph of the Bernstein polynomial

$$B_3 f(x) = 0(1-x)^3 + 3x(1-x)^2 + 3x^2(1-x) + \tfrac{2}{3}x^3$$

while the second is for the four given control points A, B, C, D.

[14]To see these pictures move, visit `https://www.math.uci.edu/~ndonalds/math140b/bezier.html`

Exercises 27

1. Show that the closed bounded interval assumption in the approximation theorem is required by giving an example of a continuous function $f : (-1,1) \to \mathbb{R}$ which is *not* the uniform limit of a sequence of polynomials.

2. If $g : [a,b] \to \mathbb{R}$ is continuous, then $f(x) := g\big((b-a)x + a\big)$ is continuous on $[0,1]$. If $P_n \to f$ uniformly on $[0,1]$, prove that $Q_n \to g$ uniformly on $[a,b]$, where

$$Q_n(x) = P_n\left(\frac{x-a}{b-a}\right)$$

3. Use the binomial theorem to check that every Bernstein polynomial for $f(x) = x$ is $B_n f(x) = x$ itself!

4. Find a parametrization of the cubic Bézier curve with control points $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$. Now sketch the curve.
(*Use a computer algebra package if you like!*)

5. (Hard) Show that the Bernstein polynomials for $f(x) = x^2$ are given by

$$B_n f(x) = \frac{1}{n}x + \frac{n-1}{n}x^2$$

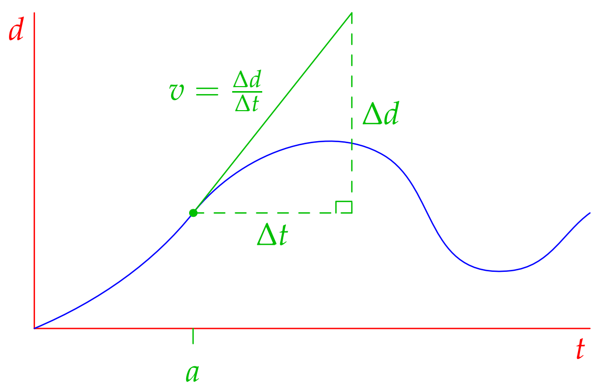
and thus verify explicitly that $B_n f \to f$ uniformly.

3 Differentiation

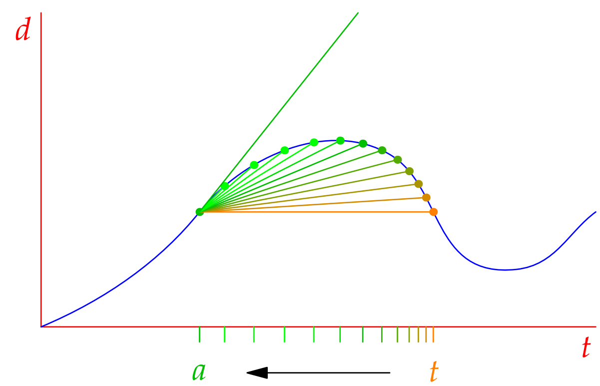
Differentiation grew out of the problem of *instantaneous velocity.* Velocity can only easily be measured as an *average* over a time interval:[15] if an object travels Δd meters in Δt seconds, then its average velocity is $v_{\text{av}} = \frac{\Delta d}{\Delta t}$ ms^{-1}. An early 'definition' (dating to the 1300's) makes the instantaneous velocity equal to the constant velocity that would be observed if a body were to stop accelerating: while useless for the purposes of measurement, this is essentially Newton's first law regarding inertial motion (1687). We also see the concept of the *tangent line* beginning to appear: if one graphs position against time, then a couple of things should be clear:

- The graph of inertial (constant speed) motion is a straight line whose slope is the velocity.
- The tangent line to a curve at a point has slope equal to the instantaneous velocity at that point.

The problem of finding, defining and computing instantaneous velocity thus morphed into the consideration of tangent lines to curves. With the advent of analytic geometry in the early 1600's, mathematicians such as Fermat and Descartes pioneered versions of the familiar *secant ('cutting') line* method for computing tangents.



Instantaneous velocity equals constant velocity corresponding to tangent line



Secant lines approximate tangent line as $t \to a$

The average velocity of the particle over the time interval $[a, t]$ is the slope of the secant line, namely

$$v_{\text{av}}(a, t) = \frac{d(t) - d(a)}{t - a}$$

Since the secant lines approximate the tangent line as t approaches a, it seems reasonable that we should compute the instantaneous velocity in this manner:

$$v(a) = \lim_{t \to a} v_{\text{av}}(a, t) = \lim_{t \to a} \frac{d(t) - d(a)}{t - a}$$

This is, of course, the modern definition of the *derivative.*

[15]Even a modern technique such as Doppler-shift compares measurements separated by the extremely small period of a light or soundwave. These are still therefore *average* velocities, albeit taken over very small time intervals.

Definition 3.1. Let $f : U \to \mathbb{R}$ and let $a \in U$. We say that f *is differentiable at* a if the following limit exists (is *finite*!)

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

We call this limit the *derivative of* f *at* a and denote its value by either $f'(a)$ or $\left.\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=a}$.

If $f'(a)$ exists for all $a \in U$ then f is *differentiable (on U)*; the derivative becomes a function $f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x}$.

Notation The contrasting styles are partly attributable, to the primary founders of calculus, Issac Newton and Gottfried Leibniz. Each has its pros and cons and you should be comfortable with both.

One-sided derivatives Since the defining limit is *two-sided*, differentiability only makes sense at *interior points* of U. *Left-* and *right-derivatives* may be defined via one-sided limits; differentiability is equivalent to these being equal. All results in this section hold for one-sided derivatives with suitable (sometimes tedious) modifications. It is quite common, though strictly incorrect, to say that f is differentiable on an interval $[a, b)$ if it is differentiable on the interior (a, b) and *right*-differentiable at a; however, we will strictly adhere to differentiable meaning *two-sided.*

Examples 3.2. 1. Let $f(x) = x^2 + 4x$. Then, for any $a \in \mathbb{R}$,

$$\begin{aligned}\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{x^2 + 4x - a^2 - 4a}{x - a} = \lim_{x \to a} \frac{(x - a)(x + a + 4)}{x - a} \\ &= \lim_{x \to a}(x + a + 4) = 2a + 4\end{aligned}$$

Note how the definition of $\lim\limits_{x \to a}$ allows us to cancel the $x - a$ terms from the numerator and denominator. We conclude that f is differentiable (on $\mathbb{R}$) and that $f'(x) = 2x + 4$.

2. Let $g(x) = \frac{x+1}{2x-3}$. Then, for any $a \neq \frac{3}{2}$,

$$\begin{aligned}\lim_{x \to a} \frac{f(x) - f(a)}{x - a} &= \lim_{x \to a} \frac{1}{x - a}\left[\frac{x + 1}{2x - 3} - \frac{a + 1}{2a - 3}\right] = \lim_{x \to a} \frac{5a - 5x}{(x - a)(2x - 3)(2a - 3)} \\ &= \lim_{x \to a} \frac{-5}{(2x - 3)(2a - 3)} = \frac{-5}{(2a - 3)^2}\end{aligned}$$

f is therefore differentiable on its domain $\mathbb{R} \setminus \{\frac{3}{2}\}$ with derivative $f'(x) = \frac{-5}{(2x-3)^2}$.

The familiar expressions

$$f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}, \qquad f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

are equivalent to the original definition (see Exercise 5). While seemingly simpler, they sometimes lead to nastier calculations: see what happens if you try the previous example in this language…

We now turn to possibly the most well-known result of Freshman Calculus.

Theorem 3.3 (Power Law). *Let $r \in \mathbb{R}$. Then $f(x) = x^r$ is differentiable with $f'(x) = rx^{r-1}$.*

The domains of f and f' depend messily on r, but the above certainly holds on the interval $(0, \infty)$. We leave a complete proof to the exercises and instead consider a few generalizable examples.

Examples 3.4. 1. If $n \in \mathbb{N}$ and $a \in \mathbb{R}$, a simple factorization yields

$$\begin{aligned}\lim_{x\to a} \frac{x^n - a^n}{x - a} &= \lim_{x\to a} \frac{(x-a)(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1})}{x - a} \qquad (*)\\ &= \lim_{x\to a}(x^{n-1} + ax^{n-2} + \cdots + a^{n-2}x + a^{n-1}) = na^{n-1}\end{aligned}$$

We conclude that $\frac{\mathrm{d}}{\mathrm{d}x} x^n = nx^{n-1}$.

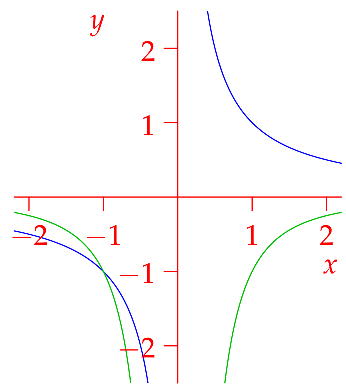
2. If $f(x) = x^{-1}$ and $a \neq 0$, then

$$\lim_{x\to a} \frac{x^{-1} - a^{-1}}{x - a} = \lim_{x\to a} \frac{a - x}{ax(x-a)} = \lim_{x\to a} \frac{-1}{ax} = -\frac{1}{a^2}$$

from which we conclude that $f'(x) = -x^{-2}$.

A similar approach followed by the factorization $(*)$ proves the power law for all negative integer exponents:

$$\frac{x^{-n} - a^{-n}}{x - a} = \frac{a^n - x^n}{a^n x^n (x-a)} = \cdots$$

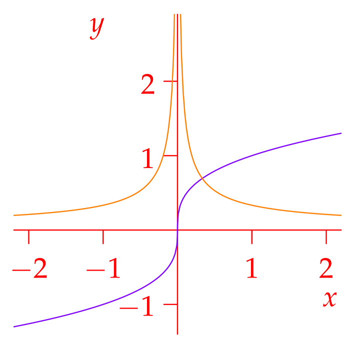


3. To differentiate $x^{1/n}$, simply substitute $x = y^n$ and observe case 1. If $g(x) = x^{1/3}$ and $a \neq 0$, then $y = x^{1/3}$ and $b = a^{1/3}$ yield

$$\lim_{x\to a} \frac{x^{1/3} - a^{1/3}}{x - a} = \lim_{y\to b} \frac{y - b}{y^3 - b^3} = \frac{1}{3b^2} = \frac{1}{3}a^{-2/3}$$

$$\implies g'(x) = \frac{1}{3}x^{-2/3}$$

Note that g is *not* differentiable at $x = 0$!



We could similarly compute the derivative for all rational exponents, though it is much easier to wait for the chain rule. The power law for irrational exponents is somewhat more ticklish.

Corollary 3.5 (Basic Transcendental Functions). *Recalling our development of power series in the previous chapter, the power law (for positive integers!) is all we need to see that*

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x), \qquad \frac{\mathrm{d}}{\mathrm{d}x}\sin x = \cos x, \qquad \frac{\mathrm{d}}{\mathrm{d}x}\cos x = -\sin x$$

It is also possible to develop these results independently of power series (see e.g. Exercise 9).

Failure of differentiability

It is instructive to consider when a function can fail to be differentiable. First a simple result shows that functions are not differentiable at discontinuities.

Theorem 3.6. *If f is differentiable at a then f is continuous at a.*

Proof. Simply take the limit (think carefully why this works!):

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}(x-a)+f(a)\right] = f'(a)(0-0)+f(a) = f(a)$$ ∎

It remains to consider situations when a function is continuous but not differentiable.

Examples 3.7. The following cover all situations where a function is continuous on an interval and differentiable everywhere *except* at a single interior point; similarly to isolated discontinuities, these are classified by considering the three ways in which the derivative limit might not exist.

1. A *vertical tangent line* occurs when the derivative is infinite. For instance, $g(x) = x^{1/3}$ at $x = 0$.

2. *Corners* occur when the one-sided derivatives are unequal (could be infinite). For instance, $f(x) = |x|$ is not differentiable at zero, the one-sided derivatives being

$$\lim_{x\to 0^+}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^+}\frac{x}{x} = 1 \neq \lim_{x\to 0^-}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^-}\frac{-x}{x} = -1$$

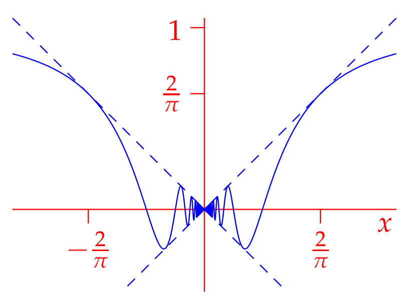
   Indeed f is differentiable everywhere except at zero, with

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

3. A *singularity* is where left- and/or right-derivatives do not exist. The standard example in this case is

$$f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   which is continuous on $\mathbb{R}$ and differentiable everywhere except at zero: the details are in Exercise 8.



Singularities and vertical tangent lines can also prevent one-sided differentiability.

More esoteric examples of non-differentiability are also possible:

- Utilizing series, we can create functions which are continuous on an interval but *nowhere differentiable*! For a classic example, see page 28.

- It is also possible to construct a function which differentiable (and thus continuous) at precisely one point; can you think of an example?

The Basic Rules of Differentiation

Theorem 3.8. *Let f, g be differentiable and k, l be constants.*

1. *(Linearity) The function $kf + lg$ is differentiable with $(kf + lg)' = kf' + lg'$.*
2. *(Product rule) The function fg is differentiable with $(fg)' = f'g + fg'$.*
3. *(Inverse functions) If f is bijective with non-zero derivative, then f^{-1} is differentiable and*

$$\frac{\mathrm{d}}{\mathrm{d}x} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. Parts 1 and 2 follow from the limit laws:

$$\lim_{x \to a} \frac{(kf + lg)(x) - (kf + lg)(a)}{x - a} = \lim_{x \to a} \left[k \frac{f(x) - f(a)}{x - a} + l \frac{g(x) - g(a)}{x - a} \right] = kf'(a) + lg'(a)$$

$$\lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} = \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} g(x) + f(a) \frac{g(x) - g(a)}{x - a} \right] = f'(a)g(a) + f(a)g'(a)$$

Note where we used the continuity of g in the second line ($\lim g(x) = g(a)$). Part 3 is an exercise. ∎

The inverse function rule is intuitive since the graphs of f and f^{-1} are related by reflection in the line $y = x$; gradients at corresponding points are therefore reciprocal. In Leibniz notation the result reads $\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$.

Examples 3.9. 1. Linearity allows us to differentiate any polynomial: for instance

$$\frac{\mathrm{d}}{\mathrm{d}x}(7x^2 + 13x^4) = 7\frac{\mathrm{d}}{\mathrm{d}x}x^2 + 13\frac{\mathrm{d}}{\mathrm{d}x}x^4 = 14x + 52x^3$$

2. The product rule extends the reach of differentiation somewhat:

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^4 \sin x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}x^4\right) \sin x + x^4 \frac{\mathrm{d}}{\mathrm{d}x} \sin x = 4x^3 \sin x - x^4 \cos x$$

3. The inverse trigonometric functions can now be differentiated. For instance,

$$y = \sin^{-1} x \implies \frac{\mathrm{d}}{\mathrm{d}x} \sin^{-1} x = \frac{\mathrm{d}y}{\mathrm{d}x} = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

4. Define natural log to be the inverse of the (bijective!) exponential function $\exp(x)$:

$$y = \ln x \iff x = \exp y$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} \ln x = \left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^{-1} = \frac{1}{\exp y} = \frac{1}{x}$$

The full details, and the justification that $\exp x = e^x$, form an optional exercise.

Theorem 3.10 (Chain Rule). *If g is differentiable at a and f is differentiable at $g(a)$ then $f \circ g$ is differentiable at a with derivative*

$$(f \circ g)'(a) = f'(g(a))\, g'(a)$$

In Leibniz notation this reads $\frac{\mathrm{d}(f\circ g)}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\mathrm{d}g}{\mathrm{d}x}$ which looks like a simple cancellation of the $\mathrm{d}g$ terms![16]

Proof. Define $\gamma : \mathrm{dom}(f) \to \mathbb{R}$ via

$$\gamma(v) = \begin{cases} \frac{f(v)-f(g(a))}{v-g(a)} & \text{if } v \neq g(a) \\ f'(g(a)) & \text{if } v = g(a) \end{cases} \tag{$*$}$$

Since f is differentiable at $g(a)$, we see that γ is continuous and $\lim_{v \to g(a)} \gamma(v) = f'(g(a))$.
Since g is differentiable at a, there exists an open interval $U \ni a$ for which $x \in U \implies g(x) \in \mathrm{dom}(f)$.
Now compute: for any $x \in U \setminus \{a\}$, let $v = g(x)$ in $(*)$, whence

$$\frac{f(g(x)) - f(g(a))}{x-a} = \gamma(g(x))\frac{g(x)-g(a)}{x-a}$$

Take limits as $x \to a$ for the result. ∎

Corollary 3.11 (Quotient Rule). *Suppose f and g are differentiable. Then $\frac{f}{g}$ is differentiable whenever $g(x) \neq 0$. Moreover*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

The proof is an exercise.

Examples 3.12. 1. By the quotient rule,

$$\frac{\mathrm{d}}{\mathrm{d}x}\tan x = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

2. We can now differentiate highly involved combinations of elementary functions:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan(e^{4x^2}) - \frac{7x}{\sin x}\right] = 8xe^{4x^2}\sec^2(e^{4x^2}) - \frac{7\sin x - 7x\cos x}{\sin^2 x}$$

[16] This is completely unjustified since $\mathrm{d}g$ does not (for us) mean anything on its own! The same problem appears in the famously faulty one-line 'proof' of the chain rule:

$$\lim_{x\to a}\frac{f(g(x)) - f(g(a))}{x-a} \overset{?}{=} \lim_{x\to a}\frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \lim_{x\to a}\frac{g(x)-g(a)}{x-a}$$

The second limit cannot exist unless $g(x) \neq g(a)$ for all x near, but not equal to, a. The faulty argument is repaired by replacing the second difference quotient with $f'(g(a))$ whenever $g(x) = g(a)$, *before* taking the limit. This is precisely what $\gamma(g(x))$ does in the correct proof.

Exercises 28 1. Use Definition 3.1 to calculate the derivatives.

