# Math 147 — Complex Analysis

Neil Donaldson

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## 1 Complex Numbers

### 1.1 Definition and Basic Algebraic Properties

In the 1500's, Italian mathematician Rafael Bombelli posited a solution to the seemingly absurd equation $x^2 = -1$. By supposing that it behaved according to the 'usual' rules of algebra, Bombelli and others were able to describe the solutions to any quadratic equation. To some extent, this was math for its own sake; Bombelli always considered his solutions to be entirely 'fictitious.'

For a modern definition, we start with the Cartesian plane $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$.

**Definition 1.1.** Given real numbers $x, y$, the *complex number* $z = x + iy$ is the point $(x, y) \in \mathbb{R}^2$. Its *real* and *imaginary parts* are the co-ordinates

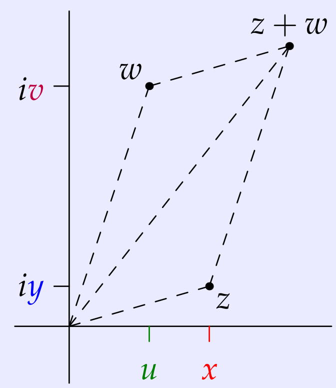
$$\operatorname{Re} z = x, \qquad \operatorname{Im} z = y$$

The *complex numbers* $\mathbb{C}$ comprise the *real* vector space $\mathbb{R}^2$ with the extra operation of *complex multiplication*: If $z = x + iy$ and $w = u + iv$, define

(Vector) Addition: $z + w := (x + u) + i(y + v)$

Complex multiplication: $zw := (xu - yv) + i(xv + yu)$

When drawn with axes, the complex plane is known as the *Argand diagram* and we refer, respectively, to the *real* and *imaginary axes*.




Since $\mathbb{C} = \mathbb{R}^2$ is a real vector space under addition, we have several immediate properties:

**Lemma 1.2 (Basic properties of complex addition).**

*Associativity:* $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

*Commutativity:* $z + w = w + z$ *(this is the parallelogram law as illustrated in the picture)*

*(Real) scalar multiplication:* $\forall \lambda \in \mathbb{R}$, $\lambda(x + iy) = \lambda x + i\lambda y$

*Additive inverse:* $-z = -(x + iy) = (-x) + i(-y) = -x - iy$

**Example 1.3.** If $z = 3 + 4i$ and $w = 2 - 7i$, then

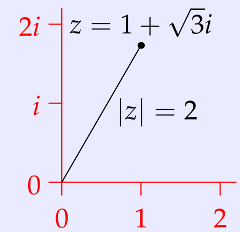
$$z - w = z + (-w) = (3 - 2) + (4 - (-7))i = 1 + 11i$$

The natural distance measure from $\mathbb{R}^2$ transfers to $\mathbb{C}$ (the complex numbers are a real metric space).

**Definition 1.4.** The *modulus* of a complex number $z = x + iy$ is the Euclidean distance of the point $(x, y)$ from the origin:

$$|z| := \sqrt{x^2 + y^2}$$

In the picture, $z = 1 + \sqrt{3}i$ has modulus $|z| = \sqrt{1+3} = 2$.




Some natural inequalities following straight from the picture in Definition 1.1.

**Lemma 1.5 (Triangle inequalities).** *For all $z, w \in \mathbb{C}$,*

$$|z + w| \le |z| + |w| \quad \text{and} \quad |z + w| \ge \big||z| - |w|\big|$$

In the second inequality, we take the *absolute value* of the difference of the moduli. Unlike in $\mathbb{R}$, these inequalities follow from an honest triangle! We can easily extend the first by induction,

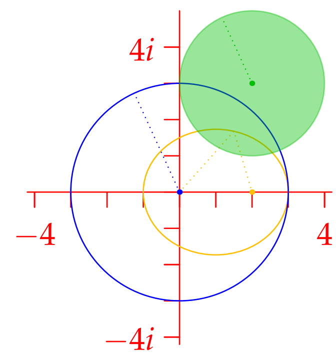
$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + \cdots + |z_n|$$

The modulus may be used to describe various curves and regions in the plane.

**Examples 1.6.**

1. $|z| = 3$ describes the circle radius 3 centered at the origin. In Cartesian co-ordinates, this becomes $x^2 + y^2 = 9$.
2. $|z - 2 - 3i| \le 2$ describes the *disk* of radius 2 centered at $2 + 3i$.
3. $|z| + |z - 2| = 4$ describes an *ellipse* with foci 0 and 2. This is more familiar after multiplying out (try it!):

$$\sqrt{x^2 + y^2} + \sqrt{(x-2)^2 + y^2} = 4 \implies (x-2)^2 + y^2 = \cdots$$
$$\implies \frac{(x-1)^2}{4} + \frac{y^2}{3} = 1$$




**Complex multiplication, division and the complex conjugate**

Multiplication is the source of all the distinct structure of the complex numbers. For starters, we instantly see that $i$ is a solution to Bombelli's absurd equation:

$$i^2 = (0 + 1i)(0 + 1i) = (0 \cdot 0 - 1 \cdot 1) + i(0 \cdot 1 + 1 \cdot 0) = -1$$

The upshot is that we can treat complex addition, subtraction and multiplication as if we are working with *linear polynomials*[1] in the abstract variable $i$; simply replace $i^2$ with $-1$ when needed.

[1]This is precisely the definition you'll see if you take a course in Rings & Fields, where $\mathbb{C}$ is the *factor ring* of real polynomials modulo the *ideal* $\langle x^2 + 1 \rangle$.

**Example 1.7.** If $z = 3 + 4i$ and $w = 2 - 7i$,

$$\begin{aligned} zw &= (3+4i)(2-7i) = 3 \cdot 2 + 4i \cdot 2 - 3 \cdot 7i - 4i \cdot 7i = 6 + 8i - 21i - 28i^2 \\ &= 6 + 8i - 21i + 28 = 34 - 13i \end{aligned}$$

The basic algebraic properties of complex multiplication are straightforward, if tedious, to verify:

**Lemma 1.8 (Basic properties of multiplication).** *For any complex numbers $z_1, z_2, z_3$,*

*Associativity:* $z_1(z_2z_3) = (z_1z_2)z_3$

*Commutativity:* $z_1z_2 = z_2z_1$

*Distributivity:* $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$

To develop division, it is is helpful to introduce a new concept.

**Definition 1.9.** The *(complex) conjugate* of $z = x + iy$ is $\overline{z} := x - iy$ (read $z$-bar). Geometrically, $\overline{z}$ is obtained by reflection in the real axis.

It is immediate that $z\overline{z} = x^2 + y^2 = |z|^2$, which helps us conclude:

**Lemma 1.10.** *Every non-zero complex number $z = x + iy$ has a unique multiplicative inverse*

$$z^{-1} = \frac{\overline{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2}$$

*Proof.* That $zz^{-1} = 1$ is trivial. For uniqueness, suppose we also have $zw = 1$ and use associativity and commutativity to conclude that

$$w = (z^{-1}z)w = z^{-1}(zw) = z^{-1}$$

∎

Division is simply multiplication by an inverse.

**Example 1.11.** Given $z = 3 + 4i$ and $w = 2 - 7i$, we compute

$$\begin{aligned} \frac{w}{z} &= \frac{2-7i}{3+4i} = wz^{-1} = \frac{w\overline{z}}{|z|^2} = \frac{(2-7i)(3-4i)}{|3+4i|^2} = \frac{2 \cdot 3 - 2 \cdot 4i - 7i \cdot 3 + 7i \cdot 4i}{3^2 + 4^2} \\ &= \frac{6 - 8i - 21i + 28i^2}{25} = \frac{-22 - 29i}{25} \end{aligned}$$

If you prefer, you can think about this as multiplying the numerator and denominator by the conjugate[2] of the denominator:

$$\frac{2-7i}{3+4i} = \frac{2-7i}{3+4i} \cdot \frac{3-4i}{3-4i} = \cdots$$

[2]Compare this approach with elementary algebra where, for example, $5 + \sqrt{3}$ is the *conjugate* of $5 - \sqrt{3}$, and we use it to compute $\frac{1}{5-\sqrt{3}} = \frac{5+\sqrt{3}}{(5-\sqrt{3})(5+\sqrt{3})} = \frac{5+\sqrt{3}}{22}$.