(a) $f(x) = x^3$ at $x = 2$

(b) $g(x) = x + 2$ at $x = a$

(c) $f(x) = x^2 \cos x$ at $x = 0$

(d) $r(x) = \dfrac{3x+4}{2x-1}$ at $x = 1$

2. Differentiate the function $f(x) = \cos\left(e^{x^5-3x}\right)$ using the chain and product rules.

3. (a) Prove the quotient rule (Corollary 3.11) by combining the chain and product rules.
   (b) Prove the inverse derivative rule (Theorem 3.8, part 3).
   (*Hint: You can't simply differentiate* $1 = \frac{dx}{dx} = \frac{d}{dx} f(f^{-1}(x))$ *using the chain rule; why not?*)

4. (a) Find the derivatives of secant, cosecant and cotangent using the quotient rule.
   (b) Why did we choose the positive square-root when computing $\frac{d}{dx}\sin^{-1} x$? What is the standard domain of arcsine, and what happens at $x = \pm 1$?
   (c) Find the derivatives of the inverse trigonometric functions using the inverse function rule.

5. Using the definition of the derivative, and supposing that f is differentiable at a, prove that

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{f(a+h) - f(a-h)}{2h}$$

6. Prove that the function $f(x) = x\,|x|$ is differentiable everywhere and compute its derivative.

7. Show that following function is differentiable everywhere and compute its derivative:

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Moreover, prove that the derivative f' is *discontinuous* at $x = 0$.

8. Show that the following function is differentiable everywhere *except* at zero:

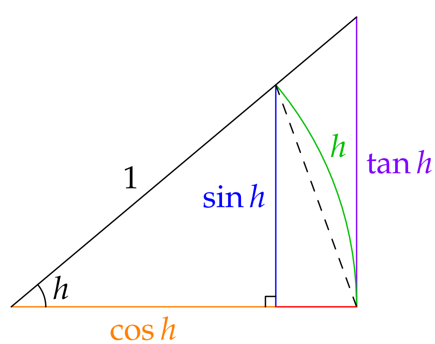
$$f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

9. (a) Suppose $0 < h < \frac{\pi}{2}$. Use the picture to show that

$$0 < \frac{1 - \cos h}{h} < \sin \frac{h}{2} \quad \text{and} \quad \sin h < h < \tan h$$

Hence conclude that $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{1-\cos h}{h} = 0$.

(b) Use part (a) to prove that $\frac{d}{dx} \sin x = \cos x$



10. (Hard) Use induction to prove the Leibniz rule (general product rule):

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

Masochists Corner (non-examinable)

We finish with two *very hard* bonus exercises, though the first is somewhat easier. If you want a challenge, give 'em a go!

The Exponential Function & the General Power Law

Consider the function $\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$ which converges for all real x.
As we saw when discussing power series, this function satisfies the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x), \qquad \exp(0) = 1$$

Define $e := \exp(1)$. Certainly e^x makes sense whenever $x \in \mathbb{Q}$. When x is irrational, define

$$e^x := \sup\{e^q : q \in \mathbb{Q},\ q < x\}$$

Our primary goal is to *prove* that $\exp(x) = e^x$. As a nice bonus we recover Bernoulli's limit identity $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ and obtain a complete proof of the power law.

(a) For all $x, y \in \mathbb{R}$, prove that $\exp(x+y) = \exp(x)\exp(y)$

(*Hint: use the binomial theorem and change the order of summation*)

(b) Show that $\exp(x)$ is always positive, *even when* $x < 0$.

(c) Prove that $\exp : \mathbb{R} \to (0, \infty)$ is bijective.

(*Hint:* $x \geq 0 \implies \exp(x) \geq 1 + x$*; take limits then apply part (a)*)

(d) Prove that $e^x = \exp(x)$. Do this in three stages:

- If $x \in \mathbb{N}$, use part (a). Now check for $x \in \mathbb{Z}^-$.
- If $x = \frac{m}{n} \in \mathbb{Q}$, first compute $\left[\exp(\frac{m}{n})\right]^n$.
- If x is irrational, start with $\exists(q_n) \subseteq \mathbb{Q}$ such that $q_n < x$ and $e^{q_n} \to e^x \ldots$

(e) Let $\ln : (0, \infty) \to \mathbb{R}$ be the inverse function of exp. Prove the logarithm laws:

$$\ln(xy) = \ln x + \ln y \quad \text{and} \quad \ln x^r = r \ln x$$

(*Just do this when* $r \in \mathbb{N}$*; another argument like part (d) is required in general*)

(f) We've already seen that $\frac{\mathrm{d}}{\mathrm{d}y}\ln y = \frac{1}{y}$. Use the fact that

$$\frac{\mathrm{d}}{\mathrm{d}y}\ln y = \lim_{h\to 0}\frac{\ln(y+h) - \ln y}{h}$$

to prove that $\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$, thus recovering Bernoulli's definition of e.

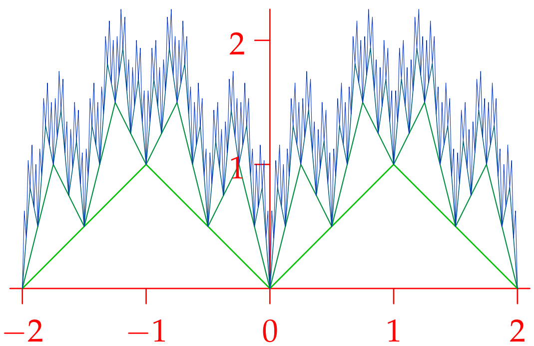
(g) For any $r \in \mathbb{R}$, *define* $x^r := \exp(r \ln x)$. Hence obtain the power law for any exponent.

A Very Strange Function

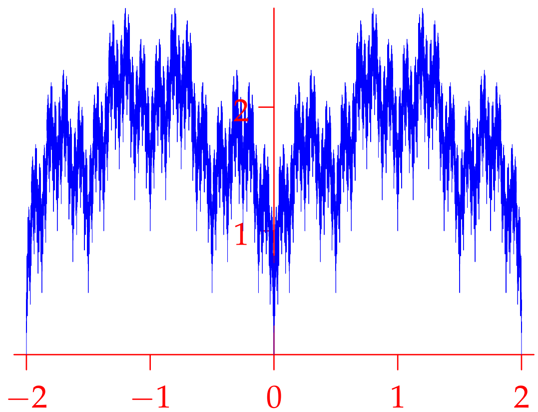
Here is a classic example of a continuous but nowhere-differentiable function!

Let f be the *sawtooth* function defined by $f(x) = |x|$ whenever $x \in [-1,1]$ and extending periodically to $\mathbb{R}$ so that $f(x+2) = f(x)$. Now define $g : \mathbb{R} \to \mathbb{R}$ via

$$g(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n f(4^n x)$$



$f(x)$ and iterations to $n = 3$



$g(x)$ (really $n = 6$, but can you tell?!)

(a) Prove that g is well-defined and continuous on $\mathbb{R}$.

(b) Let $x \in \mathbb{R}$ and $m \in \mathbb{N}$ be fixed. Define $h_m = \pm\frac{1}{2} \cdot 4^{-m}$ where the $\pm$-sign is chosen so that no integers lie strictly between $4^m x$ and $4^m(x + h_m) = 4^m x \pm \frac{1}{2}$.

For each $n \in \mathbb{N}_0$, define

$$k_n = \frac{f\big(4^n(x+h_m)\big) - f(4^n x)}{h_m}$$

Prove the following

i. $|k_n| \le 4^n$ with equality when $n = m$.

ii. $n > m \implies k_n = 0$.

(*Hint: $|f(y) - f(z)| \le |y - z|$: when is this an equality?*)

(c) Use part (b) to prove that

$$\left|\frac{g(x+h_m) - g(x)}{h_m}\right| \ge \frac{1}{2}(3^m + 1)$$

Hence conclude that g is *nowhere differentiable.*

29 The Mean Value Theorem

We now turn to one of the central results in calculus.

Theorem 3.13 (Mean Value Theorem/MVT). *Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists $\xi \in (a,b)$ such that $f'(\xi) = \frac{f(b)-f(a)}{b-a}$.*

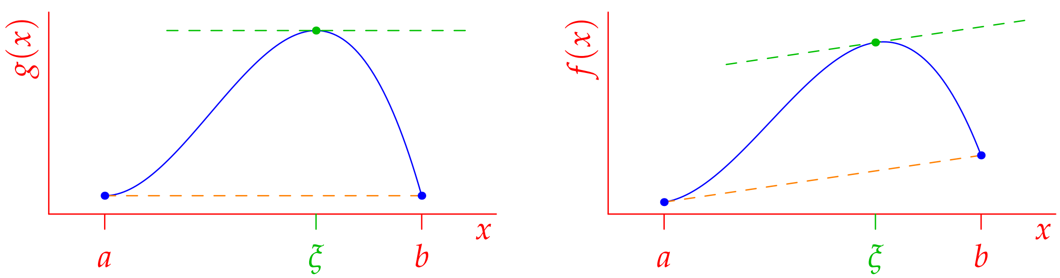
This follows easily from two lemmas.

Lemma 3.14.
1. *(Critical Points) Suppose g is bounded on (a,b) and attains its maximum or minimum at $\xi \in (a,b)$. If g is differentiable at ξ then $g'(\xi) = 0$.*
2. *(Rolle's Theorem) Suppose g is continuous on $[a,b]$, differentiable on (a,b), and that $g(a) = g(b)$. Then there exists $\xi \in (a,b)$ such that $g'(\xi) = 0$.*

The main result follows by applying Rolle's theorem to

$$g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-b)$$

and observing that $g(a) = f(b) = g(b)$ and $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$.



Critical Points/Rolle's Theorem   Mean Value Theorem

In the pictures, the orange and green lines are *parallel*: the average slope over the interval $[a,b]$ equals the gradient/derivative $f'(\xi)$.

Proof of Lemma. 1. Suppose, for a contradiction, that

$$g'(\xi) = \lim_{x\to\xi} \frac{g(x)-g(\xi)}{x-\xi} > 0$$

Let $\epsilon = g'(\xi)$ in the definition of limit: $\exists \delta > 0$ such that

$$0 < |x-\xi| < \delta \implies \left| \frac{g(x)-g(\xi)}{x-\xi} - g'(\xi) \right| < g'(\xi) \implies 0 < \frac{g(x)-g(\xi)}{x-\xi} < 2g'(\xi)$$

In particular, if $x \in (\xi, \xi+\delta)$, then $g(x) > g(\xi)$, contradicting the maximality at ξ.

The argument when $g'(\xi) < 0$ is similar. Finally, apply to $-g$ for the result at a minimum.

2. By the extreme value theorem, g is bounded and attains its bounds. If the maximum and minimum *both* occur at the endpoints a, b, then g is constant: any $\xi \in (a,b)$ satisfies the result. Otherwise, at least one extreme value occurs at some $\xi \in (a,b)$: part 1 says that $g'(\xi) = 0$. ∎

Examples 3.15. 1. Let $f(x) = (x-1)^2(4-x) + x$ on $[a,b] = [1,4]$: this is roughly the above picture illustrating the mean value theorem. We compute the average slope and the derivative:

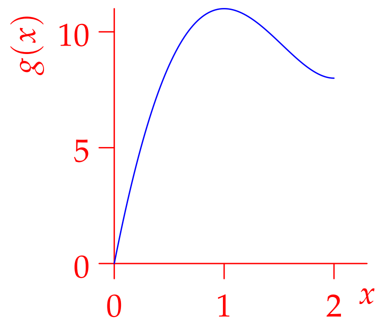
$$\frac{f(b)-f(a)}{b-a} = 1, \qquad f'(x) = 2(x-1)(4-x) - (x-1)^2 + 1 = -3x^2 + 12x - 8$$

and observe that

$$f'(\xi) = \frac{f(b)-f(a)}{b-a} \iff 3\xi^2 - 12\xi + 9 = 0 \iff \xi = 1 \text{ or } 3$$

Since only 3 lies in the interval $(1,4)$, this is the value ξ satisfying the mean value theorem.

2. We find the maximum and minimum values of $g(x) = x^4 - 14x^2 + 24x$ on the interval $[0,2]$. The function is differentiable, with



$$g'(x) = 4x^3 - 28x + 24 = 4(x-2)(x-1)(x+3)$$

By the Lemma, the locations of the extrema are either the end-points $x = 0, 2$ or locations with zero derivative ($x = 1$). Since

$$f(0) = 0, \quad f(1) = 11, \quad f(2) = 8$$

we conclude that $\max(f) = f(1) = 11$ and $\min(f) = f(0) = 0$.

Consequences of the Mean Value Theorem Several simple corollaries relate to monotonicity.

Definition 3.16. Suppose $f : I \to \mathbb{R}$ is defined on an interval I. We say that f is:

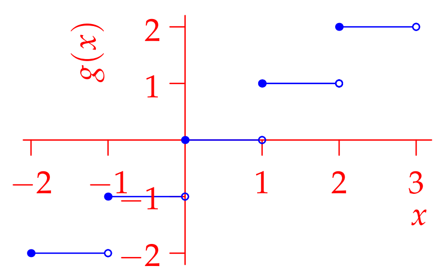
Increasing (monotone-up) on I if $x < y \implies f(x) \le f(y)$

Decreasing (monotone-down) on I if $x < y \implies f(x) \ge f(y)$

We say *strictly* increasing/decreasing if the inequalities are strict.

Examples 3.17. 1. $f : x \mapsto x^2$ is strictly increasing on $[0, \infty)$ and strictly decreasing on $(-\infty, 0]$.

2. The floor function $f : x \mapsto \lfloor x \rfloor$ (the greatest integer less than or equal to x) is increasing, but not strictly, on $\mathbb{R}$.



Corollary 3.18. *Suppose f is differentiable on an interval I, then*

1. $f' \ge 0$ on $I \iff f$ *is increasing on I*
2. $f' \le 0$ on $I \iff f$ *is decreasing on I*
3. $f' = 0$ on $I \iff f$ *is constant on I*

Proof. ($\Rightarrow$) Let $x < y$ where $x, y \in I$. By the mean value theorem, $\exists \xi \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(\xi) \quad \text{whence} \quad f'(\xi) \geq 0 \implies f(y) \geq f(x)$$

($\Leftarrow$) For the converse, use the definition of derivative: $f'(\xi) = \lim_{x \to \xi} \frac{f(x) - f(\xi)}{x - \xi}$. If f is increasing, then

$$x > \xi \implies f(x) \geq f(\xi) \implies f'(\xi) \geq 0$$

Parts 2 and 3 are similar. ∎

Corollary 3.18 yields a couple of flashbacks to elementary calculus.

Corollary 3.19. *Let I be an interval.*

1. *(Anti-derivatives on an interval) If $f'(x) = g'(x)$ on I, then $\exists c$ such that $g(x) = f(x) + c$ on I.*
2. *(First derivative test) Suppose f is continuous on I and differentiable except perhaps at ξ. If*

$$\begin{cases} f'(x) < 0 & \text{whenever } x < \xi, \text{ and} \\ f'(x) > 0 & \text{whenever } x > \xi \end{cases} \qquad \text{then } f \text{ has its minimum value at } x = \xi$$

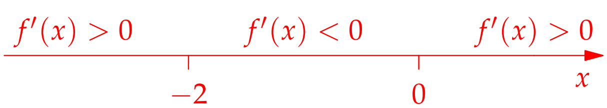
The statement for a maximum is similar.

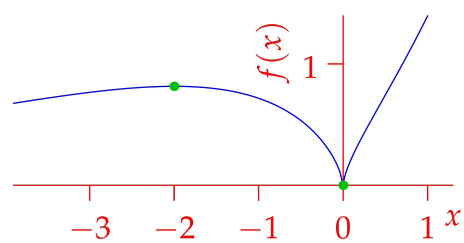
Examples 3.20. 1. Since $\frac{d}{dx} \sin(3x^2 + x) = (6x+1)\cos(3x^2 + x)$ on (the interval) $\mathbb{R}$, whence all anti-derivatives of $f(x) = (6x+1)\cos(3x^2 + x)$ are given by

$$\int f(x)\,dx = \int (6x+1)\cos(3x^2 + x)\,dx = \sin(3x^2 + x) + c$$

As is typical, we use the *indefinite integral* notation $\int f(x)\,dx$ for anti-derivatives.

2. If $f(x) = x^{2/3}e^{x/3}$, then $f'(x) = \frac{1}{3}x^{-1/3}(2+x)e^{x/3}$.
By Lemma 3.14, the only possible critical points are at $x = 0$ or -2. The sign of the derivative is also clear:





By the 1st derivative test, f has a maximum at $x = -2$ and a minimum at $x = 0$.

We finish this section by tying together the mean and intermediate value theorems.

Theorem 3.21 (IVT for Derivatives). *Suppose f is differentiable on an interval I containing $a < b$, and that L lies between $f'(a)$ and $f'(b)$. Then $\exists \xi \in (a, b)$ such that $f'(\xi) = L$.*

If $f'(x)$ is *continuous*, this is just the intermediate value theorem applied to f'. A full proof is left to the exercises; surprisingly, continuity is *not* required...

Exercises 29 1. Determine whether the conclusion of the mean value theorem holds for each function on the given interval. If so, find a suitable point ξ. If not, state which hypothesis fails.

(a) x^2 on $[-1,2]$ (b) $\sin x$ on $[0,\pi]$ (c) $|x|$ on $[-1,2]$
(d) $1/x$ on $[-1,1]$ (e) $1/x$ on $[1,3]$

2. Suppose f and g are differentiable on an open interval I, that $a<b$ and $f(a)=f(b)=0$. By considering $h(x)=f(x)e^{g(x)}$, prove that $f'(\xi)+f(\xi)g'(\xi)=0$ for some $\xi\in(a,b)$.

3. Use the Mean Value Theorem to prove the following:
   (a) $x<\tan x$ for all $x\in(0,\pi/2)$.
   (b) $\frac{x}{\sin x}$ is a strictly increasing function on $(0,\pi/2)$.
   (c) $x\le\frac{\pi}{2}\sin x$ for all $x\in[0,\pi/2]$.

4. Suppose that $|f(x)-f(y)|\le(x-y)^2$ for all $x,y\in\mathbb{R}$. Prove that f is a constant function.

5. (a) Prove that $f'>0$ on an interval $I \implies f$ is *strictly* increasing on I.
   (b) Show that the converse of part (a) is *false.*
   (c) Carefully prove the first derivative test (Corollary 3.19).

6. If f is differentiable on an interval I such that $f'(x)\neq 0$ for all $x\in I$, use the intermediate value theorem for derivatives to prove that f is either strictly increasing or strictly decreasing.