**Exercises 1.1** 1. For any $z \in \mathbb{C}$, prove that $\operatorname{Re}(iz) = -\operatorname{Im} z$ and that $\operatorname{Im}(iz) = \operatorname{Re} z$.

2. (a) Check explicitly that $z = 2 + 3i$ and its conjugate $\overline{z} = 2 - 3i$ solve the quadratic equation $z^2 - 4z + 13 = 0$.

   (b) Suppose $a, b, c \in \mathbb{R}$ where $\omega := 4ac - b^2 > 0$. Check that $z = \frac{-b+i\sqrt{\omega}}{2a}$ and its conjugate $\overline{z}$ both solve the quadratic equation $az^2 + bz + c = 0$.
   (*Since $i^2 = -1$, we write $\sqrt{-\omega} = i\sqrt{\omega}$: the quadratic formula now applies to all real quadratics*)

3. Explicitly prove the commutativity of complex multiplication (Lemma 1.8) using the vector definition of $\mathbb{C}$ (Definition 1.1).

4. Evaluate the following in the form $x + iy$:

   (a) $\dfrac{2-i}{3-5i}$ (b) $(1+i)^4$ (c) $(2+3i)^{-2} - (2-3i)^{-2}$

5. Prove the following: you should write $z = x + iy$ rather than using the vector definition.

   (a) $\overline{\overline{z}} = z$ (b) $(z^{-1})^{-1} = z$ (c) $\overline{zw} = \overline{z} \cdot \overline{w}$

6. (a) For any $z, w$, prove that $|z + w| \geq \big||z| - |w|\big|$.

   (b) What relationship between $z, w$ corresponds to *equality* here? Draw a picture!

7. Suppose that $|z| \geq 2$ and consider the polynomial $P(z) = z^3 + 3z - 1$.

   (a) Prove that $\left|\frac{3z-1}{z^3}\right| \leq \frac{7}{8}$

   (b) Write $|P(z)| = \left|z^3 + 3z - 1\right| = \left|z^3\right| \left|1 + \frac{3z-1}{z^3}\right|$. Use the extended triangle inequality (Exercise 6(a)) to prove that $|P(z)| \geq 1$.
   (*This shows that all zeros of $P(z)$ lie inside the circle $|z| < 2$.*)

8. By considering the inequality $(|x| - |y|)^2 \geq 0$, prove that

$$\sqrt{2}\,|z| \geq |\operatorname{Re} z| + |\operatorname{Im} z|$$

9. Prove that the hyperbola $x^2 - y^2 = 1$ can be written in the form $z^2 + \overline{z}^2 = 2$.

10. Draw a picture of the ellipse satisfying the equation $|z| + |z - 4i| = 6$. Find the equation of the curve in Cartesian coordinates: $\frac{(x-c)^2}{a^2} + \frac{(y-d)^2}{b^2} = 1$ where $(c, d)$ is the center of the ellipse and $a, b$ are the semi-axes.
    (*Hint: write $|z - 4i| = 6 - |z|$, square both sides, cancel $x^2, y^2$ terms and repeat...*)

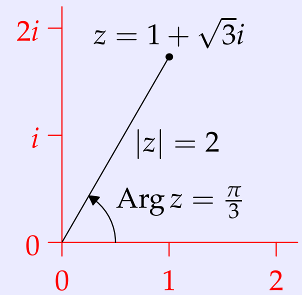
### 1.2 The Exponential or Polar Form of a Complex Number

Recall Definition 1.4 of the *modulus* of a complex number. We extend this to also consider the *angle*.