7. We prove the intermediate value theorem for derivatives. Let f,a,b and L be as in the Theorem, define $g:I\to\mathbb{R}$ by $g(x)=f(x)-Lx$, and let $\xi\in[a,b]$ be such that
$$g(\xi)=\min\{g(x):x\in[a,b]\}$$
   (a) Why can we be sure that ξ exists? If $\xi\in(a,b)$, explain why $f'(\xi)=L$.
   (b) Now assume WLOG that $f'(a)<f'(b)$. Prove that $g'(a)<0<g'(b)$. By considering $\lim\limits_{x\to a^+}\frac{g(x)-g(a)}{x-a}$, show that $\exists x>a$ for which $g(x)<g(a)$. Hence complete the proof.

8. Suppose f' exists on (a,b), and is continuous except for a discontinuity at $c\in(a,b)$.
   (a) Obtain a contradiction if $\lim\limits_{x\to c^+}f'(x)=L<f'(c)$. Hence argue that f' cannot have a *removable* or a *jump* discontinuity at $x=c$.
   (*Hint: let* $\epsilon=\frac{f'(c)-L}{2}$ *in the definition of limit then apply IVT for derivatives*)
   (b) Similarly, obtain a contradiction if $\lim\limits_{x\to c^+}f'(x)=\infty$ and conclude that f' cannot have an *infinite* discontinuity at $x=c$.
   (c) It remains to see that f' can have an essential discontinuity. Recall (Exercise 28.7) that
$$f:\mathbb{R}\to\mathbb{R}:x\mapsto\begin{cases}x^2\sin(1/x) & x\neq 0\\ 0 & x=0\end{cases}$$
   is differentiable on $\mathbb{R}$, but has discontinuous derivative at $x=0$.
      i. By considering $x_n=\frac{1}{2n\pi}$ and $y_n=\frac{1}{(2n+1)\pi}$, show that f' has an essential discontinuity at $x=0$.
      ii. Prove that if $s_n\to 0$ and $f'(s_n)$ converges to some M, then $M\in[-1,1]$.
      iii. Use IVT for derivatives to show that for any $L\in[-1,1]$, $\exists(t_n)\subseteq\mathbb{R}\setminus\{0\}$ such that $\lim\limits_{n\to\infty}f'(t_n)=L$.

30 L'Hôpital's Rule

We are often forced to consider limits known as *indeterminate forms*, which do not yield easily to the standard limits laws. For example, it is tempting to try to write

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-1}=\frac{\lim\limits_{x\to 0}\sin 2x}{\lim\limits_{x\to 0}e^{3x}-1}=\frac{0}{0} \qquad (*)$$

This is an incorrect application of the limit laws since the resulting quotient has no meaning.

Definition 3.22. An *indeterminate form* is a limit where a naïve application of the limit laws results in a meaningless expression: the primary types are $\frac{0}{0}$, $\frac{\infty}{\infty}$, $\infty-\infty$, $0\cdot\infty$, 0^0, 0^∞, and 1^∞.

Examples 3.23. 1. $\lim\limits_{x\to 7^+}(x-7)^{\frac{1}{x-7}}$ is an indeterminate form of type 0^∞.

2. The above indeterminate form $(*)$ may be evaluated using the definition of the derivative

$$\lim_{x\to 0}\frac{\sin 2x}{e^{3x}-1}=\lim_{x\to 0}\frac{\sin 2x-0}{x-0}\frac{x-0}{e^{3x}-1}=\left(\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0}\sin 2x\right)\left(\frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0}e^{3x}\right)^{-1}=\frac{2}{3}$$

By considering $\lim\limits_{x\to 0}\frac{3a\sin 2x}{2(e^{3x}-1)}$, we see that an indeterminate form of type $\frac{0}{0}$ can take *any value* a!

This approach generalizes: if $f(a)=0=g(a)$, we obtain the simplest version of l'Hôpital's rule;

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\frac{x-a}{g(x)-g(a)}=\frac{f'(a)}{g'(a)}$$

This obviously isn't rigorous. Our goal is to make it so and to extend to the following situations:

- Limits where $a=\pm\infty$.
- When the RHS cannot be cleanly evaluated: for instance $g'(a)=0$ or if the original limit is $\pm\infty$.

Covering all cases makes the proof an absolute behemoth! Because of this, and because such limits can often be evaluated more instructively using elementary methods, the rule is often discouraged in Freshman calculus. To prepare for the upcoming monster, we first generalize the MVT.

Lemma 3.24 (Extended Mean Value Theorem). *Fix $a<b$, suppose f,g are continuous on $[a,b]$ and differentiable on (a,b). Then there exists $\xi\in(a,b)$ such that*

$$\big(f(b)-f(a)\big)g'(\xi)=\big(g(b)-g(a)\big)f'(\xi)$$

Proof. Simply apply the standard mean value theorem (really Rolle's Theorem) to

$$h(t)=(f(b)-f(a))g(t)-(g(b)-g(a))f(t)$$

which satisfies $h(a)=h(b)$. ∎

Theorem 3.25 (l'Hôpital's rule). *Let $a \in \mathbb{R} \cup \{\pm\infty\}$ and suppose functions f and g satisfy:*

1. $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ *for some* $L \in \mathbb{R} \cup \{\pm\infty\}$
2. *(a)* $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, *or* *(b)* $\lim_{x \to a} g(x) = \infty$ *(no condition on f)*

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$. *The same result holds for one-sided limits.*

Examples 3.26.

1. If $f(x) = e^{4x}$ and $g(x) = 21x - 17$, then
$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} \frac{4e^{4x}}{21} = \infty \implies \lim_{x \to \infty} \frac{e^{4x}}{21x - 17} = \infty$$
This is an example of type $\frac{\infty}{\infty}$.

2. For an example of type $\frac{0}{0}$, consider $f(x) = x^2 - 9$ and $g(x) = \ln(4 - x)$:
$$\lim_{x \to 3^-} \frac{f'(x)}{g'(x)} = \lim_{x \to 3^-} \frac{2x}{-1/(4-x)} = \lim_{x \to 3^-} 2x(x-4) = -6 \implies \lim_{x \to 3^-} \frac{x^2 - 9}{\ln(4-x)} = -6$$

3. One can apply the rule repeatedly: for example
$$\lim_{x \to 0} \frac{e^{4x} - 1 - 4x}{x^2} = \lim_{x \to 0} \frac{4e^{4x} - 4}{2x} = \lim_{x \to 0} \frac{16e^{4x}}{2} = 8$$
There is an abuse of protocol here, since the existence of the first limit is dependent on the last. The approach is acceptable, though you should understand why it is an abuse. Indeed...

4. It is important that the limit $\lim \frac{f'}{g'}$ be seen to exist *before* applying l'Hôpital's rule! Consider $f(x) = x + \cos x$ and $g(x) = x$: certainly $\lim_{x \to \infty} \frac{f(x)}{g(x)}$ has type $\frac{\infty}{\infty}$, however
$$\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \lim_{x \to \infty} 1 - \sin x$$
does not exist! In this case the rule is unnecessary, since
$$\frac{f(x)}{g(x)} = 1 + \frac{\cos x}{x} \xrightarrow[x \to \infty]{} 1$$
by the squeeze theorem.

5. Finally, a short example to explain why l'Hôpital's rule is often prohibited in Freshman calculus. Consider the calculation:
$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1$$
This appears to be a legitimate application of the rule. However, recall (Exercise 28.9) that one purpose of this limit is to demonstrate that $\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x$; to use this fact to calculate the limit on which it depends is the very definition of circular logic!

Other Indeterminate Forms

The remaining indeterminate forms listed in Definition 3.22 may be modified so that l'Hôpital's rule applies. Since you've likely seen several such examples in elementary calculus, we give just a couple.

Examples 3.27. 1. An indeterminate form of type $\infty - \infty$ is transformed to one of type $\frac{0}{0}$ before applying the rule (twice):

$$\begin{aligned}\lim_{x\to 0^+} \frac{1}{e^x - 1} - \frac{1}{x} &= \lim_{x\to 0^+} \frac{x+1-e^x}{x(e^x-1)} && \text{(type } \tfrac{0}{0}\text{)}\\ &= \lim_{x\to 0^+} \frac{1-e^x}{e^x - 1 + xe^x} && \text{(still type } \tfrac{0}{0}\text{)}\\ &= \lim_{x\to 0^+} \frac{-e^x}{2e^x + xe^x} = -\frac{1}{2}\end{aligned}$$

2. For an indeterminate form of type 1^∞, we use the log laws & the continuity of the exponential:

$$\begin{aligned}\lim_{x\to 0^+} (1+\sin x)^{1/x} &= \exp\left(\lim_{x\to 0^+} \frac{1}{x}\ln(1+\sin x)\right) && \text{(type } \tfrac{0}{0}\text{)}\\ &= \exp\left(\lim_{x\to 0^+} \frac{\cos x}{1+\sin x}\right)\\ &= e^1 = e\end{aligned}$$

Proving l'Hôpital's Rule

The complete argument is very long; if you do nothing else, read the following proof of the simplest case. Everything else is a modification.

Proof (type $\frac{0}{0}$ with right limits). We prove first for *right-limits* $x \to a^+$. First observe that condition 1. forces the existence of an interval (a, b) on which f, g are differentiable and $g'(x) \neq 0$.

Assume we have a form of type $\frac{0}{0}$ (case 2. (a)) and assume additionally that a and L are finite. Everything follows from the definition of limit (condition 1.) and Lemma 3.24:

$$\text{Given } \epsilon > 0,\ \exists \delta \in (0, b-a) \text{ such that } a < \xi < a + \delta \implies \left|\frac{f'(\xi)}{g'(\xi)} - L\right| < \frac{\epsilon}{2} \qquad (*)$$

$$a < y < x < a + \delta \implies \exists \xi \in (y, x) \text{ such that } \frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(\xi)}{g'(\xi)} \qquad (\dagger)$$

Since $g' \neq 0$, the usual mean value theorem says we never divide by zero in $(\dagger)$:

$$\exists c \in (y, x) \text{ such that } g(x) - g(y) = g'(c)(x-y) \neq 0$$

Observe that $\left|\frac{f(x)-f(y)}{g(x)-g(y)} - L\right| = \left|\frac{f'(\xi)}{g'(\xi)} - L\right| < \frac{\epsilon}{2}$, let $y \to a^+$ and use 2. (a) to see that

$$\forall x \in (a, a+\delta), \quad \left|\frac{f(x)}{g(x)} - L\right| \leq \frac{\epsilon}{2} < \epsilon$$

which is the required result. ∎

We now describe some modifications.

If $a = -\infty$: Replace the blue part of $(*)$ as follows:

$$\text{Given } \epsilon > 0, \exists m \le b \text{ such that } \xi < m \implies \left| \frac{f'(\xi)}{g'(\xi)} - L \right| < \frac{\epsilon}{2}$$

The rest of the proof goes through after replacing a with $-\infty$ and $a + \delta$ with m.

If $L = \infty$: Replace the green parts of $(*)$ with Given $M > 0$ and $\frac{f'(\xi)}{g'(\xi)} > 2M$. Fixing the rest of the proof is again straightforward.

If $L = -\infty$: Replace the green parts of $(*)$ with Given $M > 0$ and $\frac{f'(\xi)}{g'(\xi)} < -2M$.

Left-limits: If f, g are differentiable on (c, a), then the blue part may be replaced with either:

- (a finite) $\exists \delta \in (0, a - c)$ such that $a - \delta < \xi < a$
- ($a = \infty$) $\exists m \ge c$ such that $\xi > m$

The blue and green parts of $(*)$ can be replaced independently. This completes the proof for all indeterminate forms of type $\frac{0}{0}$.

Proof (case 2. (b) when $\lim g(x) = \infty$*).* This requires a little more modification.[17] Since $g' \neq 0$, and $\lim_{x \to a^+} g(x) = \infty$, Exercise 29.6 says that g is *strictly decreasing* on (a, b). By replacing b by some $\tilde{b} \in (a, b)$, if necessary, we may assume that

$$a < y < x < b \implies 0 < g(x) < g(y) \tag{‡}$$

Assume a and L are finite, and obtain $(*)$ and $(\dagger)$ as before. Let $x \in (a, a + \delta)$ be fixed and multiply $(\dagger)$ by $\frac{g(y) - g(x)}{g(y)}$ (this is *positive* by (‡)): a little algebra and the triangle inequality tell us that

$$\begin{aligned} a < y < x &\implies \frac{f(y)}{g(y)} = \frac{f'(\xi)}{g'(\xi)} + \frac{f(x)}{g(y)} - \frac{g(x)}{g(y)} \cdot \frac{f'(\xi)}{g'(\xi)} \\ &\implies \left| \frac{f(y)}{g(y)} - L \right| \le \left| \frac{f'(\xi)}{g'(\xi)} - L \right| + \frac{1}{g(y)} \left(|f(x)| + |g(x)| \left(L + \frac{\epsilon}{2} \right) \right) \end{aligned}$$

Since $\lim_{y \to a^+} g(y) = \infty$ and x is fixed, we see that there exists $\eta \le x - a < \delta$ such that

$$y \in (a, a + \eta) \implies \frac{1}{g(y)} \left(|f(x)| + |g(x)| \left(L + \frac{\epsilon}{2} \right) \right) < \frac{\epsilon}{2}$$

Finally combine with $(*)$: given $\epsilon > 0, \exists \eta > 0$ such that $y \in (a, a + \eta) \implies \left| \frac{f(y)}{g(y)} - L \right| < \epsilon$.
The same modifications listed previously complete the proof. ∎

[17] *Forms of type* $\frac{\infty}{\infty}$*?* Instead of assumption 2. (b), why not simply assume $\lim f = \lim g = \infty$ and write $\frac{f}{g} = \frac{1/g}{1/f}$ to obtain a form of type $\frac{0}{0}$? The problem is that the derivative of the 'new' denominator $\frac{d}{dx} \frac{1}{f} = \frac{-f'}{f^2}$ need not be non-zero on any interval (a, b) and so condition 1. need not hold. We could modify this, but it would make for a weaker theorem. Example 3.26.4 illustrates this: $f'(x) = 1 + \sin x$ has zeros on any unbounded interval.

After the 2. (b) case is proved and we know that $\lim \frac{f}{g} = L$, it is then clear that $\lim f$ must also be infinite (unless $L = 0$ in which case $\lim f$ could be anything and need not exist). This situation therefore really does deal with forms of type $\frac{\infty}{\infty}$.

Exercises 30 1. Evaluate the following limits, if they exist:

(a) $\lim_{x\to 0} \dfrac{x^3}{\sin x - x}$ (b) $\lim_{x\to \frac{\pi}{2}^-} \tan x - \dfrac{2}{\pi - 2x}$

(c) $\lim_{x\to 0} (\cos x)^{1/x^2}$ (d) $\lim_{x\to 0} (1+2x)^{1/x}$

(e) $\lim_{x\to\infty} (e^x + x)^{1/x}$

2. Let f be differentiable on (c,∞) and suppose that $\lim_{x\to\infty}[f(x)+f'(x)] = L$ is finite.

   (a) Prove that $\lim_{x\to\infty} f(x) = L$ and that $\lim_{x\to\infty} f'(x) = 0$.
   (*Hint: write* $f(x) = \frac{f(x)e^x}{e^x}$)

   (b) Does anything change if L exists and is *infinite*?

3. If $p_n(x)$ is a polynomial of degree n, use induction to prove that $\lim_{x\to\infty} p_n(x)e^{-x} = 0$

4. Let $f(x) = x + \sin x \cos x$, $g(x) = e^{\sin x} f(x)$ and $h(x) = \dfrac{2\cos x}{e^{\sin x}(f(x) + 2\cos x)}$

   (a) Prove that $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$ but that $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

   (b) If $\cos x \neq 0$, and x is large, show that $\frac{f'(x)}{g'(x)} = h(x)$.

   (c) Prove that $\lim_{x\to\infty} h(x) = 0$. Explain why this does not contradict part (a)!

31 Taylor's Theorem

A primary goal of power series is the approximation of functions. As such, there are two natural questions to ask of a given function f:

1. Given $c \in \text{dom}(f)$, is there a series $\sum a_n(x-c)^n$ which equals $f(x)$ on an interval containing c?
2. If we take the first n terms of such a series, how accurate is this polynomial approximation?

Example 3.28. Recall the geometric series

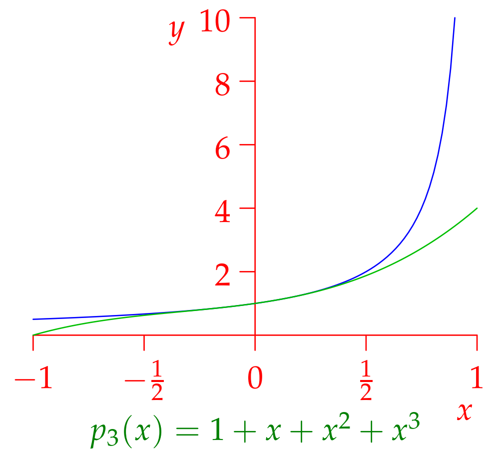
$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ whenever } -1 < x < 1$$

The polynomial approximation

$$p_n(x) = \sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{1-x^{n+1}}{1-x}$$

has error

$$R_n(x) = f(x) - p_n(x) = \frac{x^{n+1}}{1-x}$$



If x is close to 0, this is likely very small; for instance if $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, then

$$|R_n(x)| \le \frac{1}{1-\frac{1}{2}} \left(\frac{1}{2}\right)^{n+1} = 2^{-n}$$

However, when x is close to 1, the error is unbounded!

The behavior in the Example occurs in general: the truncated polynomial approximations are better near the center of the series. To see this, we first need to consider higher-order derivatives.

Definition 3.29. We write f'' for the *second derivative* of f, namely the derivative of its derivative

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a}$$

The existence of $f''(a)$ presupposes that f' exists on an (open) interval containing a. We can similarly consider third, fourth, and higher-order derivatives. As a function, the n^{th} derivative is written

$$f^{(n)}(x) = \frac{\mathrm{d}^n f}{\mathrm{d}x^n}$$

By convention, the *zeroth derivative* is the function itself $f^{(0)}(x) = f(x)$. We say that f is *n times differentiable* at a if $f^{(n)}(a)$ exists, and *infinitely differentiable* (or *smooth*) if derivatives of all orders exist.

Example 3.30. $f(x) = x^2\,|x|$ is twice differentiable, with $f''(x) = 6\,|x|$. It is smooth everywhere except at $x = 0$, where third (and higher-order) derivatives do not exist.