**Definition 1.12.** A complex number can be written in polar co-ordinates:

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

Plainly $r = |z|$ is the modulus. The angle $\arg z = \theta$ is the *argument* of $z$.
The argument is multi-valued in that we also have $\arg z = \theta + 2\pi n$ for any integer $n$. We therefore distinguish the *principal argument* $\operatorname{Arg} z$ by insisting that $-\pi < \operatorname{Arg} z \le \pi$.




Note that 0 has no argument: it is the only complex number without an argument!

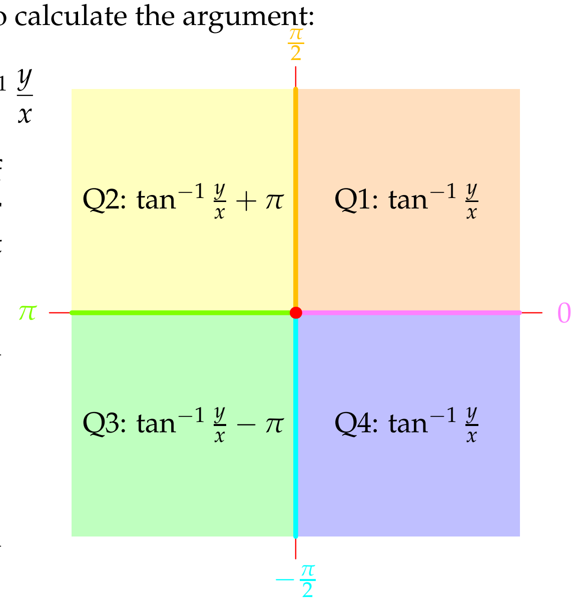
**Example 1.13.** In the above picture, $z = 1 + \sqrt{3}i$ has principal argument $\operatorname{Arg} z = \frac{\pi}{3}$. You can write the argument either as many different values, or as a set:[3] all the following are legitimate

$$\arg z = \{\tfrac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\}, \quad \text{or} \quad \arg z = \tfrac{\pi}{3}, \quad \text{or} \quad \arg z = \tfrac{7\pi}{3}$$

Provided $x \neq 0$, the following is (almost) all we need to calculate the argument:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \tan\theta = \frac{y}{x} \overset{???}{\implies} \arg z = \tan^{-1}\frac{y}{x}$$

This isn't quite right. Since arctan has range $(-\frac{\pi}{2}, \frac{\pi}{2})$, if $z$ lies in the second or third quadrants, an addition or subtraction of $\pi$ might be required to find the correct value of the principal argument $\operatorname{Arg} z$.




**Example 1.14.** Since $z = -3 - 3i$ lies in the third quadrant, we have

$$\operatorname{Arg} z = \tan^{-1}\frac{-3}{-3} - \pi = -\frac{3\pi}{4}$$

If we preferred the argument to be positive, we could instead choose a non-principal argument $\arg z = \frac{5\pi}{4}$.

The polar form allows us to define a crucial new function.

**Definition 1.15.** Given $\theta \in \mathbb{R}$, the *exponential* $e^{i\theta}$ is defined by *Euler's formula*

$$e^{i\theta} := \cos\theta + i\sin\theta$$

The *polar form* of a complex number can now be written $z = re^{i\theta}$ where $r = |z|$ and $\theta = \arg z$.
If $w = u + iv$ is complex, then its exponential is defined by $e^w := e^u e^{iv} = e^u(\cos v + i\sin v)$.

[3] This is merely the common mathematical fudge of denoting an equivalence class $\{\frac{\pi}{3} + 2\pi n : n \in \mathbb{Z}\}$ by any of its representatives, e.g. $\frac{\pi}{3}$ and $\frac{7\pi}{3}$.