Definition 3.31. Suppose f is n times differentiable at $x = c$. The n^{th} *Taylor polynomial* p_n of f centered at c is

$$p_n(x) := \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

The *remainder* $R_n(x)$ is the error in the polynomial approximation

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{j=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

If f is infinitely differentiable at $x = c$, then its *Taylor series* centered at $x = c$ is the power series

$$\mathrm{T}f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

When $c = 0$ this is known as a *Maclaurin series*.[18]

For simplicity we'll most often work with Maclaurin series, with general cases hopefully being clear.

Examples 3.32. 1. If $f(x) = e^{3x}$, then $f^{(n)}(x) = 3^n e^x$, from which the Maclaurin series is

$$\mathrm{T}f(x) = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$$

2. If $g(x) = \sin 7x$, then the sequence of derivatives is

$$7\cos 7x, \quad -7^2 \sin 7x, \quad -7^3 \cos 7x, \quad 7^4 \sin 7x, \quad 7^5 \cos 7x, \quad -7^6 \sin 7x, \ldots$$

At $x = 0$, every even derivative is zero, while the odd derivatives alternate in sign; the Maclaurin series is easily seen to be

$$\mathrm{T}g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 7^{2n+1}}{(2n+1)!}x^{2n+1}$$

3. If $h(x) = \sqrt{x}$, then $h'(x) = \frac{1}{2}x^{-1/2}$, $h''(x) = \frac{-1}{2^2}x^{-3/2}$, and $h'''(x) = \frac{3}{2^3}x^{-5/2}$, from which the third Taylor polynomial centered at $c = 1$ is

$$\begin{aligned} p_2(x) &= h(1) + h'(1)(x-1) + \frac{h''(1)}{2}(x-1)^2 + \frac{h'''(1)}{6}(x-1)^3 \\ &= 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 \end{aligned}$$

Rather than compute more examples, we develop a little theory that makes verifying Taylor series much easier.

[18]Named for Englishman Brook Taylor (1685–1731) and Scotsman Colin Maclaurin (1698–1746). Taylor's general method expanded on examples discovered by James Gregory and Issac Newton in the mid-to-late 1600's.

Differentiation of Taylor Polynomials and Series

Suppose $P(x) = \sum a_j x^j$ is a power series with radius of convergence $R > 0$. As we discovered previously, this is differentiable term-by-term on $(-R, R)$. Indeed

$$\begin{aligned}
P'(x) &= \sum_{j=1}^{\infty} a_j j x^{j-1} \implies P'(0) = a_1 \\
P''(x) &= \sum_{j=2}^{\infty} a_j j(j-1) x^{j-2} \implies P''(0) = 2a_2 \\
P'''(x) &= \sum_{j=3}^{\infty} a_j j(j-1)(j-2) x^{j-3} \implies P'''(0) = 3!a_3 \\
&\vdots \\
P^{(k)}(x) &= \sum_{j=k}^{\infty} a_j j(j-1)\cdots(j-k+1) x^{j-k} = \sum_{j=k}^{\infty} \frac{j!a_j}{(j-k)!} x^{j-k} \implies P^{(k)}(0) = k!a_k
\end{aligned}$$

Otherwise said, P is its own Maclaurin series! The same discussion holds for polynomials: indeed if $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is a polynomial, then for all $k \le n$,

$$P^{(k)}(0) = f^{(k)}(0) \iff a_k = \frac{f^{(k)}(0)}{k!}$$

If this holds for *all* $k \le n$, then P must be the Taylor polynomial of f! With a little modification, we've proved the following:

Theorem 3.33. 1. *If $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ on a neighborhood of c, then $\sum_{n=0}^{\infty} a_n (x-c)^n$ is the Taylor series of f.*

2. *The n^{th} Taylor polynomial of f centered at $x = c$ is the unique polynomial p_n of degree $\le n$ whose value and first n derivatives agree with those of f at $x = c$: that is*

$$\forall k \le n, \ \ p_n^{(k)}(c) = f^{(k)}(c)$$

This answers our first motivating question: a function can equal at most one power series with a given center. The second question requires a careful study of the *remainder*: we'll do this shortly.

Examples 3.34 (Common Maclaurin Series). These should be familiar from elementary calculus. Each of these functions equals the given series by our previous discussion of power series: by the Theorem, each series is therefore the Maclaurin series of the given function with no requirement to calculate directly!

$$\begin{aligned}
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && x \in \mathbb{R} && \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n && x \in (-1,1) \\
\sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} && x \in \mathbb{R} && \ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n && x \in (-1,1] \\
\cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} && x \in \mathbb{R} && \tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} && x \in [-1,1]
\end{aligned}$$

Examples 3.35 (Modifying Maclaurin Series). By substituting for x in a common series, we quickly obtain new ones.

1. Substitute $x \mapsto 7x$ in the Maclaurin series for $\sin x$, to recover our earlier example

$$\sin 7x = \sum_{n=0}^{\infty} \frac{(-1)^n 7^{2n+1}}{(2n+1)!} x^{2n+1}, \quad x \in \mathbb{R}$$

   Note how this requires almost no calculation: since the function equals a series, the Theorem says we have the Maclaurin series for $\sin 7x$!

2. Substitute $x \mapsto x^2$ in the Maclaurin series for e^x to obtain

$$e^{x^2} = \exp(x^2) = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}, \quad x \in \mathbb{R}$$

   This would be disgusting to verify directly, given the difficulty of repeatedly differentiating e^{x^2}.

3. We find the Taylor series for $f(x) = \frac{1}{5-x}$ centered at $x = 2$:

$$f(x) = \frac{1}{3+2-x} = \frac{1}{3(1-\frac{2-x}{3})} = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2-x}{3}\right)^n$$

   which is valid whenever $-1 < \frac{2-x}{3} < 1 \iff -1 < x < 5$.

4. Fix $c \in \mathbb{R}$ and observe that, for all $x \in \mathbb{R}$,

$$e^x = e^{c+x-c} = e^c e^{x-c} = \sum_{n=0}^{\infty} \frac{e^c}{n!}(x-c)^n$$

   We conclude that the series is the Taylor series of e^x centered at $x = c$. Of course this is easily verified using the definition, since $\frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big|_{x=c} e^x = e^c$.

5. Combining the Theorem with the multiple-angle formula, we obtain the Taylor series for $\sin x$ centered at $x = c$:

$$\begin{aligned}\sin x &= \sin(c+x-c) = \sin c\cos(x-c) + \cos x \sin(x-c)\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n \sin c}{(2n)!}(x-c)^{2n} + \sum_{n=0}^{\infty}\frac{(-1)^n \cos c}{(2n+1)!}(x-c)^{2n+1}\end{aligned}$$

Definition 3.36. A function is *analytic* on a domain if for each c there exists a neighborhood of c on which the function equals its Taylor series centered at c.

All the examples we've so far seen are analytic on their domains; indeed the last two of Examples 3.35 *prove* this for the exponential and sine functions. Every analytic function is automatically smooth (infinitely differentiable), however the converse is *false* in that not every smooth function is analytic (see Exercise 10). Analyticity is of greater importance in complex analysis where it is seen to be equivalent to complex-differentiability.

Accuracy of Taylor Approximations

Our final goal is to estimate the accuracy of a Taylor polynomial as an approximation to its generating function. Otherwise said, we want to estimate the size of the remainder $R_n(x) = f(x) - p_n(x)$.

Theorem 3.37 (Taylor's Theorem: Lagrange's form). *Suppose f is $n+1$ times differentiable on an open interval I containing c and let $x \in I \setminus \{c\}$. Then there exists some ξ between c and x for which the remainder centered at c satisfies*

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

Proof. For simplicity let $c = 0$. Fix $x \neq 0$, define a constant M_x and a function $g : I \to \mathbb{R}$ by

$$R_n(x) = \frac{M_x}{(n+1)!}x^{n+1} \quad \text{and} \quad g(t) = \frac{M_x}{(n+1)!}t^{n+1} + p_n(t) - f(t) = \frac{M_x}{(n+1)!}t^{n+1} - R_n(t)$$

Observe that

$$\begin{aligned} k \leq n+1 \implies g^{(k)}(x) &= \frac{M_x}{(n+1-k)!}t^{n+1-k} + p_n^{(k)}(t) - f^{(k)}(t) \qquad (*) \\ \implies g^{(k)}(0) &= p_n^{(k)}(0) - f^{(k)}(0) = 0 \quad \text{if } k \leq n \end{aligned}$$

where we invoked Theorem 3.33.
Apply Rolle's Theorem repeatedly (WLOG assume $x > 0$):

- $\exists \xi_1$ between 0 and x such that $g'(\xi_1) = 0$.
- $\exists \xi_2$ between 0 and ξ_1 such that $g''(\xi_2) = 0$, etc.
- Iterate to obtain a sequence (ξ_k) such that

$$0 < \xi_{n+1} < \xi_n < \cdots < \xi_1 < x \quad \text{and} \quad g^{(k)}(\xi_k) = 0$$

Take $\xi = \xi_{n+1}$ and consider $(*)$: since $\deg p_n \leq n$, we see that

$$0 = g^{(n+1)}(\xi) = M_x - f^{(n+1)}(\xi) \implies R_n(x) = f(x) - p_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$$

■

Corollary 3.38. *Suppose f is smooth on an open interval I containing c and that all derivatives $f^{(n)}$ of all orders are bounded on I. Then f equals its Taylor series (centered at c) on I.*

Proof. For simplicity, let $c = 0$. Suppose $\left|f^{(n+1)}(\xi)\right| \leq K$ for all $\xi \in I$. Choose any $N > |x|$ and observe that

$$n > N \implies |R_n(x)| \leq \frac{K|x|^{n+1}}{(n+1)!} = \frac{K|x|^{n+1}}{N!(N+1)\cdots(n+1)} \leq \frac{K|x|^N}{N!}\left(\frac{|x|}{N}\right)^{n+1-N} \xrightarrow[n\to\infty]{} 0$$

■

Examples 3.39. 1. The functions sine and cosine have derivatives bounded by 1 on $\mathbb{R}$, and thus both functions equal their Maclaurin series on $\mathbb{R}$. This removes the need to have previously justified these facts using the theory of differential equations.

2. The exponential function does not have bounded derivatives, however we can still apply Taylor's Theorem. For any fixed x, $\exists\xi$ between 0 and x such that

$$|R_n(x)| = \left|\frac{e^\xi}{(n+1)!}x^{n+1}\right| \xrightarrow[n\to\infty]{} 0$$

by the same argument in the Corollary. Thus e^x equals its Maclaurin series on the real line.

3. Extending Example 3.32.3, we see that the function $h(x) = \sqrt{x}$ has the following linear approximation (1st Taylor polynomial) centered at $c = 9$

$$p_1(x) = h(9) + h'(9)(x-9) = 3 + \frac{1}{6}(x-9)$$

This yields the simple approximation

$$\sqrt{10} \approx p_1(10) = 3 + \frac{1}{6} = \frac{19}{6}$$

Taylor's Theorem can be used to estimate its accuracy (remember to shift the center to 9!):

$$R_1(10) = \frac{h''(\xi)}{2!}(10-9)^2 = -\frac{1}{2^2\cdot 2!}\xi^{-3/2} = -\frac{1}{8\xi^{3/2}} \quad \text{for some} \quad \xi \in (9,10)$$

Certainly $\xi^{-3/2} < 9^{-3/2} = \frac{1}{27}$, whence

$$-\frac{1}{216} < R_1(10) < 0 \implies \frac{19}{6} - \frac{1}{216} = \frac{683}{216} < \sqrt{10} < \frac{684}{216} = \frac{19}{6}$$

$\frac{19}{6}$ is therefore an overestimate for $\sqrt{10}$, but is accurate to within $\frac{1}{216} < 0.005$.

Alternative Versions of Taylor's Theorem

The two other common expressions for the remainder are typically less easy to use than Lagrange's form, but can sometimes provide sharper estimates for the remainder, particularly when x is far from the center of the series.

Corollary 3.40. *Suppose $f^{(n+1)}$ is continuous on an open interval I containing c, let $x \in I \setminus \{c\}$, and let $R_n(x) = f(x) - p_n(x)$ be the remainder for the Taylor polynomial centered at c. Then:*

1. *(Integral Remainder)* $R_n(x) = \displaystyle\int_c^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t$

2. *(Cauchy's Form)* $\exists\xi$ *between c and x such that* $R_n(x) = \dfrac{(x-\xi)^n}{n!}(x-c)f^{(n+1)}(\xi)$

Using these expressions it is possible to explicitly prove Newton's binomial series formula:

Theorem 3.41. *If $\alpha \in \mathbb{R}$ and $|x| < 1$, then*

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^n$$
$$= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \frac{\alpha(\alpha-1)(\alpha-2)(\alpha-3)}{4!}x^4 + \cdots$$

If $\alpha \in \mathbb{N}_0$, this is the usual binomial theorem. Otherwise it is more interesting, for instance,

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$$
$$\frac{1}{(1+x)^3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots$$

Of course this last could easily be obtained from $\frac{1}{1+x} = \sum(-1)^n x^n$ by differentiating twice!

Exercises 31 1. Compute the Maclaurin series for $\cos x$ directly from the definition and use Taylor's Theorem to indicate why it converges to $\cos x$ for all $x \in \mathbb{R}$.

2. Repeat the previous exercise for $\sinh x = \frac{1}{2}(e^x - e^{-x})$ and $\cosh x = \frac{1}{2}(e^x + e^{-x})$.

3. Find the Maclaurin series for the function $\sin(3x^2)$. How do you know you are correct?

4. Find the Taylor series of $f(x) = x^4 - 3x^2 + 2x - 5$ centered at $x = 2$ and show that $\mathrm{T}f(x) = f(x)$.

5. Find a rational approximation to $\sqrt[3]{9}$ using the first Taylor polynomial for $f(x) = \sqrt[3]{x}$. Now use Taylor's Theorem to estimate its accuracy.

6. If $c \neq 1$, use the fact that $1 - x = (1-c)\left(1 - \frac{x-c}{1-c}\right)$ to obtain the Taylor series of $\frac{1}{1-x}$ centered at c. Hence conclude that $\frac{1}{1-x}$ is analytic on its domain $\mathbb{R} \setminus \{1\}$.

7. We use Taylor's Theorem to prove that the Maclaurin series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$ converges to $\ln(1+x)$ whenever $0 < x \leq 1$.

   (a) Explicitly compute $\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} \ln(1+x)$.

   (b) Suppose $0 < x \leq 1$. Using Taylor's Theorem, prove that $\lim_{n\to\infty} R_n(x) = 0$.

   (*If* $-1 < x < 0$, *the argument is tougher, being similar to Exercise 11*)

8. Why can't we use Taylor's Theorem to approximate the error in $\frac{1}{1-x} = 1 + x + R_1(x)$ when $x \geq 1$? Try it when $x = 2$, what happens? What about when $x = -2$?

9. Prove Taylor's Theorem with integral remainder when $c = 0$ by using the following as an induction step: for each $n \in \mathbb{N}$, define

$$A_n(x) = \int_0^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,\mathrm{d}t$$

   and use integration by parts to prove that $A_{n+1} = A_n - \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(0)$.

   (*The Cauchy form follows from the intermediate value theorem for integrals which we'll see later*)

10. Consider the function

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Prove by induction that there exists a degree $2n$ polynomial q_n for which

$$f^{(n)}(x) = q_n\left(\tfrac{1}{x}\right) e^{-1/x} \text{ whenever } x > 0$$

(b) Prove that f is infinitely differentiable at $x = 0$ with $f^{(n)}(0) = 0$ (*use Exercise 30.3*).

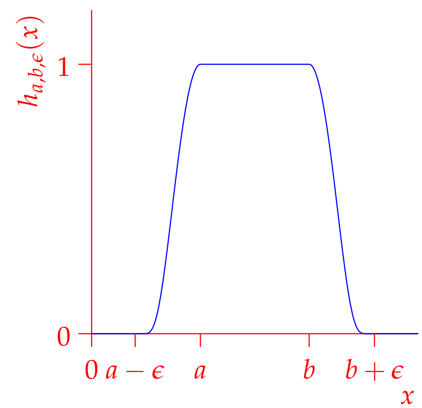
The Maclaurin series of f is identically zero! Moreover, f is smooth (infinitely differentiable) on $\mathbb{R}$ but non-analytic at zero since it does not equal its Taylor series on any open interval containing zero.
A modification allows us to create bump functions, *which find wide use in analysis. If $a < b$, define*

$$g_{a,b} : x \mapsto f(x-a)f(b-x)$$

This is smooth on $\mathbb{R}$ but non-zero only on the interval (a, b). A further modification involving two such functions $g_{a,b}$ creates a smooth function on $\mathbb{R}$ which satisfies

$$h_{a,b,\epsilon}(x) = \begin{cases} 0 & \text{if } x \le a-\epsilon \text{ or } x \ge b+\epsilon \\ 1 & \text{if } a \le x \le b \end{cases}$$

This 'switches on' rapidly from 0 to 1 near a and switches off similarly near b. By letting ϵ be small, we smoothly (but not uniformly) approximate the indicator function on $[a, b]$.



11. (Hard) We prove the binomial series formula. Let $f(x) = (1+x)^\alpha$ and $g(x) = 1 + \sum_{n=1}^{\infty} a_n x^n$ where $a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$. Our goal is to prove that $f = g$ on the interval $(-1, 1)$.

(a) Check that $f^{(n)}(0) = n!a_n$ so that g really is the Maclaurin series of f.

(b) i. Prove that the radius of convergence of g is 1.

ii. Prove that $\lim_{n\to\infty} na_n x^n = 0$ whenever $|x| < 1$.

iii. If $|x| < 1$ and ξ lies between 0 and x, prove that $\left|\frac{x-\xi}{1+\xi}\right| \le |x|$.
(*Hint: $\xi = tx$ for some $t \in (0,1)\ldots$*)

(c) Use Taylor's Theorem with Cauchy remainder to prove that

$$|R_n(x)| < (n+1)\,|a_{n+1}|\,|x|^{n+1}\,(1+\xi)^{\alpha-1}$$

Hence conclude that $g = f$ whenever $|x| < 1$.

(d) Here is an alternative argument:

i. Show that $(n+1)a_{n+1} + na_n = \alpha a_n$.

ii. Differentiate term-by-term to prove directly that g satisfies the differential equation $(1+x)g'(x) = \alpha g(x)$. Solve this to show that $g = f$ whenever $|x| < 1$.

4 Integration

The theory of infinite series addresses the problem of summing infinitely many *finite* quantities. By contrast, integration is the business of summing infinitely many *infinitesimal* quantities. Mathematicians have attempted to do both for well over 2000 years, and the philosophical objections are just as old.[19] The development and increased application of calculus from the late 1600s spurred mathematicians to try to put the theory on a firmer footing, though from Newton and Leibniz it took another 150 years before Bernhard Riemann (1856) provided a thorough development of the integral.

32 The Riemann Integral

The basic idea behind Riemann integration is to approximate area using a sequence of rectangles whose width tends to zero. The following discussion is hopefully familiar.

Example 4.1. Consider $f(x) = x^2$ defined on $[0, 1]$.
For each $n \in \mathbb{N}$, let $\Delta x = \frac{1}{n}$ and define $x_i = i\Delta x$.
Above each *subinterval* $[x_{i-1}, x_i]$, raise a rectangle of height $f(x_i) = x_i^2$.
The sum of the areas of these rectangles is the *Riemann sum with right-endpoints*[20]

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x = \sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{3} + \frac{3n+1}{6n^2}$$

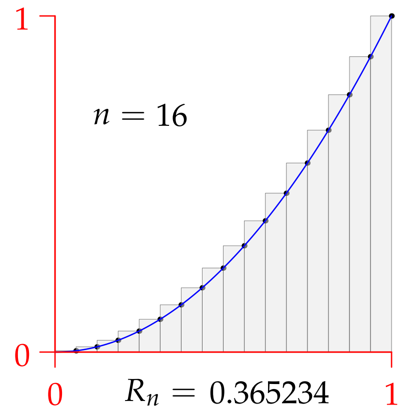
The *Riemann sum with left-endpoints* is defined similarly:

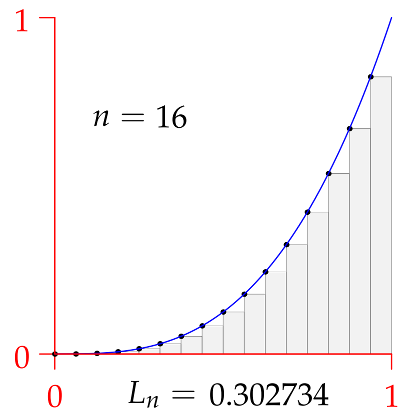
$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x = \sum_{i=1}^{n} \frac{(i-1)^2}{n^3} = \frac{1}{3} - \frac{3n-1}{6n^2}$$

Since f is increasing, the area A under the curve satisfies

$$L_n \le A \le R_n$$

and the squeeze theorem allows us to conclude that $A = \frac{1}{3}$.





The example contains the essential idea, but more flexibility is needed. To get further, we must properly define the concepts of partition and Riemann sum.

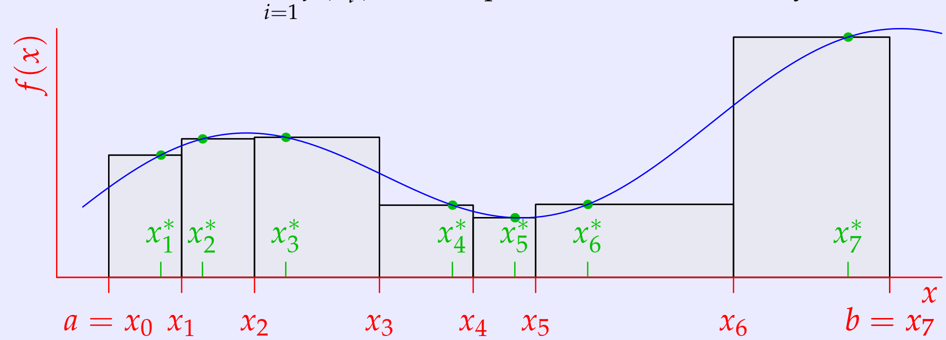
[19] Two of Zeno's ancient paradoxes are relevant here: Achilles and the Tortoise concerns a convergent infinite series, while the Arrow Paradox discusses a difficulty with integration by questioning whether time can be considered as a sum of instants. Perhaps the most famous contemporary criticism comes from Bishop George Berkeley, who gave his name to the Californian city and thus the first UC campus: in *The Analyst* (1734), Berkeley savaged the foundations of calculus, describing the infinitesimal increments required in Newton's theory of *fluxions* (derivatives) as merely the "ghosts of departed quantities."

[20] Now is a good time to review some identities: $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$, $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$, $\sum_{i=1}^{n} i^3 = \frac{1}{4}n^2(n+1)^2$

Definition 4.2. A *partition* $P = \{x_0, \ldots, x_n\}$ of an interval $[a, b]$ is a finite sequence such that

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

For each $1 \leq i \leq n$, define $\Delta x_i = x_i - x_{i-1}$. The *mesh* of the partition is $\text{mesh}(P) := \max \Delta x_i$.
Choose a *sample point* x_i^* in each *subinterval* $[x_{i-1}, x_i]$.
If $f : [a, b] \to \mathbb{R}$, the *Riemann sum* $\sum_{i=1}^{n} f(x_i^*) \Delta x_i$ computes the area of a family of n rectangles.



In elementary calculus, one typically computes Riemann sums for *equally-spaced* partitions with *left*, *right* or *middle* sample points. The double freedom of partition & sample points makes applying the definition a challenge, so instead we consider two special families of rectangles.

Definition 4.3. Given a partition P of $[a, b]$ and a bounded function f on $[a, b]$, define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \qquad U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i$$

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) \qquad L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i$$

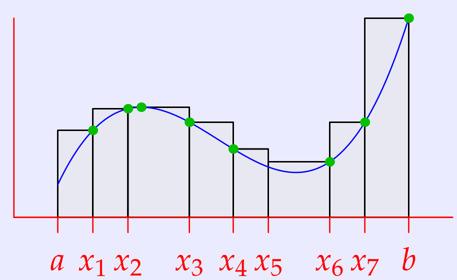
$U(f, P)$ and $L(f, P)$ are the *upper* and *lower Darboux sums* for f with respect to P. The *upper* and *lower Darboux integrals* are

$$U(f) = \inf U(f, P) \qquad L(f) = \sup L(f, P)$$

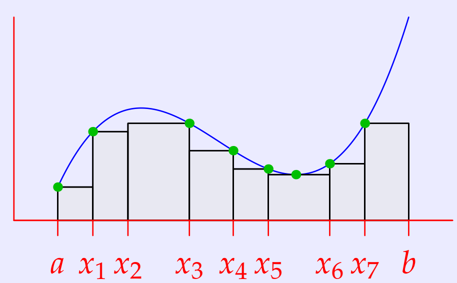
where the supremum and infimum are over all partitions.
We say that f is *(Riemann) integrable* on $[a, b]$ if $U(f) = L(f)$ and denote this value by

$$\int_a^b f \quad \text{or} \quad \int_a^b f(x) \, dx$$

If the interval is understood or irrelevant, it is common just to say that f is integrable and write $\int f$.



Upper Darboux sum $U(f, P)$



Lower Darboux sum $L(f, P)$

Intuitively, $L(f, P)$ is the sum of the areas of rectangles built on P which just fit under the graph of f. It is also the infimum of all Riemann sums on P. If f is discontinuous, then $L(f, P)$ need not be a Riemann sum; there might not be suitable sample points!

Examples 4.4. 1. We revisit Example 4.1 in this language.

Given a partition $Q = \{x_0, \ldots, x_n\}$ of $[0,1]$ and sample points $x_i^* \in [x_{i-1}, x_i]$, we compute the Riemann sum for $f(x) = x^2$

$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x_i = \sum_{i=1}^{n} (x_i^*)^2 (x_i - x_{i-1})$$

Since f is increasing, we have $x_{i-1}^2 \le (x_i^*)^2 \le x_i^2$ on each interval, whence

$$L(f,Q) = \sum_{i=1}^{n} (x_{i-1})^2 (x_i - x_{i-1}) \le \sum_{i=1}^{n} (x_i^*)^2 (x_i - x_{i-1}) \le \sum_{i=1}^{n} (x_i)^2 (x_i - x_{i-1}) = U(f,Q)$$

The Darboux sums are therefore the Riemann sums for right and left endpoints.

If we take Q_n to be the partition with subintervals of equal width $\Delta x = \frac{1}{n}$, then

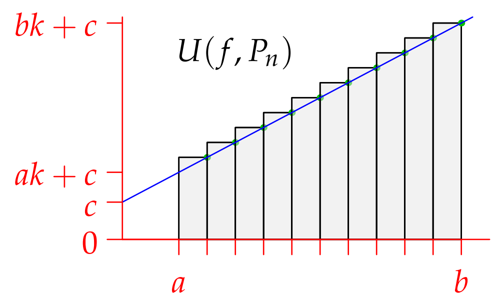
$$U(f) = \inf_P U(f,P) \le U(f,Q_n) = \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \Delta x = R_n$$

is the right Riemann sum discussed originally. Similarly $L(f) \ge L_n$. Since L_n and R_n both converge to $\frac{1}{3}$ as $n \to \infty$, the squeeze theorem forces

$$L_n \le L(f) \le U(f) \le R_n \implies L(f) = U(f) = \frac{1}{3}$$

whence f is Riemann integrable on $[0,1]$ with $\int_0^1 x^2\,dx = \frac{1}{3}$.

2. Suppose $f(x) = kx + c$ on the interval $[a,b]$ where $k > 0$. Take the evenly spaced partition P_n where $x_i = a + \frac{b-a}{n} i$ with $\Delta x_i = \frac{b-a}{n}$. Since f is increasing, the upper Darboux sum is again the Riemann sum with right-endpoints:



$$\begin{aligned} U(f,P_n) = R_n &= \sum_{i=1}^{n} f(x_i)\Delta x \\ &= \frac{b-a}{n} \sum_{i=1}^{n} \frac{k(b-a)}{n} i + ak + c \\ &= \frac{b-a}{n}\left[\frac{k(b-a)}{n} \cdot \frac{1}{2} n(n+1) + (ak+c)n\right] \\ &\xrightarrow[n\to\infty]{} \frac{1}{2}k(b-a)^2 + (b-a)(ak+c) = \frac{k}{2}(b^2-a^2) + c(b-a) \end{aligned}$$

We similarly see that the lower Darboux sum is given by the Riemann sum with left endpoints, and that

$$L(f,P_n) = L_n = \frac{b-a}{n}\left[\frac{k(b-a)}{n} \cdot \frac{1}{2} n(n-1) + (ak+c)n\right] \xrightarrow[n\to\infty]{} \frac{k}{2}(b^2-a^2) + c(b-a)$$

By the same argument as above, $L_n \le L(f) \le U(f) \le R_n$ and the squeeze theorem show that f is integrable with $\int_a^b f = \frac{k}{2}(b^2 - a^2) + c(b-a)$.

Following the examples, a few remarks are in order.

Riemann versus Darboux Definition 4.3 is really that of the *Darboux integral.* Riemann's definition is as follows: for $f[a,b] \to \mathbb{R}$ to be integrable with integral $\int_a^b f$ means

$$\forall \epsilon > 0,\ \exists \delta \text{ such that } \forall P, x_i^*,\ \text{mesh}(P) < \delta \implies \left| \sum_{i=1}^{n} f(x_i^*)\Delta x_i - \int_a^b f \right| < \epsilon$$

It can be shown that this is equivalent to the Darboux integral. We won't pursue Riemann's formulation further, except to observe that *if* a function is integrable and $\text{mesh}(P_n) \to 0$, then $\int_a^b f = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$: this allows us to approximate integrals using any sample points we choose, hence why *right* endpoints ($x_i^* = x_i$) are so common in Freshman calculus.

Monotone Functions Darboux sums are particularly easy to compute for monotone functions. As in the examples, if f is increasing, then each $M_i = f(x_i)$, from which $U(f,P)$ is the Riemann sum with *right-endpoints.* Similarly, $L(f,P)$ is the Riemann sum with *left-endpoints.* The roles reverse if f is decreasing.

Area If f is positive and continuous,[21] the Riemann integral $\int_a^b f$ serves as a *definition* for the area under the curve $y = f(x)$. This should make intuitive sense:

1. In the second example where we have a straight line, we obtain the same value for the area by computing directly as the sum of a rectangle and a triangle!
2. If the area under the curve is to make sense, then, for any partition P, it plainly satisfies the inequalities

   $$L(f,P) \le \text{Area} \le U(f,P)$$

   But these are exactly the same as those satisfied by the integral itself:

   $$L(f,P) \le L(f) = \int_a^b f = U(f) \le U(f,P)$$

In the examples we exhibited a sequence of partitions (P_n) where $U(f,P_n)$ and $L(f,P_n)$ both converged to the same limit. The next results develop some basic properties of partitions and make this process rigorous.

Lemma 4.5. *Suppose $f : [a,b] \to \mathbb{R}$ is bounded and suppose P, Q are partitions of $[a,b]$.*

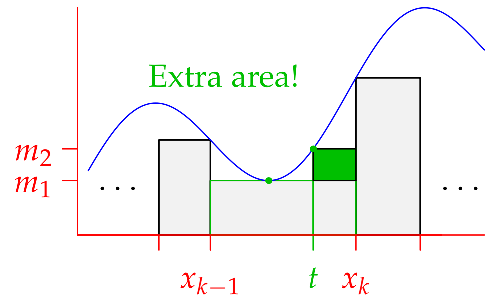
1. *If Q is a* refinement *of P, that is $P \subseteq Q$, then*

   $$L(f,P) \le L(f,Q) \le U(f,Q) \le U(f,P)$$

2. *For any partitions P, Q, we have $L(f,P) \le U(f,Q)$*
3. *$L(f) \le U(f)$*

[21] We'll see later (Theorem 4.16) that every continuous function is integrable.

Proof. 1. We prove inductively. First suppose that $Q = P \cup \{t\}$ contains exactly one additional point $t \in (x_{k-1}, x_k)$. Write



$$\begin{aligned} m_1 &= \inf\{f(x) : x \in [x_{k-1}, t]\} \\ m_2 &= \inf\{f(x) : x \in [t, x_{k-1}]\} \\ m &= \inf\{f(x) : x \in [x_{k-1}, x_k]\} = \min\{m_1, m_2\} \end{aligned}$$

The Darboux sums $L(f, P)$ and $L(f, Q)$ are identical except for the terms involving t; indeed

$$\begin{aligned} L(f, Q) - L(f, P) &= m_1(t - x_{k-1}) + m_2(x_k - t) - m(x_k - x_{k-1}) \\ &= (m_1 - m)(t - x_{k-1}) + (m_2 - m)(x_k - t) \geq 0 \end{aligned}$$

Since partitions are finite sets, by induction we see that $P \subseteq Q \implies L(f, P) \leq L(f, Q)$.

The argument for $U(f, Q) \leq U(f, P)$ is similar, and the middle inequality is trivial.

2. If P and Q are partitions, then $P \cup Q$ is a refinement of both P and Q. By part 1,

$$L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q) \tag{$*$}$$

3. We leave this as an exercise.

■

Theorem 4.6 (Cauchy criterion for integrability). *Suppose $f : [a, b] \to \mathbb{R}$ is bounded.*

1. *f is integrable $\iff$ $\forall \epsilon > 0, \exists P$ such that $U(f, P) - L(f, P) < \epsilon$*
2. *f is integrable $\iff$ $\exists (P_n)_{n \in \mathbb{N}}$ such that $U(f, P_n) - L(f, P_n) \to 0$. Moreover, in such a case both sequences $U(f, P_n)$ and $L(f, P_n)$ converge to $\int_a^b f$.*

We call this a Cauchy criterion since integrability is demonstrated without mention of the integral!

Proof. 1. $(\Rightarrow)$ Suppose f is integrable. Since $\inf U(f, Q) = \int f = \sup L(f, R)$, $\exists Q, R$ such that

$$U(f, Q) - \int f < \frac{\epsilon}{2} \quad \text{and} \quad \int f - L(f, R) < \frac{\epsilon}{2}$$

Let $P = Q \cup R$ and apply $(*)$: $L(f, R) \leq L(f, P) \leq U(f, P) \leq U(f, Q)$. But then

$$U(f, P) - L(f, P) \leq U(f, Q) - L(f, R) = U(f, Q) - \int f + \int f - L(f, R) < \epsilon$$

$(\Leftarrow)$ For every partition, $L(f, P) \leq L(f) \leq U(f) \leq U(f, P)$. Thus

$$0 \leq U(f) - L(f) \leq U(f, P) - L(f, P) < \epsilon$$

Since this holds for all $\epsilon > 0$, we see that $U(f) = L(f)$.

2. This is an exercise.

■

Examples 4.7. 1. The freedom to choose a partition can be very useful. Consider $f(x) = \sqrt{x}$ on the interval $[0, b]$. We choose a partition that evaluates nicely when fed to this function:

$$P_n = \{x_0, \ldots, x_n\} \quad \text{where} \quad x_i = \left(\frac{i}{n}\right)^2 b$$
$$\implies \Delta x_i = x_i - x_{i-1} = \frac{b}{n^2}\left(i^2 - (i-1)^2\right) = \frac{(2i-1)b}{n^2}$$

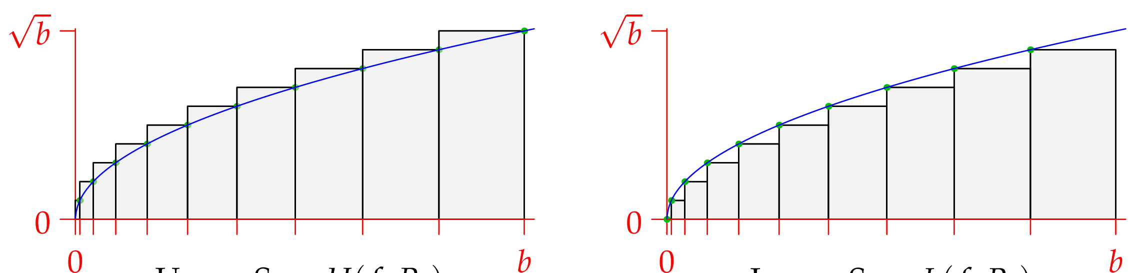
Since f is increasing on $[0, b]$, we see that

$$U(f, P_n) = \sum_{i=1}^{n} f(x_i)\Delta x_i = \sum_{i=1}^{n} \frac{i\sqrt{b}}{n} \cdot \frac{(2i-1)b}{n^2} = \frac{b^{3/2}}{n^3} \sum_{i=1}^{n} 2i^2 - i$$
$$= \frac{b^{3/2}}{n^3}\left[\frac{1}{3}n(n+1)(2n+1) - \frac{1}{2}n(n+1)\right] \xrightarrow[n\to\infty]{} \frac{2}{3}b^{3/2}$$

Similarly

$$L(f, P_n) = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i = \sum_{i=1}^{n} \frac{(i-1)\sqrt{b}}{n} \cdot \frac{(2i-1)b}{n^2} = \frac{b^{3/2}}{n^3} \sum_{i=1}^{n} 2i^2 - 3i + 1$$
$$= \frac{b^{3/2}}{n^3}\left[\frac{1}{3}n(n+1)(2n+1) - \frac{3}{2}n(n+1) + n\right] \xrightarrow[n\to\infty]{} \frac{2}{3}b^{3/2}$$

Since these limits are equal, we conclude that f is integrable and that $\int_0^b \sqrt{x}\,dx = \frac{2}{3}b^{3/2}$.