There are several reasons why Euler's formula provides a sensible definition of $e^{i\theta}$: in particular it fits with two common definitions of the exponential in real analysis:

1. If $k \in \mathbb{R}$, then $e^{k\theta}$ is the solution to the initial value problem $y' = ky$ with $y(0) = 1$. Assuming that differentiation works when $k = i$, Euler's formula satisfies this criterion

$$\frac{\mathrm{d}}{\mathrm{d}\theta}e^{i\theta} = \frac{\mathrm{d}}{\mathrm{d}\theta}(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = ie^{i\theta}$$

2. The real and imaginary parts of the Maclaurin series $\exp z = \sum \frac{z^n}{n!}$ evaluated at $z = i\theta$ are, respectively, the Maclaurin series of $\cos\theta$ and $\sin\theta$.

Another reason is that the definition satisfies the usual exponential laws.

**Lemma 1.16 (Exponential laws).** *Let $z = re^{i\theta}$ and $w = se^{i\psi}$ be written in polar form. Then*

1. $zw = rse^{i(\theta+\psi)}$, *in particular* $|zw| = |z|\,|w|$ *and* $\arg zw = \arg z + \arg w$
2. $\frac{z}{w} = \frac{r}{s}e^{i(\theta-\psi)}$
3. $z^n = r^n e^{in\theta}$, $n \in \mathbb{Z}$

Note that the *principal argument* might not behave so nicely for products; the best we can say is that

$$\operatorname{Arg} zw = \operatorname{Arg} z + \operatorname{Arg} w + 2\pi n \text{ for some } n = 0, \pm 1$$

*Proof.* Part 1 follows from the multiple-angle formulæ for sine and cosine:

$$\begin{aligned} e^{i(\theta+\psi)} &= \cos(\theta+\psi) + i\sin(\theta+\psi) = \cos\theta\cos\psi - \sin\theta\sin\psi + i(\sin\theta\cos\psi + \cos\theta\sin\psi) \\ &= (\cos\theta + i\sin\theta)(\cos\psi + i\sin\psi) = e^{i\theta}e^{i\psi} \end{aligned}$$

Parts 2 and 3 are now straightforward. ∎

**Examples 1.17.** 1. Given $z = -7 + i$ and $w = 3 + 4i$, we find the modulus and argument of $zw$ in two ways:

(a) First find the polar forms of $z, w$, then apply the Lemma:

$$z = |z|\,e^{i\arg z} = 5\sqrt{2}\exp\left(i(\pi - \tan^{-1}\tfrac{1}{7})\right), \quad w = |w|\,e^{i\arg w} = 5\exp\left(i\tan^{-1}\tfrac{4}{3}\right)$$
$$\implies |zw| = |z|\,|w| = 25\sqrt{2}, \quad \arg zw = \arg z + \arg w = \pi - \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{4}{3}$$

(b) First find $zw = (-7+i)(3+4i) = -25 - 25i$ then compute its polar form:

$$zw = 25\sqrt{2}e^{-\frac{3\pi i}{4}} \implies |zw| = 25\sqrt{2}, \qquad \operatorname{Arg} zw = -\frac{3\pi}{4}$$

The first answer is plainly uglier. It is usually better to use the second approach unless the arguments of $z, w$ are exactly computable. In Exercise 7, we check that these values correspond.

2. We compute $z^{10}$ when $z=\sqrt{3}-i$. First observe that $z=2e^{-\frac{\pi i}{6}}$, from which

$$z^{10}=2^{10}e^{-\frac{5\pi i}{3}}=1024e^{\frac{\pi i}{3}}=512(1+\sqrt{3}i)$$

3. The identity $(e^{i\theta})^n=e^{in\theta}$ is known as *de Moivre's formula*; it is usually written

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

Many trigonometric identities follow from this by taking real or imaginary parts. For instance, when $n=3$,

$$\cos 3\theta+i\sin 3\theta=(\cos\theta+i\sin\theta)^3=\cos^3\theta+3i\cos^2\theta\sin\theta-3\cos\theta\sin^2\theta-i\sin^3\theta$$
$$\implies \cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta=4\cos^3\theta-3\cos\theta$$

**Exercises 1.2** 1. Use induction to prove that for any $n\in\mathbb{N}_{\geq 2}$ we have

$$e^{i\theta_1}e^{i\theta_2}\cdots e^{i\theta_n}=e^{i(\theta_1+\theta_2+\cdots+\theta_n)}$$

2. Find the principal argument of $(1+i)^{2022}$.

3. Prove that $\left|e^{i\theta}\right|=1$ and that $\overline{e^{i\theta}}=e^{-i\theta}$.

4. (a) Show that if $\operatorname{Re} z>0$ and $\operatorname{Re} w>0$, then $\operatorname{Arg} zw=\operatorname{Arg} z+\operatorname{Arg} w$.
   (b) If $z$ and $w$ both lie in quadrant 2, explain why $\operatorname{Arg} zw=\operatorname{Arg} z+\operatorname{Arg} w-2\pi$.

5. Prove that non-zero $z,w\in\mathbb{C}$ have the same modulus if and only if $\exists p,q\in\mathbb{C}$ such that $z=pq$ and $w=p\overline{q}$.

6. Use de Moivre's formula to establish the identity

$$\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$$

7. (a) Let $\alpha=\tan^{-1}\frac{4}{3}$ and $\beta=\tan^{-1}\frac{1}{7}$. Use right-triangles to show that

$$\cos\alpha=\frac{3}{5},\qquad \sin\alpha=\frac{4}{5},\qquad \cos\beta=\frac{7}{\sqrt{50}},\qquad \sin\beta=\frac{1}{\sqrt{50}}$$

   Now use the cosine multiple-angle formula to check that $\alpha-\beta=\frac{\pi}{4}$.
   (*This shows that* $\arg zw=\frac{5\pi}{4}$ *in Example 1.17(a)*)
   (b) Generalize the approach in part (a) to prove the multiple-angle formula for tangent

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

   when $\alpha,\beta$ are acute angles.

8. The polar form of a complex number is well-suited to describing *circles.* For instance the circle centered at $i$ with radius 3 may be parametrized by $z=i+3e^{i\theta}$ where $\theta\in(-\pi,\pi]$.
   (a) Describe the circle centered at $z_0=3+4i$ with radius 2.
   (b) Show that the points $z=re^{i\theta}$ for which $r=2a\cos\theta$ describe a circle.
   (*Hint: Multiply by* $r$)

## 1.3 Roots of Complex Numbers

A naïve approach to taking roots in $\mathbb{C}$ is very messy.

**Example 1.18.** To find $c$ such that $c^2 = -5 + 12i$, we need to solve an equation:

$$-5 + 12i = c^2 = (x + iy)^2 = x^2 - y^2 + 2ixy \iff \begin{cases} x^2 - y^2 = -5 \\ xy = 6 \end{cases}$$

Substituting $y = 6x^{-1}$ into the first equation yields a quadratic in $x^2$:

$$x^4 + 5x^2 - 36 = (x^2 - 4)(x^2 + 9)$$

from which we conclude that $x = \pm 2$ and obtain the square roots $\pm c = \pm(2 + 3i)$.

The example is reassuring in that we obtain precisely two square roots. However, attempting to extend the method to cube, or higher, roots is utterly doomed! Instead we use the polar form. Suppose $n \in \mathbb{N}$ and that $c, z$ satisfy $z = c^n$. In polar form

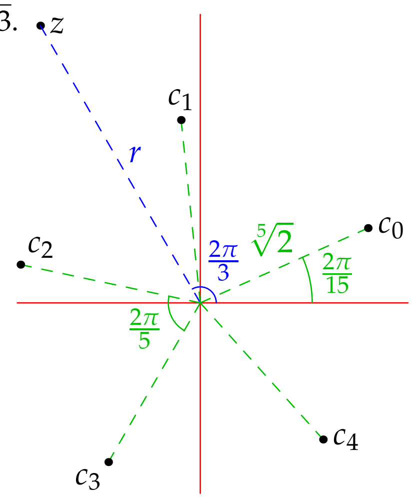
$$z = re^{i\theta}, \quad c = se^{i\psi} \implies re^{i\theta} = s^n e^{in\psi}$$

By equating moduli and arguments, we conclude that

$$r = s^n, \qquad n\psi = \theta + 2\pi k \tag{$*$}$$

where $k$ is some integer. We'll shortly put this together to obtain a proper definition, but we already have enough for a calculation.