Upper Sum $U(f, P_n)$ Lower Sum $L(f, P_n)$

2. We finish this section with the classic example of a non-integrable function. Let $f : [a, b] \to \mathbb{R}$ to be the indicator function of the irrational numbers,

$$f(x) = \begin{cases} 1 & \text{if } x \notin \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

Since any interval of positive length contains both rational and irrational numbers, we see that

$$\sup\{f(x) : x \in [x_{i-1}, x_i]\} = 1 \quad \text{and} \quad \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$$

for *any* partition $P = \{x_0, \ldots, x_n\}$. We conclude that

$$U(f, P) = \sum_{i=1}^{n}(x_i - x_{i-1}) = b - a \implies U(f) = b - a \quad \text{and}$$
$$L(f, P) = 0 \implies L(f) = 0$$

Since the upper and lower integrals are unequal, f is not Riemann integrable.

As any freshman calculus student can attest, if you can find an anti-derivative, then the fundamental theorem of calculus (Section 34) makes evaluating integrals far easier. For instance, you are probably desperate to write

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{3}x^{3/2} = x^{1/2} \implies \int_0^b \sqrt{x}\,\mathrm{d}x = \frac{2}{3}x^{3/2}\Big|_0^b = \frac{2}{3}b^{3/2}$$

rather than computing Riemann/Darboux sums as in the previous example! In most practical cases, however, no easy-to-compute anti-derivative exists, so the best we can do is approximate integrals by evaluating Riemann sums for progressively finer partitions. Thankfully computers excel at such tedious work!

Exercises 32 1. For each function on the given interval, use partitions to find the upper and lower Darboux integrals. Hence prove that the function is integrable and compute its integral.

(a) $f(x) = x^3$ on $[0, b]$ for any $b > 0$.

(b) $g(x) = \sqrt[3]{x}$ on $[0, b]$.

(*Hint: mimic Example 4.7.1*)

2. Repeat question 1 for the following two functions. You *cannot* simply compute Riemann sums for left and right endpoints and take limits: why not?

(a) $h(x) = x(2-x)$ on $[0, 2]$

(*Hint: choose a partition with* $2n$ *points such that* $x_n = 1$ *and observe that* $h(2-x) = h(x)$)

(b) $k(x) = \begin{cases} 2x & \text{if } x \le 1 \\ 5-x & \text{if } x > 1 \end{cases}$ on $[0, 3]$.

(*Hint: this time try a partition with* $3n$ *points...*)

3. Let $f(x) = x$ for rational x and $f(x) = 0$ for irrational x.

(a) Calculate the upper and lower Darboux integrals for f on the interval $[0, b]$.

(b) Is f integrable on $[0, b]$?

4. Prove part 3 of Lemma 4.5: $L(f) \le U(f)$.

5. Prove part 2 of Theorem 4.6.

$$f \text{ is integrable} \iff \exists (P_n)_{n\in\mathbb{N}} \text{ such that } U(f, P_n) - L(f, P_n) \to 0.$$

Moreover, both $U(f, P_n)$ and $L(f, P_n)$ converge to $\int f$.

6. (a) Reread Definition 4.3. What happens if we allow $f : [a, b] \to \mathbb{R}$ to be *unbounded*?

(b) Read "*Riemann versus Darboux*" on page 73. Explain why being *Riemann* integrable also forces f to be bounded.

7. (If you like coding) Write a short program to estimate $\int_a^b f(x)\,\mathrm{d}x$ using Riemann sums. This can be very simple (equal partitions with right endpoints), or more complex (random partition and sample points given a mesh). Apply your program to estimate $\int_0^5 \sin(x^2 e^{-\sqrt{x}})\,\mathrm{d}x$.

33 Properties of the Riemann Integral

The rough take-away of this long section is that everything you think is integrable probably is! There will not be many examples since we have not established many explicit values for integrals.

Theorem 4.8 (Linearity). *If f, g are integrable and k, l are constant, then $kf + lg$ is integrable and*

$$\int kf + lg = k\int f + l\int g$$

Example 4.9. Thanks to examples in the previous section, we can now calculate, for instance

$$\int_0^2 5x^3 - 3\sqrt{x}\,\mathrm{d}x = 5\cdot\frac{1}{4}\cdot 2^4 - 3\cdot\frac{2}{3}\cdot 2^{3/2} = 20 - 4\sqrt{2}$$

Proof. Suppose $\epsilon > 0$ is given. By Theorem 4.6 part 3, there exist partitions R, S such that

$$U(f,R) - L(f,R) < \frac{\epsilon}{2} \quad \text{and} \quad U(g,S) - L(g,S) < \frac{\epsilon}{2}$$

By Theorem 4.6 part 1, if $P := R \cup S$, then both inequalities are satisfied by P. On each subinterval,

$$\inf f(x) + \inf g(x) \le \inf(f(x) + g(x)) \quad \text{and} \quad \sup(f(x) + g(x)) \le \sup f(x) + \sup g(x)$$

since the individual suprema/infima could be 'evaluated' at different places. Thus

$$L(f,P) + L(g,P) \le L(f+g,P) \le U(f+g,P) \le U(f,P) + U(g,P)$$

whence $U(f+g,P) - L(f+g,P) < \epsilon$ and $f+g$ is integrable. Moreover,

$$\int (f+g) - \int f - \int g \le \Big(U(f,P) - \int f\Big) + \Big(U(g,P) - \int g\Big) < \epsilon$$

Using lower Darboux integrals similarly, we see that

$$-\epsilon < \int (f+g) - \int f - \int g < \epsilon$$

Since this holds for all $\epsilon > 0$, we conclude that $\int (f+g) = \int f + \int g$.
That kf is integrable with $\int kf = k\int f$ is an exercise. Put these together for the result. ■

Corollary 4.10 (Changing endvalues). *Suppose g is integrable on $[a,b]$ and that $f : [a,b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ on (a,b). Then f is integrable on $[a,b]$ and $\int_a^b f = \int_a^b g$.*

Definition 4.11 (Integration on an open interval). A *bounded* function $f : (a,b) \to \mathbb{R}$ is *integrable* if it has an integrable extension $g : [a,b] \to \mathbb{R}$ where $f(x) = g(x)$ on (a,b). In such a case, we define $\int_a^b f := \int_a^b g$.

The Corollary (its proof is an exercise) shows that the choice of extension is irrelevant.

Theorem 4.12 (Basic Comparisons). *Suppose f and g are integrable on $[a,b]$.*

1. *If $f(x) \leq g(x)$, then $\int f \leq \int g$.*
2. *If $m \leq f(x) \leq M$ then $m(b-a) \leq \int_a^b f \leq M(b-a)$.*
3. *fg is integrable.*
4. *$|f|$ is integrable and $\left|\int f\right| \leq \int |f|$.*
5. *$\max(f,g)$ and $\min(f,g)$ are integrable.*

Part 3 is *not* integration by parts and does *not* tell us how $\int fg$ relates to $\int f$ and $\int g$!

Proof.

1. Since $g(x) - f(x) \geq 0$ is integrable, $L(g-f, P) \geq 0$ for all partitions P, and so

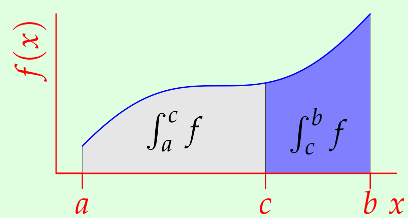
$$0 \leq L(g-f) = \int g - f = \int g - \int f$$

2. Apply part 1 twice.
3. This is an exercise.
4. The integrability is an exercise. For the comparison, apply part 1 to $-|f| \leq f \leq |f|$.
5. $\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$, etc.

∎

Theorem 4.13 (Domain splitting). *Suppose that $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. If f is integrable on both $[a,c]$ and $[c,b]$, then it is integrable on $[a,b]$ and*

$$\int_a^b f = \int_a^c f + \int_c^b f$$

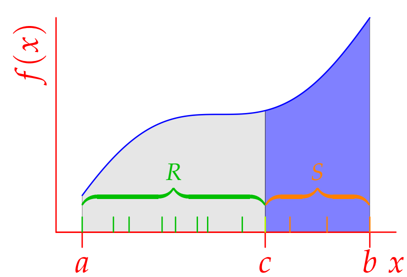


In light of this result, it is conventional to allow integral limits to be reversed:

$$\int_b^a f := -\int_a^b f \quad \text{is consistent with} \quad \int_a^a f = 0$$

Proof. Let $\epsilon > 0$ be given, then $\exists R, S$ partitions of $[a,c]$, $[c,b]$ such that

$$U(f,R) - L(f,R) < \frac{\epsilon}{2}, \qquad U(f,S) - L(f,S) < \frac{\epsilon}{2}$$



Choose $P = R \cup S$ to partition $[a,b]$, then

$$U(f,P) - L(f,P) = U(f,R) + U(f,S) - L(f,R) - L(f,S) < \epsilon$$

Moreover

$$\int_a^b f - \int_a^c f - \int_c^b f \leq U(f,P) - L(f,R) - L(f,S) = U(f,P) - L(f,P) < \epsilon$$

The other side is similar.

∎

Example 4.14. If $f(x) = \sqrt{x}$ on $[0,1]$ and $f(x) = 1$ on $[1,2]$, then

$$\int_0^2 f = \int_0^1 \sqrt{x}\,dx + \int_1^2 1\,dx = \frac{2}{3} + 1 = \frac{5}{3}$$

Monotonic & Continuous Functions

We are now in a position to establish the integrability of two large classes of functions.

Definition 4.15. A function $f : [a,b] \to \mathbb{R}$ is:

Monotonic if it is either *increasing* ($x < y \implies f(x) \le f(y)$) or *decreasing*.

Piecewise monotonic if there is a partition $P = \{x_0, \ldots, x_n\}$ of $[a,b]$ such that f is monotonic on each open subinterval (x_{k-1}, x_k).

Piecewise continuous if there is a partition such that f is *uniformly* continuous on each (x_{k-1}, x_k).

Theorem 4.16. *If f is* monotonic *or* continuous *on $[a,b]$, then it is integrable.*

Examples 4.17. 1. Since sine is continuous, we can approximate via a sequence of Riemann sums

$$\int_0^\pi \sin x\,dx = \frac{\pi}{n} \lim_{n\to\infty} \sum_{i=1}^n \sin \frac{\pi i}{n}$$

Evaluating this limit is another matter entirely, one best handled in the next section...

2. Similarly, $e^{\sqrt{x}}$ can be integrated and therefore approximated via Riemann sums:

$$\int_0^1 e^{\sqrt{x}}\,dx = \frac{1}{n} \lim_{n\to\infty} \sum_{i=1}^n \exp\sqrt{\frac{i}{n}} = \lim_{n\to\infty} \sum_{i=1}^n \frac{2j-1}{n} \exp \frac{j}{n}$$

Both sums use right endpoints; the first has equal subintervals and the second is analogous to Example 4.7.1. These limits would typically be estimated using a computer.

Proof. Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Since $[a,b]$ is closed and bounded, f is *uniformly* continuous. Let $\epsilon > 0$ be given:

$$\exists \delta > 0 \text{ such that } \forall x, y \in [a,b],\ |x - y| < \delta \implies |f(x) - f(y)| < \frac{\epsilon}{b-a}$$

Let P be a partition with mesh $P < \delta$. Since f attains its bounds on each $[x_{i-1}, x_i]$ (extreme value theorem),

$$\exists x_i^*, y_i^* \in [x_{i-1}, x_i] \quad \text{such that} \quad M_i - m_i = f(x_i^*) - f(y_i^*) < \frac{\epsilon}{b-a}$$

from which

$$U(f,P) - L(f,P) < \sum_{i=1}^n \frac{\epsilon}{b-a}(x_i - x_{i-1}) = \epsilon$$

The monotonicity argument is an exercise. ∎

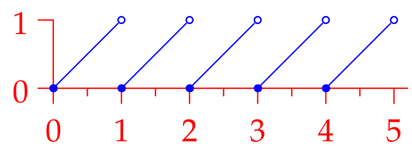
Corollary 4.18. *Piecewise continuous and* bounded *piecewise monotonic functions are integrable.*

Proof. If f is piecewise continuous, then the restriction of f to (x_{k-1}, x_k) has a continuous extension $g_k : [x_{k-1}, x_k] \to \mathbb{R}$; integrable by Theorem 4.16. By Corollary 4.10, f is integrable on $[x_{k-1}, x_k]$ with $\int_{x_{k-1}}^{x_k} f = \int_{x_{k-1}}^{x_k} g_k$. Several applications of Theorem 4.13 finish things off:

$$\int_a^b f = \sum_{k=1}^{n} \int_{x_{k-1}}^{x_k} f$$

The argument for piecewise monotonicity is similar. ∎

Example 4.19. The 'fractional part' function $f(x) = x - \lfloor x \rfloor$ is both piecewise continuous and piecewise monotone on any bounded interval. It is therefore integrable on any $[a, b]$.



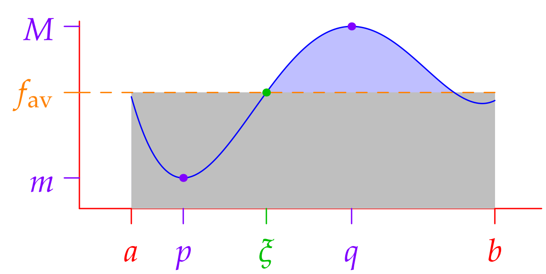
We finish with the final incarnation of the intermediate value theorem.

Corollary 4.20 (IVT for integrals). *If f is continuous on $[a, b]$, then $\exists \xi \in (a, b)$ for which*

$$f(\xi) = \frac{1}{b-a} \int_a^b f$$

Proof. Since f is continuous, it is integrable on $[a, b]$. By the extreme value theorem it is also bounded and attains its bounds: $\exists p, q \in [a, b]$ such that

$$f(p) := \inf_{x \in [a,b]} f(x), \qquad f(q) = \sup_{x \in [a,b]} f(x)$$



Applying Theorem 4.12, part 2, with $m = f(p)$ and $M = f(q)$, we see that

$$(b-a) f(p) \le \int_a^b f \le (b-a) f(q)$$

Now divide by $b - a$ and apply the usual intermediate value theorem for f to see that the required ξ exists between p and q. ∎

In the picture, when f is positive and continuous, the grey area equals that under the curve; imagine levelling off the blue hill with a bulldozer... The notation $f_{\text{av}} = \frac{1}{b-a} \int_a^b f$ is short for the average value of f on $[a, b]$: to see why this interpretation is sensible, approach $\int f$ via a sequence of Riemann sums on equally-spaced partitions P_n, then

$$\frac{1}{b-a} \int_a^b f = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} \frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

is the limit of a sequence of *averages* of equally-spaced samples $f(x_i^*)$.

What can/can't be integrated? (non-examinable)

We now know a great many examples of integrable functions: essentially

- Piecewise continuous & monotonic functions are integrable.
- Linear combinations, products, absolute values, maximums and minimums of (already) integrable functions.

After so many positive integrability conditions, it is reasonable to ask precisely which functions are Riemann integrable. There is a precise answer, though it is quite tricky to understand.

Theorem 4.21 (Lebesgue). *Suppose $f : [a,b] \to \mathbb{R}$ is bounded. Then*

$$f \text{ is Riemann integrable} \iff \text{it is continuous except on a set of measure zero}$$

Naïvely, the *measure* of a set is the sum of the lengths of its maximal subintervals; though unfortunately this doesn't make for a very useful definition.[22] Any countable subset has measure zero; Lebesgue's result is almost as if we can extend Corollary 4.18 to allow for infinite sums. Indeed you might have encountered a function which is continuous only on the irrationals; such a function is Riemann integrable. There are also some uncountable sets with measure zero such as Cantor's middle-third set: if f is the indicator function of Cantor's set

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

then f is continuous except on $\mathcal{C}$, and is Riemann integrable with $\int_0^1 f(x)\,dx = 0$.

Exercises 33

1. Explain why $\int_0^{2\pi} x^2 \sin^8(e^x)\,dx \le \frac{8}{3}\pi^3$
2. If f is integrable on $[a,b]$ prove that it is integrable on any interval $[c,d] \subseteq [a,b]$.
3. We complete the proof of Theorem 4.8 (linearity of integration).
   (a) Suppose $k > 0$, let $A \subseteq \mathbb{R}$ and define $kA := \{kx : x \in A\}$. Prove that $\sup kA = k \sup A$ and $\inf kA = k \inf A$.
   (b) If $k > 0$ prove that kf is integrable on any interval and that $\int kf = k \int f$.
   (c) How should you modify your argument if $k < 0$?
4. Give an example of an integrable but *discontinuous* function on a closed bounded interval $[a,b]$ for which the conclusion of the Intermediate Value Theorem for Integrals is *false.*

[22] Formally, the *length* of an open interval (a,b) is $b-a$ and a set $A \subseteq \mathbb{R}$ has *measure zero* if

$$\forall \epsilon > 0,\ \exists \text{ open intervals } I_n \text{ such that } A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ and } \sum_{i=1}^{\infty} \text{length}(I_n) < \epsilon$$

More generally, the *measure* of a set (subject to a technical condition) is the infimum of the sum of the lengths of any countable collection of open covering intervals. A rigorous discussion of *measure theory* is properly a matter for graduate analysis. Somewhat surprisingly, there exist sets with positive measure that contain no subintervals, and even sets which are non-measurable!

5. Explicitly compute the value of the integral $\int_{1/2}^{15/2} x - \lfloor x \rfloor \, dx$ (recall Example 4.19).

6. We prove and extend Corollary 4.10. Suppose f is integrable on $[a, b]$.

   (a) If $g : [a, b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ for all $x \in (a, b)$, prove that g is integrable and $\int_a^b g = \int_a^b f$.