**Example 1.19.** We compute the fifth roots of $z = 2e^{\frac{2\pi i}{3}} = -1 + i\sqrt{3}$.




In the above language,

$$s = \sqrt[5]{2}, \quad \psi = \frac{1}{5}\left(\frac{2\pi}{3} + 2\pi k\right) = \frac{2\pi}{15}(1 + 3k)$$

which results in the fifth roots

$$c_0 = \sqrt[5]{2}e^{\frac{2\pi i}{15}} \qquad c_1 = \sqrt[5]{2}e^{\frac{8\pi i}{15}} \qquad c_2 = \sqrt[5]{2}e^{\frac{14\pi i}{15}}$$

$$c_3 = \sqrt[5]{2}e^{\frac{20\pi i}{15}} = \sqrt[5]{2}e^{-\frac{10\pi i}{15}} \qquad c_4 = \sqrt[5]{2}e^{\frac{26\pi i}{15}} = \sqrt[5]{2}e^{-\frac{4\pi i}{15}}$$

Note how there are precisely *five* fifth roots: once $k \geq n (= 5)$ in $(*)$, the roots start repeating. The last two roots were written both with positive and principal arguments: both have advantages!

Observe also how the fifth roots form the vertices of a regular pentagon, equally spaced around the circle of radius $\sqrt[5]{2}$.

Finally, note how essential the polar form was to this calculation. We could convert back to rectangular form, but since $\cos \frac{2\pi}{15}$ and $\sin \frac{2\pi}{15}$ do not have friendly values, this is of limited utility.

**Definition 1.20.** Given a non zero complex number $z = re^{i\theta}$ and a positive integer $n$, the $n^{th}$ *roots* of $z$ are the $n$ complex numbers

$$c_k = \sqrt[n]{r}\exp\frac{(\theta + 2k\pi)i}{n} \qquad k = 0, 1, \ldots, n-1$$

where $\sqrt[n]{r}$ is the usual real (positive!) $n^{\text{th}}$ root.
There are some conventions to follow. Let $\theta = \operatorname{Arg} z$ be the *principal argument* of $z$:

(a) The *principal $n^{th}$ root* is $\sqrt[n]{z} := \sqrt[n]{r}e^{\frac{i\theta}{n}}$.

(b) The *set of $n^{th}$ roots* is $z^{\frac{1}{n}} := \{c_0, \ldots, c_{n-1}\}$.

Denote by $\omega_n = e^{\frac{2\pi i}{n}}$ a *primitive $n^{th}$ root of unity,* then the full set of $n^{th}$ *roots of unity* is

$$1^{\frac{1}{n}} = \{\omega_n^k : k = 0, \ldots, n-1\} = \{e^{\frac{2\pi k i}{n}} : k = 0, \ldots, n-1\}$$

The $n^{\text{th}}$ roots of $z$ may be written in terms of the principal root and the $n^{\text{th}}$ roots of unity

$$z^{\frac{1}{n}} = \sqrt[n]{z}\,1^{\frac{1}{n}} = \{\sqrt[n]{z}\omega_n^k : k = 0, \ldots, n-1\}$$

The geometric effect of multiplying by $\omega_n^k = e^{\frac{2\pi k i}{n}}$ is to rotate counter-clockwise by $\frac{2\pi k}{n}$ radians:

$$\arg\sqrt[n]{z}\omega_n^k = \arg\sqrt[n]{z} + \arg\omega_n^k = \arg\sqrt[n]{z} + \frac{2\pi k}{n}$$

It follows that the $n^{\text{th}}$ roots of $z = re^{i\theta}$ form the vertices of a regular $n$-gon spaced equally round the circle of radius $\sqrt[n]{r}$. Compare this with the previous example.

**Examples 1.21.** 1. First compare what happens when $z = r = 16$ and $n = 4$.

- $\sqrt[4]{16} = 2$ is the principal sixth root.
- In the real numbers, we have *two* fourth roots: $16^{\frac{1}{4}} = \pm 2$.
- In complex analysis, there are *four* fourth roots: $16^{\frac{1}{4}} = \{2, 2i, -2, -2i\}$ where $i = \omega_4 = e^{\frac{i\pi}{2}}$ is a primitive fourth root of unity.

2. We compute the fourth roots of $z = 8\sqrt{2}(1+i)$.

First we write in polar form: $z = 16e^{\frac{\pi i}{4}}$. Since $\operatorname{Arg} z = \frac{\pi}{4}$, the principal fourth root is

$$\sqrt[4]{8\sqrt{2}(1+i)} = 2e^{\frac{\pi i}{16}}$$

To find all fourth roots, simply multiply by the fourth roots of unity $1^{\frac{1}{4}} = \{1, i, -1, -i\}$:

$$\left(8\sqrt{2}(1+i)\right)^{\frac{1}{4}} = \{\pm 2e^{\frac{\pi i}{16}}, \pm 2ie^{\frac{\pi i}{16}}\} = \{2e^{\frac{\pi i}{16}}, 2e^{\frac{9\pi i}{16}}, 2e^{-\frac{15\pi i}{16}}, 2e^{-\frac{7\pi i}{16}}\}$$

Evaluating these in rectangular form is messy but possible (see Exercise 7). In practice it is better to leave such expressions in polar form.

**Exercises 1.3** 1. Find the square roots of $-\sqrt{3}+i$ and express them in rectangular co-ordinates. (*Hint: you may find it useful that* $(\sqrt{3}-1)^2 = 4-2\sqrt{3}$)

2. Find the sixth roots of $i$ in polar co-ordinates. Which is the principal root?

3. Use the fact that the cube roots of unity are $1, \omega_3 = \frac{-1+\sqrt{3}i}{2}$ and $\omega_3^2 = \frac{-1-\sqrt{3}i}{2}$ to evaluate the cube roots of $-27$ in rectangular co-ordinates.

4. We previously found the fourth roots of 16. Use these to find the fourth roots of $-16$. Hence factorize the equation $z^4+16=0$ as a product of two quadratic equations with real coefficients.

5. If $\omega$ is an $n^{\text{th}}$ root of unity *other than 1,* prove that $\sum_{k=0}^{n-1} \omega^k = 0$.

   (*Hint: recall geometric series*)

6. (a) Suppose that $a, b, c \in \mathbb{C}$ with $a \neq 0$ and suppose that $z$ satisfies the quadratic equation $az^2+bz+c=0$. Prove the quadratic formula:

   $$z = \frac{-b+(b^2-4ac)^{1/2}}{2a}$$

   Note that $(b^2-4ac)^{1/2}$ is the *set* of square roots of $b^2-4ac$, so that this provides *two* solutions whenever $b^2-4ac \neq 0$.

   (b) Find the roots of the equation $iz^2+(1+i)z+3=0$ in rectangular form.

7. Use the half-angle formula $\cos^2\frac{\alpha}{2} = \frac{1}{2}(1+\cos\alpha)$ to explicitly evaluate $\cos\frac{\pi}{8}$ and then $\cos\frac{\pi}{16}$. Hence find an expression for the rectangular form of $\sqrt[4]{8\sqrt{2}(1+i)} = 2e^{\frac{\pi i}{16}}$ using only square roots.

8. Recall Example 1.18. Verify that the method in Definition 1.20 gives the same value for the principal square root $\sqrt{-5+12i}$.

   (*You'll need some trig identities...*)