   (*Hint: consider $h = f - g$ and show that $\int h = 0$*)

   (b) Now suppose $g : [a, b] \to \mathbb{R}$ satisfies $f(x) = g(x)$ for all $x \in [a, b]$ except at finitely many points. Prove that g is integrable and $\int_a^b g = \int_a^b f$.

7. Show that an increasing function on $[a, b]$ is integrable and thus complete Theorem 4.16.

   (*Hint: Choose a partition P with* mesh $P < \frac{\epsilon}{f(b) - f(a)}$)

8. Suppose f and g are integrable on $[a, b]$.

   (a) Define $h(x) = (f(x))^2$. We know:

   - f is bounded: $\exists K$ such that $|f(x)| \le K$ on $[a, b]$.
   - Given $\epsilon > 0$, $\exists P$ such that $U(f, P) - L(f, P) < \frac{\epsilon}{2K}$. For each subinterval $[x_{i-1}, x_i]$, let

   $$M_i = \sup f(x), \qquad m_i = \inf f(x), \qquad \overline{M}_i = \sup h(x), \qquad \overline{m}_i = \inf h(x)$$

   Prove that $\overline{M}_i - \overline{m}_i \le 2(M_i - m_i)K$ and use this to conclude that h is integrable.

   (b) Prove that fg is integrable.

   (*Hint: $fg = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2$*)

   (c) Prove that $U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P)$ for any partition P. Hence conclude that $|f|$ is integrable.

   (*One can extend these arguments—it's a bit harder!—to show that if j is continuous, then $j \circ f$ is integrable. Parts (a) and (c) correspond, respectively, to $j(x) = x^2$ and $j(x) = |x|$.*)

9. (Hard) Let $f(x) = \begin{cases} x & \text{if } x \neq 0 \text{ and } \sin \frac{1}{x} > 0 \\ -x & \text{if } x \neq 0 \text{ and } \sin \frac{1}{x} < 0 \\ 0 & \text{if } x = 0 \end{cases}$

   (a) Show that f is not piecewise continuous on $[0, 1]$.

   (b) Show that f is not piecewise monotonic on $[0, 1]$.

   (c) Show that f is integrable on $[0, 1]$.

   (*Hint: given ϵ, hunt for a suitable partition to make $U(f, P) - L(f, P) < \epsilon$ by considering $[0, x_1]$ differently to the other subintervals*)

   (d) Make a similar argument to show that $g = \sin \frac{1}{x}$ is integrable on $(0, 1]$, where

   $$g(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

   Note that neither argument *evaluates* the integrals!

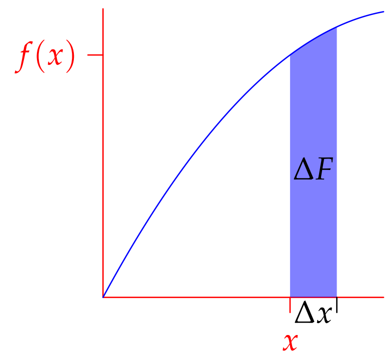
34 The Fundamental Theorem of Calculus

The key result linking integration and differentiation is usually presented in two parts.[23] While there are significant subtleties, the rough statements are as follows:

Part I Differentiation reverses integration: $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$

Part II Integration reverses differentiation: $\int_a^b F'(x)\,\mathrm{d}x = F(b) - F(a)$

These facts seemed intuitively obvious to early practitioners of calculus: indeed, given a continuous positive function f:



- Let $F(x)$ denote the area under the curve between 0 and x.
- A small increase Δx results in the area increasing by ΔF.
- $\Delta F \approx f(x)\Delta x$ is approximately the area of a rectangle, whence $\frac{\Delta F}{\Delta x} \approx f(x)$. This is part I.
- $F(b) - F(a) \approx \sum \Delta F_i \approx \sum f(x_i)\Delta x_i$. Since $F' = f$, this is part II.

In fact when Leibniz introduced the symbols $\int$ and d in the late 1600's, it was partly to reflect the fundamental theorem.[24] If you're happy with non-rigorous notions of limit, rate of change, area, and (infinite) sums, the above is all you need!

Of course, we are very much concerned with the details: What must we assume regarding f and F, and how are these properties used in the proof?

Theorem 4.22 (FTC, part I). *Suppose f is integrable on $[a,b]$. For any $x \in [a,b]$, define*

$$F(x) := \int_a^x f(t)\,\mathrm{d}t$$

Then:

1. *F is uniformly continuous on $[a,b]$;*
2. *If f is continuous at $c \in [a,b]$, then F is differentiable at c with $F'(c) = f(c)$.*

As ever, the condition at $c = a$ should be *right*-continuous and the conclusion *right*-differentiable, etc.

Compare this with the naïve version above where we assumed f was continuous. We now require only the *integrability* of f, and its continuity at *one point* for the full result.

[23] We follow the traditional numbering; some authors reverse these.

[24] $\int$ is a stylized S for *sum*, while d stands for *difference*. Given a sequence $F = (F_0, F_1, F_2, \ldots, F_n)$, construct a new sequence of *differences*

$$\mathrm{d}F = (F_1 - F_0, F_2 - F_1, \ldots, F_n - F_{n-1})$$

which can then be summed:

$$\int \mathrm{d}F = (F_1 - F_0) + (F_2 - F_1) + \cdots (F_n - F_{n-1}) = F_n - F_0 \qquad (*)$$

Viewing a function as an 'infinite sequence' of values spaced along an interval, dF becomes a sequence of *infinitesimals* and $(*)$ is essentially the fundamental theorem: $\int \mathrm{d}F = F(b) - F(a)$. It is the conception of a function that is suspect here, not the essential relationship between sums and differences.

Examples 4.23. You should have seen many examples in an elementary calculus course.

1. Since $f(x) = \sin^2(x^3 - 7)$ is continuous on any bounded interval, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_4^x \sin^2(t^3-7)\,\mathrm{d}t = \sin^2(x^3-7)$$

If one follows Theorem 4.13 and its resulting conventions, then this is valid for all $x \in \mathbb{R}$.

2. The chain rule permits more complicated examples. For instance, since $f(t) = \sin\sqrt{t}$ is continuous on its domain $[0,\infty)$ and $y(x) = x^2 + 3$ has range $[3,\infty) \subseteq \operatorname{dom}(f)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{x^2+3} \sin\sqrt{t}\,\mathrm{d}t = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}y}\int_0^y \sin\sqrt{t}\,\mathrm{d}t = 2x\sin\sqrt{x^2+3}$$

3. For a final positive example, observe that

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{\sin x}^{e^x} \tan(t^2)\,\mathrm{d}t = e^x\tan(e^{2x}) - \cos x\tan(\sin^2 x)$$

To evaluate this, one first chooses any constant a and writes

$$\int_{\sin x}^{e^x} = \int_a^{e^x} + \int_{\sin x}^a = \int_a^{e^x} - \int_a^{\sin x}$$

before differentiating. This is valid provided $\sin x$, e^x and a all lie in the same subinterval of

$$\operatorname{dom}\tan(t^2) = \mathbb{R} \setminus \{\pm\sqrt{\tfrac{\pi}{2}}, \pm\sqrt{\tfrac{3\pi}{2}}, \pm\sqrt{\tfrac{5\pi}{2}}, \ldots\}$$

Since $|\sin x| \le 1 < \sqrt{\frac{\pi}{2}}$, this requires

$$\left|e^{2x}\right| < \frac{\pi}{2} \iff x < \frac{1}{2}\ln\frac{\pi}{2}$$

Choosing $a = 1$ would certainly suffice.

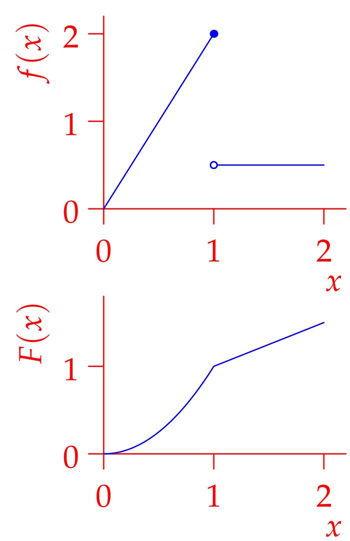
4. Now consider why the theorem requires continuity. The piecewise continuous function

$$f : [0,2] \to \mathbb{R} : x \mapsto \begin{cases} 2x & \text{if } x \le 1 \\ \frac{1}{2} & \text{if } x > 1 \end{cases}$$

has a jump discontinuity at $x = 1$. We can still compute

$$F(x) = \begin{cases} \int_0^x 2t\,\mathrm{d}t = x^2 & \text{if } x \le 1 \\ \int_0^1 2t\,\mathrm{d}t + \int_1^x \frac{1}{2}\mathrm{d}t = \frac{1}{2}(x+1) & \text{if } x > 1 \end{cases}$$

This is continuous, indeed uniformly so. However the discontinuity of f results in F having a *corner* and thus being *non-differentiable* at $x = 1$. Indeed, $F'(x) = f(x)$ for all $x \ne 1$; that is, at all values of x where f is continuous.



Proving FTC I Neither half of the theorem is particularly difficult once you write down what you know and what you need to prove. Here are the key ingredients:

1. F uniformly continuous means controlling the size of

$$|F(y) - F(x)| = \left| \int_a^y f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t \right| = \left| \int_x^y f(t)\,\mathrm{d}t \right| \le \int_x^y |f(t)|\,\mathrm{d}t$$

   But the boundedness of f allows us to bound this last integral...

2. $F'(c) = f(c)$ means showing that $\lim_{x\to c} \frac{F(x)-F(c)}{x-c} = f(c)$, which means controlling the size of

$$\left| \frac{F(x) - F(c)}{x - c} - f(c) \right| = \left| \frac{1}{x-c} \int_c^x f(t)\,\mathrm{d}t - f(c) \right|$$

   The trick here will be to bring the *constant* $f(c)$ inside the integral as $\frac{1}{x-c}\int_c^x f(c)\,\mathrm{d}t$ so that what we really have to control is the size of $\frac{1}{|x-c|}\int_c^x |f(t) - f(c)|\,\mathrm{d}t$. This is where the continuity of f comes in...

Proof. 1. Since f is integrable, it is bounded: $\exists M > 0$ such that $|f(x)| \le M$ for all x.

Let $\epsilon > 0$ be given and define $\delta = \frac{\epsilon}{M}$. Then, for any $x, y \in [a, b]$,

$$\begin{aligned} 0 < y - x < \delta \implies |F(y) - F(x)| &= \left| \int_x^y f(t)\,\mathrm{d}t \right| \le \int_x^y |f(t)|\,\mathrm{d}t && \text{(Theorem 4.12, part 4)} \\ &\le M(y - x) && \text{(Theorem 4.12, part 2)} \\ &< M\delta = \epsilon \end{aligned}$$

We conclude that F is uniformly continuous on $[a, b]$.

2. Let $\epsilon > 0$ be given. Since f is continuous at c, $\exists \delta > 0$ such that, for all $t \in [a, b]$,

$$|t - c| < \delta \implies |f(t) - f(c)| < \frac{\epsilon}{2}$$

Now for all $x \in [a, b]$ (except c),

$$\begin{aligned} 0 < |x - c| < \delta \implies \left| \frac{F(x) - F(c)}{x - c} - f(c) \right| &= \left| \frac{1}{x-c} \int_c^x f(t) - f(c)\,\mathrm{d}t \right| && \text{(Theorem 4.8)} \\ &\le \frac{1}{|x-c|} \int_c^x |f(t) - f(c)|\,\mathrm{d}t && \text{(Theorem 4.12)} \\ &\le \frac{1}{|x-c|} \frac{\epsilon}{2} |x - c| = \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Clearly $\lim_{x\to c} \frac{F(x)-F(c)}{x-c} = f(c)$, and so F is differentiable at c with $F'(c) = f(c)$. ∎

The Fundamental Theorem, part II As with part I, the *formulaic* part of the result should be familiar, though we are more interested in the assumptions and where they are needed.

Theorem 4.24 (FTC, part II). *Suppose g is continuous on $[a,b]$, differentiable on (a,b), and that g' is integrable*[25] *on (a,b). Then,*

$$\int_a^b g' = g(b) - g(a)$$

Part II is often expressed in terms of *anti-derivatives*: F being an anti-derivative of f if $F' = f$. Combined with FTC I, we recover the familiar '$+c$' result and a simpler version of the fundamental theorem often seen in elementary calculus.

Corollary 4.25. *Let f be continuous on $[a,b]$.*

- *If F is an anti-derivative of f, then $\int_a^b f = F(b) - F(a)$.*
- *Every anti-derivative has the form $F(x) = \int_a^x f(t)\,\mathrm{d}t + c$ for some constant c.*

Examples 4.26. Again, basic examples should be familiar.

1. Plainly $g(x) = x^2 + 2x^{3/2}$ is continuous on $[1,4]$ and differentiable on $(1,4)$ with derivative $g'(x) = 2x + 3\sqrt{x}$; this last is continuous (and thus integrable) on $(1,4)$. We conclude that

$$\int_1^4 2x + 3\sqrt{x}\,\mathrm{d}x = x^2 + 2x^{3/2}\Big|_1^4 = (16+16) - (1+2) = 29$$

2. If $g(x) = \sin(3x^2)$, then $g'(x) = 6x\cos(3x^2)$. Certainly g satisfies the hypotheses of the theorem on any bounded interval $[a,b]$. We conclude

$$\int_a^b 6x\cos(3x^2)\,\mathrm{d}x = \sin(3b^2) - \sin(3a^2)$$

Moreover, every anti-derivative of $f(x) = 6x\cos(3x^2)$ has the form $F(x) = \sin(3x^2) + c$.

3. Recall Example 4.23.4 where we saw that the discontinuity of f led to the *non-differentiability* of $F(x) = \int_0^x f(t)\,\mathrm{d}t$ at $x = 1$. The function F therefore fails the hypotheses of FTC II on the interval $[0,2]$.

However, except at $x = 1$, F is an anti-derivative of f and moreover $\int_0^2 f(x)\,\mathrm{d}x = F(2) - F(0)$, so we *appear* to have the formulaic conclusion of FTC II, though this is tautological given the definition of F!

The way out of this conundrum is to note that other anti-derivatives $\hat{F}$ of f exist (except at $x = 1$), and which fail to satisfy the conclusion. For instance

$$\hat{F}(x) = \begin{cases} x^2 & \text{if } x < 1 \\ \frac{1}{2}x & \text{if } x > 1 \end{cases} \implies \hat{F}(2) - \hat{F}(0) = 1 \neq \frac{3}{2} = \int_0^2 f(x)\,\mathrm{d}x$$

[25] See Definition 4.11 if you're unsure what it means for g' to be integrable on a bounded *open* interval.

Proving FTC II See Exercise 10 for a relatively easy proof when $g' = f$ is continuous. For the real McCoy, we can only rely on the *integrability* of g': the trick is to use the mean value theorem to write $g(b) - g(a)$ as a Riemann sum over a suitable partition.

Proof. Let $\epsilon > 0$ be given and choose a partition P such that $U(g', P) - L(g', P) < \epsilon$. Since g satisfies the mean value theorem on each subinterval of the partition P, we see that

$$\exists \xi_i \in (x_{i-1}, x_i) \quad \text{such that} \quad g'(\xi_i) = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}$$

from which

$$g(b) - g(a) = \sum_{i=1}^{n} g(x_i) - g(x_{i-1}) = \sum_{i=1}^{n} g'(\xi_i)(x_i - x_{i-1})$$

This is a Riemann sum for g' associated to the partition P, hence,

$$L(g', P) \leq g(b) - g(a) \leq U(g', P)$$

However we also have $L(g', P) \leq \int_a^b g' \leq U(g', P)$. Since these hold for all ϵ, the proof is complete.

■

While we certainly used the integrability of g' in the proof, it might seem strange that we assumed it at all: shouldn't every derivative be integrable? Perhaps surprisingly, the answer is no! If you want a challenge, look up the *Volterra function*, which is differentiable everywhere, but whose derivative is *non-integrable* (on, for instance, $[0, 1]$)!

The Rules of Integration

If one wants to *evaluate* an integral, rather than merely show it exists, there are really only two options:

1. Evaluate Riemann sums and take limits: often difficult if not impossible to do explicitly.

2. Use FTC II. The problem now becomes the finding of *anti-derivatives*, for which the core method is essentially *guess and differentiate.* To obtain general rules, we attempt to reverse the rules of differentiation.

Integration by Parts First consider the product rule: the product $g = uv$ of two differentiable functions is differentiable with $g' = u'v + uv'$. Now apply Theorems 4.8, 4.12 and FTC II.

Corollary 4.27 (Integration by Parts). *Suppose u, v are continuous on $[a, b]$, differentiable on (a, b), and that u', v' are integrable on (a, b). Then*

$$\int_a^b u'(x)v(x)\mathrm{d}x = u(b)v(b) - u(a)v(a) - \int_a^b u(x)v'(x)\mathrm{d}x$$

This is significantly less useful than the product rule since it is only capable of transforming the integral of a product into another such integral.

Examples 4.28. You should have seen myriad examples in a previous course. With practice, there is no need to explicitly state u and v.

1. Let $u(x) = x$ and $v'(x) = \cos x$. Then $u'(x) = 1$ and $v(x) = \sin x$, whence

$$\int_0^{\pi/2} x\cos x \, dx = [x\sin x]_0^{\pi/2} - \int_0^{\pi/2} \sin x \, dx = \frac{\pi}{2}\sin\frac{\pi}{2} - 0 - [-\cos x]_0^{\pi/2}$$
$$= \frac{\pi}{2} + \cos\frac{\pi}{2} - \cos 0 = \frac{\pi}{2} - 1$$

2. Let $u(x) = \ln x$ and $v'(x) = 1$. Then $u'(x) = \frac{1}{x}$ and $v(x) = x$, whence

$$\int_e^{e^2} \ln x \, dx = [x\ln x]_e^{e^2} - \int_e^{e^2} \frac{x}{x} \, dx = e^2 \ln e^2 - e\ln e - [x]_e^{e^2}$$
$$= 2e^2 - e - e^2 + e = e^2$$

Change of Variables/Substitution We now turn our attention to the chain rule. If $g(x) = F\big(u(x)\big)$, where F and u are differentiable, then g is differentiable with

$$g'(x) = \frac{dg}{dx} = \frac{dF}{du}\frac{du}{dx} = F'\big(u(x)\big)u'(x)$$

Now integrate both sides; the only issue is what assumptions are needed to invoke FTC II.

Theorem 4.29 (Substitution Rule). *Suppose we have two continuous functions: $u : [a,b] \to \mathbb{R}$ and $f : \text{range}(u) \to \mathbb{R}$. Suppose also that u is differentiable on (a,b) with integrable derivative u'. Then*

$$\int_a^b f\big(u(x)\big)\, u'(x) \, dx = \int_{u(a)}^{u(b)} f(u) \, du$$

This is the famous 'u-sub' formula from elementary calculus.

Proof. We leave as an exercise the verification that both integrals exist. We may also assume that range(u) is an interval of positive length[26]for otherwise both integrals are trivially zero.
Choose any $c \in \text{range}(u)$ and define

$$F : \text{range}(u) \to \mathbb{R} \text{ by } F(v) := \int_c^v f(t) \, dt$$

Since f is continuous, by FTC I we see that F is differentiable with $F'(u) = f(u)$. But now

$$\begin{aligned}\int_a^b f\big(u(x)\big)u'(x) \, dx &= \int_a^b \left[\frac{d}{dx}F\big(u(x)\big)\right] dx && \text{(chain rule)}\\ &= F\big(u(b)\big) - F\big(u(a)\big) && \text{(FTC II)}\\ &= \int_{u(a)}^{u(b)} f(u) \, du\end{aligned}$$

■

[26]By the intermediate and extreme value theorems, range(u) is already a closed bounded interval.

Examples 4.30. Reading the theorem is bad enough; its *application* often requires significant creativity in order to recognize a suitable substitution.[27]

1. To evaluate the integral $\int_0^{\sqrt{\pi}} 2x \sin x^2 \, \mathrm{d}x$, consider the substitution $u(x) = x^2$ defined on $[0, \sqrt{\pi}]$. Certainly u is continuous, and its derivative $u'(x) = 2x$ is integrable on $(0, \sqrt{\pi})$. Finally $f(u) = \sin u$ is continuous on range$(u) = [0, \pi]$. The hypotheses are satisfied, whence

$$\int_0^{\sqrt{\pi}} 2x \sin x^2 \, \mathrm{d}x = \int_0^{\sqrt{\pi}} f\big(u(x)\big) u'(x) \, \mathrm{d}x = \int_0^{\pi} \sin u \, \mathrm{d}u = -\cos u \Big|_0^{\pi} = 2$$

2. For the following integral with $f(u) = \frac{1}{u^2+1}$, we make the substitution $u(x) = x^2 - 2$. Note that $u : [\sqrt{2}, \sqrt{3}] \to [0, 1]$ and that $u'(x) = 2x$ is integrable; moreover, $f(u)$ is continuous on range$(u) = [0, 1]$. We conclude that

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2x}{x^4 - 4x^2 + 5} \, \mathrm{d}x = \int_{\sqrt{2}}^{\sqrt{3}} \frac{2x}{(x^2-2)^2 + 1} \, \mathrm{d}x = \int_0^1 \frac{1}{u^2+1} \, \mathrm{d}u = \arctan u \Big|_0^1 = \frac{\pi}{4}$$

3. The hypotheses on u really are all that is necessary. In particular, u doesn't need to be left-/right-differentiable at the endpoints of $[a, b]$! For instance, with $f(u) = u^2$ and $u(x) = \sqrt{x}$ on $[0, 4]$, we easily verify

$$\frac{8}{3} = \int_0^4 \frac{1}{2}\sqrt{x} \, \mathrm{d}x = \int_0^4 \frac{x}{2\sqrt{x}} \, \mathrm{d}x = \int_0^4 f\big(u(x)\big) u'(x) \, \mathrm{d}x = \int_0^2 f(u) \, \mathrm{d}u = \int_0^2 u^2 \, \mathrm{d}u = \frac{8}{3}$$

4. Sloppy use of the substitution rule might lead to utter nonsense. For instance, consider the 'substitution' $u = x^2$ in the following:

$$\int_{-1}^2 \frac{1}{x} \, \mathrm{d}x = \int_{-1}^2 \frac{1}{2x^2} 2x \, \mathrm{d}x = \int_1^4 \frac{1}{2u} \, \mathrm{d}u = \frac{1}{2}(\ln 4 - \ln 1) = \ln 2$$

   Of course the left hand integral does not exist since $\frac{1}{x}$ is undefined at $0 \in (-1, 2)$, so the conclusion is false. In the language of the substitution rule, $f(u) = \frac{1}{2u}$ is not continuous on range$(u) = [0, 4]$: it is not even defined at $u = 0$! You are very unlikely to make precisely this mistake since the first integral is so clearly undefined, but for more complicated functions…

[27] Hence the old adage, "Differentiation is a science; integration an art." To illustrate via an example, consider the function $f(x) = \tan(e^x \cos(3x^2) + 4x^3)$. The product and chain rules allow one to explicitly compute the derivative

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{1 + (e^x \cos(3x^2) + 4x^3)^2} \left(e^x \cos(3x^2) - 6xe^x \sin(3x^2) + 12x^2 \right)$$

By contrast, the integration analogues (integration by parts/substitution) are essentially useless in attempting to find an *explicit* anti-derivative facilitating the integration of the same function via FTC II; for instance, the integral

$$\int_0^1 \tan(e^x \cos(3x^2) + 4x^3) \, \mathrm{d}x$$

is likely impossible to evaluate explicitly and can only be approximated (e.g. via Riemann sums).

Exercises 34 1. Calculate the following limits:

(a) $\lim_{x\to 0} \frac{1}{x} \int_0^x e^{t^2} \mathrm{d}t$ (b) $\lim_{h\to 0} \frac{1}{h} \int_3^{3+h} e^{t^2} \mathrm{d}t$

2. Let $f(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \le t \le 1 \\ 4 & \text{if } t > 1 \end{cases}$

   (a) Determine the function $F(x) = \int_0^x f(t)\,\mathrm{d}t$.
   (b) Sketch F. Where is F continuous?
   (c) Where is F differentiable? Calculate F' at the points of differentiability.

3. Let f be a continuous function on $\mathbb{R}$.

   (a) Define $F(x) = \int_{x-1}^{x+1} f(t)\,\mathrm{d}t$. Carefully show that F is differentiable on $\mathbb{R}$ and compute F'.
   (b) Repeat for the function $G(x) = \int_0^{\sin x} f(t)\,\mathrm{d}t$.

4. Recall Examples 4.23.4 and 4.26.3. Find all anti-derivatives F of f on $[0,1) \cup (1,2]$. How many satisfy $\int_0^2 f(x)\,\mathrm{d}x = F(2) - F(0)$?

5. Consider integration by parts. Plainly $\int_a^x u'(t)v(t)\,\mathrm{d}t$ is an anti-derivative of $u'(x)v(x)$ by FTC I: what does integration by parts say is another?

6. Use change of variables to integrate $\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}x$

7. Use integration by parts and the substitution rule to evaluate $\int_0^b \arcsin x\,\mathrm{d}x$ for any $b < 1$.

8. Use integration by parts to evaluate $\int_0^b x \arctan x\,\mathrm{d}x$for any $b > 0$

9. Check that the assumptions of int by subs guarantee that both integrals are well-defined (i.e. that $(f \circ u)u'$ and f are integrable on the required intervals.

10. We prove a simpler version of the fundamental theorem of calculus.

    (a) Suppose f is continuous on $[a,b]$ and define $F(x) = \int_a^x f(t)\,\mathrm{d}t$. For any $c, x \in [a,b]$ where $c \neq x$, prove that

    $$m \le \frac{F(x) - F(c)}{x - c} \le M$$

    where m, M are the maximum and minimum values of $f(t)$ on the closed interval bounded by c, x. Make sure to explain why m, M exist, and use this to deduce that $F'(c) = f(c)$.

    (b) Suppose f is continuous on $[a,b]$ and that F is any anti-derivative of f on a, b (that is, $F' = f$). Use part (a) and the mean value theorem to prove that $\int_a^b f(t)\,\mathrm{d}t = F(b) - F(a)$.

36 Improper Integrals

The Riemann integral has several limitations. Even allowing for functions to be integrable on open intervals (Exercise 32.6), the definition of $\int_a^b f(x)\,\mathrm{d}x$ requires the following:

- That (a,b) be a *bounded* interval.
- That f be *bounded* on (a,b).

There is a natural way to extend the Riemann integral to unbounded intervals and functions: *limits.*

Definition 4.31. Suppose $f:[a,b)\to\mathbb{R}$ satisfies the following properties:

- f is integrable on every closed bounded subinterval $[a,t]\subseteq[a,b)$.
- Either $b=\infty$, or b is finite and f is unbounded at b,

The *improper integral* of f on $[a,b)$ is defined to be

$$\int_a^b f(x)\,\mathrm{d}x := \lim_{t\to b^-}\int_a^t f(x)\,\mathrm{d}x$$

This is *convergent* or *divergent* in the same manner as the limit.
If an integral is improper at its lower limit then $\int_a^b f(x)\,\mathrm{d}x := \lim_{s\to a^+}\int_s^b f(x)\,\mathrm{d}x$.
If an integral is improper at both ends, choose any $c\in(a,b)$ and define

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{s\to a^+}\int_s^c f(x)\,\mathrm{d}x + \lim_{t\to b^-}\int_c^t f(x)\,\mathrm{d}x$$

provided *both* one-sided improper integrals exist and the limit sum makes sense.

Theorem 4.13 says that the choice of c for a doubly-improper integral is irrelevant.

Many properties of the Riemann integral transfer to improper integrals, though not all. For example, part 1 of Theorem 4.12 extends:

Theorem 4.32. *If $0\le f(x)\le g(x)$ on $[a,b)$, then $\int_a^b f\le\int_a^b g$, whenever the integrals exist (standard or improper). In particular:*

- $\int_a^b f=\infty \implies \int_a^b g=\infty$
- $\int_a^b g$ *converges* $\implies$ $\int_a^b f$ *converges to a value* $\le\int_a^b g$.

We leave some of the detail to Exercise 36.7.

Examples 4.33.

1. $\int_0^t x^2 \, dx = \frac{1}{3}t^3$ for any $t > 0$. Clearly

$$\int_0^\infty x^2 \, dx = \lim_{t\to\infty} \frac{1}{3}t^3 = \infty$$

More formally, the improper integral $\int_0^\infty x^2 \, dx$ diverges to infinity.

2. With $f(x) = x^{-4/3}$ defined on $[1, \infty)$,

$$\int_1^\infty x^{-4/3} \, dx = \lim_{t\to\infty} \int_1^t x^{-4/3} \, dx = \lim_{t\to\infty} \left[-3x^{-1/3}\right]_1^t = \lim_{t\to\infty} 3 - 3t^{-1/3} = 3$$

3. Consider $f(x) = |x| \, e^{-x^2/2}$ on $(-\infty, \infty)$. On a bounded interval $[0, t)$, we have

$$\int_0^t f(x) \, dx = \int_0^t xe^{-x^2/2} \, dx = \left[-e^{-x^2/2}\right]_0^t = 1 - e^{-t^2/2} \xrightarrow[t\to\infty]{} 1$$

By symmetry, we conclude that

$$\int_{-\infty}^\infty |x| \, e^{-x^2/2} \, dx = 1 + 1 = 2$$

This example is important in probability: multiplying by $\frac{1}{\sqrt{2\pi}}$, we have computed the the expectation of $|X|$ when X is a normally-distributed random variable

$$\mathbb{E}(|X|) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} |x| \, e^{-x^2/2} \, dx = \sqrt{\frac{2}{\pi}}$$

4. If $t \in [0, 1)$, we can use our knowledge of derivatives $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$ to evaluate

$$\int_0^1 \frac{1}{\sqrt{1-x^2}} \, dx = \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt{1-x^2}} \, dx = \lim_{t\to 1^-} \sin^{-1} t = \frac{\pi}{2}$$

and that, moreover $\int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \, dx = \pi$. By comparison, we see that

$$\frac{1}{\sqrt{1-x^4}} \le \frac{1}{\sqrt{1-x^2}} \implies \int_{-1}^1 \frac{1}{\sqrt{1-x^4}} \, dx \le \int_{-1}^1 \frac{1}{\sqrt{1-x^2}} \, dx = \pi$$

5. Improper integrals need not exist. For instance,

$$\lim_{t\to\infty} \int_0^t \sin x \, dx = \lim_{t\to\infty} 1 - \cos t$$

diverges by oscillation.

Exercises 36 1. Use your answers from the previous section to decide whether the improper integrals $\int_0^1 \arcsin x \, dx$ and $\int_0^\infty x \arctan x \, dx$ exist. If so, what are their values?

2. Let p be a positive constant. Prove the following:

$$\int_0^1 \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{1-p} & \text{if } p < 1 \\ \infty & \text{if } p \geq 1 \end{cases} \qquad \int_1^\infty \frac{1}{x^p} \, dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1 \end{cases}$$

3. Explain why $\int_a^b f(x) \, dx = \lim_{t \to b^-} \int_a^t f(x) \, dx$ holds, even when f is integrable on $[a, b]$.

4. State a version of integration by parts modified for when $\int_a^b u'(x)v(x) \, dx$ is an improper integral. Now evaluate $\int_0^\infty xe^{-4x} \, dx$.

5. What is wrong with the following calculation?

$$\int_{-\infty}^{\infty} x \, dx = \lim_{t \to \infty} \frac{1}{2}x^2 \Big|_{-t}^{t} = \lim_{t \to \infty} \frac{1}{2}(t^2 - t^2) = \lim_{t \to \infty} 0 = 0$$

6. Prove or disprove: if $\int f$ and $\int g$ are convergent improper integrals, so is $\int fg$.

7. Prove part of Theorem 4.32. Suppose $0 \leq f(x) \leq g(x)$ for all $x \in [a, b)$, and that $\int_a^b g$ is a convergent improper integral. Prove that $\int_a^b f$ converges and that $\int_a^b f \leq \int_a^b g$.

Generalizing the Riemann Integral (non-examinable)

In the 1890's, Thomas Stieltjes[28] offered a generalization of the Riemann integral.

Definition 4.34. Let α be a monotonically increasing function on an interval $[a, b]$. Given a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and a function f, define the differences

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$

The *upper/lower Darboux–Stieltjes sums/integrals* are defined analogously to the pure Riemann case:

$$U(f, P, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \qquad L(f, P, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$
$$U(f, \alpha) = \inf U(f, P, \alpha) \qquad L(f, \alpha) = \sup L(f, P, \alpha)$$

f is *Riemann–Stieltjes integrable* of class $\mathcal{R}(\alpha)$ if $U(f, \alpha) = L(f, \alpha)$: we denote this value $\int_a^b f(x)\,\mathrm{d}\alpha$.

The standard Riemann integral corresponds to $\alpha(x) = x$. It is the ability to choose other functions α that makes the Riemann–Stieltjes integral both powerful and applicable.

Standard Properties Most results in sections 32 and 33 hold with suitable modifications, as does the discussion of improper integrals. For instance,

$$f \in \mathcal{R}(\alpha) \iff \exists P \text{ such that } U(f, P, \alpha) - L(f, P, \alpha) < \epsilon$$

The result regarding piecewise continuity of f is a notable exception: if f and α are simultaneously piecewise continuous then f might not lie in $\mathcal{R}(\alpha)$.

Weighted integrals If α is differentiable, then we obtain a standard Riemann integral

$$\int_a^b f(x)\,\mathrm{d}\alpha = \int_a^b f(x)\alpha'(x)\,\mathrm{d}x$$

weighted so that $f(x)$ contributes more when α is increasing rapidly.

Probability If $\alpha(a) = 0$ and $\alpha(b) = 1$, then α may be viewed as a *probability distribution function.*Its derivative α' is the corresponding *probability density function.* For example:

1. The *uniform distribution* on $[a, b]$ has $\alpha = \frac{1}{b-a}(x - a)$ so that

   $$\int_a^b f(x)\,\mathrm{d}\alpha = \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x$$

   Since α' is constant, the integrals weigh all values of x *uniformly.*
2. The standard *normal distribution* has $\alpha(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\,\mathrm{d}t$. The fact that $\alpha' = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ is maximal when $x = 0$ reflects the fact that a normally distributed variable is clustered near its mean.

In all cases, $\int f(x)\,\mathrm{d}\alpha = \mathbb{E}(f(X))$ computes an expectation (see, for instance, Example 4.33.3).

[28]Stieltjes was Dutch; for the pronunciation try 'steelchez.'

Non-differentiable α A major flexibility comes when we allow α to be non-differentiable, or even discontinuous! For example, given a partition $Q = \{s_0, \ldots, s_n\}$ of $[a, b]$, and a positive sequence $(c_k)_{k=1}^n$, define

$$\alpha(x) = \begin{cases} 0 & \text{if } x = a \\ \sum_{i=1}^{k} c_i & \text{if } x \in (s_{k-1}, s_k] \end{cases}$$

This is an increasing step function on $[a, b]$. The Riemann–Stieltjes integral becomes a weighted sum

$$\int_a^b f(x)\,\mathrm{d}\alpha = \sum_{i=1}^{n} c_i f(s_i)$$

Taking instead an infinite sequence $(s_n) \subseteq [a, b]$ results in an infinite series, which helps explain why so many results for series and integrals look similar!

This also touches on probability. For example, let $p \in [0, 1]$, $n \in \mathbb{N}$, and $s_k = k$ on the interval $[0, n]$. If $c_k = \binom{n}{k} p^k (1-p)^{n-k}$, then

$$\int f(x)\,\mathrm{d}\alpha = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} f(x) = \mathbb{E}(f(X))$$

is the expectation of $f(X)$ when $X \sim B(n, p)$ is a binomially distributed random variable.

Integrals and Convergence

The *Lebesgue integral* is another common generalization. Its main purpose is to permit the transfer of integrability to the limit of a sequence of integrable functions.[29] To see the problem, consider the sequence

$$f_n : [0, 1] \to \mathbb{R} : x \mapsto \begin{cases} 1 & \text{if } x = \frac{p}{q} \in \mathbb{Q} \text{ with } q \leq n \\ 0 & \text{otherwise} \end{cases}$$

Each f_n is piecewise continuous and thus Riemann integrable with $\int_0^1 f_n(x)\,\mathrm{d}x = 0$. However, the pointwise limit of f_n is the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

which is *not* Riemann integrable. In the Lebesgue theory, the limit f turns out to be integrable with integral 0, so that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^1 \lim_{n\to\infty} f_n(x)\,\mathrm{d}x$$

Recall that the interchange of limits and integrals would be automatic *if* the convergence $f_n \to f$ were *uniform*: of course the convergence isn't uniform here.

[29] Recall how *uniform convergence* does this for continuity.